Ontology Languages (COMP321)
Solutions for Exercise 3

1. Recall the syntax of the Description Logics $\mathcal{EL}$, DL-Lite, and $\mathcal{ALC}$. Suppose $\mathsf{Manager}$ and $\mathsf{Project}$ are concept names and $\mathsf{manages}$ is a role name.

   Let $\mathcal{E}$ be any of the following expressions:

   (a) $\top \sqsubseteq \bot$

   (b) $\exists \mathsf{manages}.\mathsf{Project} \sqsubseteq \mathsf{Manager}$

   (c) $\forall \mathsf{manages}.\mathsf{Project}$

   (d) $\exists \mathsf{Project}.\mathsf{manages}$

   (e) $\exists \mathsf{manages}^{-}.\top \sqsubseteq \mathsf{Project} \sqcup \mathsf{Department}$

   (f) $\mathsf{Manager} \sqsubseteq \exists \mathsf{manages}.\top$

   (g) $\mathsf{Manager} \sqsubseteq \exists \mathsf{manages}.\bot$

   (h) $(\geq 7\ \mathsf{manages}.\top) \sqsubseteq \mathsf{Manager}$

   (i) $(\geq 8\ \mathsf{manages}.\mathsf{Project}) \sqsubseteq \mathsf{Manager}$

   (j) $\forall \mathsf{manages}.\top \sqsubseteq \exists \mathsf{manages}.\mathsf{Project}$

   (k) $\exists \mathsf{manages}.\top \sqsubseteq (\geq 4\ \mathsf{manages}.\top)$.

   (l) $(\geq 4\ \mathsf{manages}.\top) \sqsubseteq \exists \mathsf{manages}.\top$.

   - Translate $\mathcal{E}$ into natural language;
   - State whether
     - it is a $\mathcal{EL}$ concept;
     - a $\mathcal{EL}$ concept inclusion;
     - a DL-Lite concept
     - a DL-Lite concept inclusion
     - a $\mathcal{ALC}$ concept
     - a $\mathcal{ALC}$ concept inclusion.
     - none of the above.

- if $\mathcal{E}$ is a concept inclusion, check whether $\mathcal{E}$ follows from the empty TBox (i.e., $\emptyset \models \mathcal{E}$). If this is not the case, define an interpretation $\mathcal{I}$ such that $\mathcal{I} \not\models \mathcal{E}$.
- if $\mathcal{E}$ is a concept, check whether $\mathcal{E}$ is satisfiable. If this is the case, define an interpretation $\mathcal{I}$ such that $\mathcal{E}^{\mathcal{I}} \neq \emptyset$.

2. Show that every $\mathcal{EL}$-TBox is satisfiable.

Solution for 1:

(a) $\top \sqsubseteq \bot$ says "the domain is empty". It is a DL-Lite and $\mathcal{ALC}$ concept inclusion and nothing else. $\emptyset \not\models \top \sqsubseteq \bot$; in fact, one can take an arbitrary interpretation $\mathcal{I}$ and $\mathcal{I} \not\models \top \sqsubseteq \bot$ holds because $\Delta^{\mathcal{I}} \neq \emptyset$.

(b) $\exists \mathsf{manages}.\mathsf{Project} \sqsubseteq \mathsf{Manager}$ says "everybody who manages a project is a manager". It is a $\mathcal{EL}$ and $\mathcal{ALC}$ concept inclusion and nothing else. The empty TBox does not imply

$$\exists \mathsf{manages}.\mathsf{Project} \sqsubseteq \mathsf{Manager}$$

To show this consider the interpretation $\mathcal{I}$ defined by

- $\Delta^{\mathcal{I}} = \{a\}$;
- $\mathsf{manages}^{\mathcal{I}} = \{(a, a)\}$;
- $\mathsf{Project}^{\mathcal{I}} = \{a\}$;
- $\mathsf{Manager}^{\mathcal{I}} = \emptyset$.

