INSTRUCTIONS TO CANDIDATES

Answer ALL questions.
1. Transform the following two formulas into CNF (conjunctive normal form).

• \(((\neg p \land \neg q) \lor r)\);
• \((\neg\neg(p \lor q) \land \neg(p \land q))\).

In your answer, state which laws (de Morgan, distributive law, complement law, etc) are applied.

(16 marks)

2. Consider the formulas

• \(P = (\neg(q \land p) \land (\neg p \land \neg q))\).
• \(Q = (\neg p \land (p \lor \neg q))\).

Check satisfiability of both \(P\) and \(Q\) using (a) truth tables and (b) the tableau algorithm from the lecture notes. In your answer for (b), show how the rules \(\Rightarrow \land\), \(\Rightarrow \lor\), \(\Rightarrow \neg\), \(\Rightarrow \neg\land\) and \(\Rightarrow \neg\lor\) from the lecture notes are applied.

(24 marks)

3. Let \(S\) be the signature consisting of the unary predicate symbols City and Country, the binary predicate symbol located_in, and the individual constants Liverpool, Madrid, Spain, UK. Translate the following sentences into first-order predicate logic sentences over \(S\):

• Liverpool is a city.
• Liverpool is not a country.
• Madrid is located in Spain.
• Liverpool is located in a country.
• Every city is located somewhere.
• Every city is located in a country.
• Some city is located in Spain.
• Nothing is located in Spain and located in the UK.

(25 marks)

4. Let \(S = \{R, Q, c\}\) be a signature, where \(R\) is a binary predicate symbol, \(Q\) is a unary predicate symbol, and \(a, b\) are individual constants. Let

\[\mathcal{F} = (D^F, R^F, Q^F, a^F, b^F)\]

be the \(S\)-structure defined by

• \(D^F = \{c, d, e, f\}\);
• \(R^F = \{(c, d), (d, e), (e, f)\}\);
• \(Q^F = \{d, f\}\);
• \(a^F = c\);
• $b^F = d$.

(a) Does $F \models \neg Q(a)$ hold? Explain your answer.
(b) Does $F \models \neg R(a, b)$ hold? Explain your answer.
(c) Does $F \models \exists x. Q(x)$ hold? Explain your answer.
(d) Does $F \models \forall x. Q(x)$ hold? Explain your answer.
(e) Does $F \models \forall x. (Q(x) \rightarrow \exists y. R(y, x))$ hold? Explain your answer.
(f) Does $F \models \forall x. (R(x, x) \rightarrow Q(x))$ hold? Explain your answer.
(g) Does $F \models \forall x. \exists y. R(x, y)$ hold? Explain your answer.
(h) Does $F \models \exists x. \exists y. \exists z. (R(x, y) \land R(y, z))$ hold? Explain your answer.

(20 marks)

5. Let $P$ and $Q$ be unary predicate symbols. Which of the following two statements are true:

- $\{ \forall x. (P(x) \land Q(x)) \} \models (\forall x. P(x) \land \forall x. Q(x))$;
- $\{ \forall x. (P(x) \lor Q(x)) \} \models (\forall x. P(x) \lor \forall x. Q(x))$.

Explain your answers by providing either a proof of the statement or a counterexample to the statement. (10 marks)

6. We define a function $F$ from the set of propositional formulas to the set of natural numbers by structural induction as follows:

- $F(p_i) = 0$ for all atomic formulas $p_i$;
- $F(P \land Q) = 1 + F(P) + F(Q)$;
- $F(P \lor Q) = 0$;
- $F(\neg P) = 0$.

Describe the function $F$ in your own words. (5 marks)
Solutions

1. The transformations are as follows:
   - By complement law
     \[ ((\neg p \land \neg q) \lor r) \equiv ((\neg p \land q) \lor r). \]
   - By distributive law
     \[ ((\neg p \land q) \lor r) \equiv ((\neg p \lor r) \land (q \lor r)) \]
     which is in CNF.
   - By complement law,
     \[ ((\neg \neg p) \land \neg (p \land q)) \equiv ((\neg (p \lor q) \land (p \land q)). \]
   - By de Morgan,
     \[ ((\neg p \lor q) \land \neg (p \land q)) \equiv ((\neg (p \land q) \land (p \lor q)). \]
   - By de Morgan,
     \[ ((\neg p \land \neg q) \land \neg (p \land q)) \equiv ((\neg p \land \neg q) \land (p \lor \neg q)). \]
     which is in CNF.

2. (a) We start with the truth tables:

\[
<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>q \land p</th>
<th>\neg (q \land p)</th>
<th>\neg q</th>
<th>\neg p</th>
<th>\neg (p \land q)</th>
<th>\neg (\neg p \land \neg q)</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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\]

It follows that \( P \) is satisfiable.

For the tableau, let \( S_0 = \{ P \} \), where \( P = (\neg (q \land p) \land (\neg p \land \neg q)). \) The rule applications are as follows:
- An application of \( \Rightarrow \) to \( S_0 \) gives
  \[ S_1 = S_0 \cup \{ \neg (q \land p), \neg (\neg p \land \neg q) \}. \]
- One possible application of \( \Rightarrow \) to \( S_1 \) and \( (q \land p) \) gives:
  \[ S_2 = S_1 \cup \{ \neg q \} \]
- One possible application of \( \Rightarrow \) to \( S_2 \) and \( (\neg p \land \neg q) \) gives:
  \[ S_3 = S_2 \cup \{ \neg p \} \]
An application of $\implies \neg$ to $S_3$ gives:

$$S_4 = S_3 \cup \{p\}$$

$S_4$ contains no clash and is complete. Thus $P$ is satisfiable.