Then $a \in (\exists \mathsf{manages}.\mathsf{Project})^{\mathcal{I}}$ but $a \notin \mathsf{Manager}^{\mathcal{I}}$. So

$$\mathcal{I} \not\models \exists \mathsf{manages}.\mathsf{Project} \sqsubseteq \mathsf{Manager}$$

(c) $\forall \mathsf{manages}.\mathsf{Project}$ denotes the class of objects which "only manage projects, and possibly nothing". It is a $\mathcal{ALC}$ concept and nothing else. It is satisfiable. An interpretation $\mathcal{I}$ with $(\forall \mathsf{manages}.\mathsf{Project})^{\mathcal{I}} \neq \emptyset$ is given by

- $\Delta^{\mathcal{I}} = \{a\}$;
- $\mathsf{manages}^{\mathcal{I}} = \{(a, a)\}$;

- $\mathsf{Project}^\mathcal{I} = \{a\}$.

Another interpretation $\mathcal{J}$ with $(\forall \mathsf{manages}.\mathsf{Project})^\mathcal{J} \neq \emptyset$ is given by

- $\Delta^\mathcal{I} = \{a\}$;
- $\mathsf{manages}^\mathcal{I} = \emptyset$;
- $\mathsf{Project}^\mathcal{I} = \emptyset$.

(d) $\exists \mathsf{Project}.\mathsf{manages}$ is none of the above because one cannot write a concept names directly after $\exists$.

(e) $\exists \mathsf{manages}^-.\top \sqsubseteq \mathsf{Project} \sqcup \mathsf{Department}$ states "everything that is managed is a project or a department". It is a DL-Lite with "or" concept inclusion and nothing else.

$$\exists \mathsf{manages}^-.\top \sqsubseteq \mathsf{Project} \sqcup \mathsf{Department}$$

does not follow from the empty TBox. An interpretation $\mathcal{I}$ in which it is false is given by

- $\Delta^\mathcal{I} = \{a\}$;
- $\mathsf{manages}^\mathcal{I} = \{(a, a)\}$;
- $\mathsf{Project}^\mathcal{I} = \emptyset$;
- $\mathsf{Department}^\mathcal{I} = \emptyset$.

A possibly more intuitive interpretation $\mathcal{J}$ in which it is false is given by

- $\Delta^\mathcal{I} = \{a, b\}$;
- $\mathsf{manages}^\mathcal{I} = \{(a, b)\}$;
- $\mathsf{Project}^\mathcal{I} = \emptyset$;
- $\mathsf{Department}^\mathcal{I} = \emptyset$.

Then $b \in (\exists \mathsf{manages}^{-}.\top)^{\mathcal{J}}$ but $b \notin (\mathsf{Project} \sqcup \mathsf{Department})^{\mathcal{J}}$.

(f) $\mathsf{Manager} \sqsubseteq \exists \mathsf{manages}.\top$ states "every manager manages something". It is a $\mathcal{EL}$, DL-Lite, and $\mathcal{ALC}$ concept inclusion and nothing else.

$$\mathsf{Manager} \sqsubseteq \exists \mathsf{manages}.\top$$

does not follow from the empty TBox. An interpretation $\mathcal{I}$ showing this is given by

- $\Delta^{\mathcal{I}} = \{a\}$;
- $\mathsf{Manager}^{\mathcal{I}} = \{a\}$;
- $\mathsf{manages}^{\mathcal{I}} = \emptyset$.

Then $a \in \mathsf{Manager}^{\mathcal{I}}$, but $a \notin (\exists \mathsf{manages}.\top)^{\mathcal{I}}$.

(g) $\mathsf{Manager} \sqsubseteq \exists \mathsf{manages}.\bot$ states "every Manager manages nothing". It is a $\mathcal{ALC}$ concept inclusion and nothing else.

$$\mathsf{Manager} \sqsubseteq \exists \mathsf{manages}.\bot$$

does not follow from the empty TBox. An interpretation $\mathcal{I}$ showing this is given by

- $\Delta^{\mathcal{I}} = \{a, b\}$;
- $\mathsf{Manager}^{\mathcal{I}} = \{a\}$;
- $\mathsf{manages}^{\mathcal{I}} = \{(a, b)\}$.

Then $a \in \mathsf{Manager}^{\mathcal{I}}$, but $a \notin (\exists \mathsf{manages}.\bot)^{\mathcal{I}}$.

In fact, $\exists \mathsf{manager}.\bot$ is unsatisfiable (always empty!), and so no interpretation in which "Manager" is non-empty can be a model of $\mathsf{Manager} \sqsubseteq \exists \mathsf{manages}.\bot$.

(h) $(\geq 7\ \mathsf{manages}.\top) \sqsubseteq \mathsf{Manager}$ states "if something manages at least seven things, then it is a manager". It is a DL-Lite concept inclusion and nothing else. $(\geq 7\ \mathsf{manages}.\top) \sqsubseteq \mathsf{Manager}$ does not follow from the empty TBox. An interpretation $\mathcal{I}$ showing this is given by

- $\Delta^{\mathcal{I}} = \{a, b_1, \ldots, b_7\}$;

- $\mathsf{Manager}^{\mathcal{I}} = \emptyset$;
- $\mathsf{manages}^{\mathcal{I}} = \{(a, b_1), \ldots, (a, b_7)\}$.

(i) $(\geq 8\ \mathsf{manages.Project}) \sqsubseteq \mathsf{Manager}$ states "if something manages at least eight projects, then it is a manager". It is none of the above. However, it is a concept inclusion in an extension of $\mathcal{ALC}$. The semantics should be clear. $(\geq 8\ \mathsf{manages.Project}) \sqsubseteq \mathsf{Manager}$ does not follow from the empty TBox. An interpretation $\mathcal{I}$ showing this is given by

- $\Delta^{\mathcal{I}} = \{a, b_1, \ldots, b_8\}$;
- $\mathsf{Manager}^{\mathcal{I}} = \emptyset$;
- $\mathsf{manages}^{\mathcal{I}} = \{(a, b_1), \ldots, (a, b_8)\}$.
- $\mathsf{Project}^{\mathcal{I}} = \{b_1, \ldots, b_8\}$.

(j) $\forall \mathsf{manages}.\top \sqsubseteq \exists \mathsf{manages.Project}$ states "everything manages a project". (The reason is that $(\forall \mathsf{manages}.\top)^{\mathcal{I}} = \Delta^{\mathcal{I}}$ for all interpretations $\mathcal{I}$!) It is a $\mathcal{ALC}$ concept inclusion and nothing else. It does not follow from the empty TBox. An interpretation $\mathcal{I}$ showing this is given by

- $\Delta^{\mathcal{I}} = \{a\}$;
- $\mathsf{manages}^{\mathcal{I}} = \emptyset$;
- $\mathsf{Project}^{\mathcal{I}} = \emptyset$

(k) $\exists \mathsf{manages}.\top \sqsubseteq (\geq 4\ \mathsf{manages}.\top)$ states "everybody managing something manages at least four things". It is a DL-Lite concept inclusion and nothing else. It does not follow from the empty TBox. An interpretation $\mathcal{I}$ showing this is given by

- $\Delta^{\mathcal{I}} = \{a, b\}$;
- $\mathsf{manages}^{\mathcal{I}} = \{(a, b)\}$.

Then $a \in (\exists \mathsf{manages}.\top)^{\mathcal{I}}$ but $a \notin (\geq 4\ \mathsf{manages}.\top)^{\mathcal{I}}$.

(l) $(\geq 4\ \mathsf{manages}.\top) \sqsubseteq \exists \mathsf{manages}.\top$ states "everybody managing at least four things manages something". It is a DL-Lite concept inclusion and nothing else. It follows from the empty TBox.

Solution for 2: Assume $\mathcal{T}$ is an $\mathcal{EL}$-TBox. Define an interpretation $\mathcal{I}$ by setting

- $\Delta^{\mathcal{I}} = \{a\}$;
- $A^{\mathcal{I}} = \{a\}$ for all concept names A;
- $r^{\mathcal{I}} = \{(a, a)\}$ for all role names r.

We have $C^{\mathcal{I}} = \{a\}$ for all $\mathcal{EL}$-concepts C. Hence $\mathcal{I} \models C \sqsubseteq D$ holds for all $\mathcal{EL}$-concept inclusions $C \sqsubseteq D$. Hence $\mathcal{I}$ is a model of $\mathcal{T}$.