(b) We start with the truth tables:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$\neg p$</th>
<th>$\neg q$</th>
<th>$(p \lor \neg q)$</th>
<th>$Q$</th>
</tr>
</thead>
<tbody>
<tr>
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</table>

It follows that $Q$ is satisfiable. For the tableau, let $S_0 = \{Q\}$, where $Q = (\neg p \land (p \lor \neg q))$. The rule applications are as follows:

- An application of $\implies \land$ to $S_0$ gives

$$S_1 = S_0 \cup \{\neg p, (p \lor \neg q)\}.$$  

- One possible application of $\implies \lor$ to $S_1$ gives:

$$S_2 = S_1 \cup \{p\}$$

which contains a clash

- The other possible application of $\implies \neg \lor$ to $S_1$ gives:

$$S_3 = S_1 \cup \{\neg q\}$$

$S_3$ does not contain a clash and is complete. Thus $Q$ is satisfiable.

3. The translations are as follows:

- Liverpool is a city: $\text{City}(Liverpool)$
- Liverpool is not a country: $\neg \text{Country}(Liverpool)$
- Madrid is located in Spain: $\text{located\_in}(\text{Madrid, Spain})$
- Liverpool is located in a country: $\exists x.(\text{located\_in}(\text{Liverpool, } x) \land \text{Country}(x))$
- Every city is located in somewhere: $\forall x.(\text{City}(x) \rightarrow \exists y.\text{located\_in}(x, y))$
- Every city is located in a country: $\forall x.(\text{City}(x) \rightarrow \exists y.(\text{Country}(y) \land \text{located\_in}(x, y)))$
- Some city is located in Spain: $\exists x.(\text{City}(x) \land \text{located\_in}(x, \text{Spain}))$
- Nothing is located in Spain and located in the UK: $\neg \exists x.(\text{located\_in}(x, \text{Spain}) \land \text{located\_in}(x, \text{UK}))$.

4. (a) $\mathcal{F} \models \neg Q(a)$ holds since $a^\mathcal{F} = c \notin Q^\mathcal{F}$.

(b) $\mathcal{F} \models \neg R(a, b)$ does not hold since $(a^\mathcal{F}, b^\mathcal{F}) = (c, d) \in R^\mathcal{F}$.

(c) $\mathcal{F} \models \exists x.Q(x)$ holds since $Q^\mathcal{F} \neq \emptyset$.

(d) $\mathcal{F} \models \forall x.Q(x)$ does not hold since $Q^\mathcal{F} \neq D^\mathcal{F}$.
(e) \( \mathcal{F} \models \forall x. (Q(x) \rightarrow \exists y. R(y, x)) \) holds since for all members \( v \) of \( Q^\mathcal{F} \) (namely \( d, f \)) there exists an elements \( w \) of \( D^\mathcal{F} \) (namely \( c \) and \( e \), respectively) such that \( (w, v) \in R^\mathcal{F} \).

(f) \( \mathcal{F} \models \forall x. (R(x, x) \rightarrow Q(x)) \) holds since there does not exist any \( d \) with \( (d, d) \in R^\mathcal{F} \).

(g) \( \mathcal{F} \models \exists x. \exists y. R(x, y) \) does not hold since there exists no \( w \) with \( (f, w) \in R^\mathcal{F} \).

(h) \( \mathcal{F} \models \exists x. \exists y. \exists z. (R(x, y) \land R(y, z)) \) holds: use \( a(x) = c, a(y) = d, a(z) = e \). Then \( (\mathcal{F}, a) \models (R(x, y) \land R(y, z)) \).

5. Let \( P \) and \( Q \) be unary predicate symbols. Which of the following two statements are true:

- \( \{ \forall x. (P(x) \land Q(x)) \} \models \forall x. P(x) \land \forall x. Q(x) \) holds: let \( \mathcal{F} \) be a structure such that \( \mathcal{F} \models \forall x. (P(x) \land Q(x)) \). Then \( P^\mathcal{F} \cap Q^\mathcal{F} = D^\mathcal{F} \). Hence \( P^\mathcal{F} = D^\mathcal{F} \) and \( Q^\mathcal{F} = D^\mathcal{F} \). Hence \( \mathcal{F} \models \forall x. P(x) \land \forall x. Q(x) \), as required.

- \( \{ \forall x. (P(x) \lor Q(x)) \} \models (\forall x. P(x) \lor \forall x. Q(x)) \) does not hold. Let \( \mathcal{F} = (D^\mathcal{F}, P^\mathcal{F}, Q^\mathcal{F}) \) be defined by setting
  - \( D^\mathcal{F} = \{1, 2\} \);
  - \( P^\mathcal{F} = \{1\} \);
  - \( Q^\mathcal{F} = \{2\} \).

Then \( \mathcal{F} \models (\forall x. P(x) \lor \forall x. Q(x)) \) and \( \mathcal{F} \not\models (\forall x. P(x) \lor \forall x. Q(x)) \), as required.

6. We define a function \( F \) from the set of propositional formulas to the set of natural numbers by structural induction as follows:

- \( F(p_i) = 0 \) for all atomic formulas \( p_i \);
- \( F(P \land Q) = 1 + F(P) + F(Q) \);
- \( F(P \lor Q) = 0 \);
- \( F(\neg P) = 0 \).

Describe the function \( F \) in your own words: \( F(P) \) is the number of occurrences of \( \land \) in \( P \) that are not within the scope of \( \lor \) or \( \neg \).