1. Let \( S = \{ P, R, c \} \) be a signature in which \( P \) is a unary predicate symbol, \( R \) is a binary predicate symbol, and \( c \) is an individual constant. For which definition of 
\[
F = (D^F, P^F, R^F, c^F)
\]
does one obtain an \( S \)-structure:

(a) \( D^F = \{ a, b, c \}, P^F = \{ a \}, R^F = \{ (a, b) \}, c^F = \emptyset \).
(b) \( D^F = \{ a, b, c \}, P^F = \emptyset, R^F = \{ (a, b), (a, a) \}, c^F = a \).
(c) \( D^F = \{ a, b, c \}, P^F = \{ a \}, R^F = \emptyset, c^F = b \).
(d) \( D^F = \{ a, b, c \}, P^F = \{ a \}, R^F = \{ (a, b), (c, d) \}, c^F = a \).
(e) \( D^F = \{ a, b, c \}, P^F = \{ a \}, R^F = \{ (a, b), (a, a) \}, c^F = \{ a \} \).
(f) \( D^F = \{ a, b, c \}, P^F = \{ a \}, R^F = \{ a \}, c^F = a \).

2. Let \( S = \{ P, R, c_1, c_2, c_3, c_4 \} \) be a signature in which \( P \) is a unary predicate symbol, \( R \) is a binary predicate symbol, and \( c_1, c_2, c_3, c_4 \) are individual constants. Let 
\[
F = (D^F, P^F, R^F, c_1^F, c_2^F, c_3^F, c_4^F)
\]
be an \( S \)-structure with

- \( D^F = \{ a, b, c, d \} \),
- \( P^F = \{ a, b \} \),
- \( R^F = \{ (a, b), (c, d) \} \),
- \( c_1^F = a \);
- \( c_2^F = b \);
- \( c_3^F = c \);
- \( c_4^F = d \).

Which of the following statements are true?
(a) $\mathcal{F} \models P(c_1)$;
(b) $\mathcal{F} \models P(c_3)$;
(c) $\mathcal{F} \models R(c_1, c_3)$;
(d) $\mathcal{F} \models \neg R(c_1, c_3)$;
(e) $\mathcal{F} \models R(c_4, c_3)$;
(f) $\mathcal{F} \models R(c_1, c_2) \lor P(c_3)$;

3. Let $P_1$ be a unary predicate symbol, $P_2$ a binary predicate symbols, and $c_1, c_2$ individual constants. Let $S = \{P_1, P_2, c_1, c_2\}$ be a signature.

- Define an $S$-structure $\mathcal{F}$ such that $\mathcal{F} \models (P_1(c_1) \land \neg P_1(c_2))$;
- Define an $S$-structure $\mathcal{F}$ such that $\mathcal{F} \models (P_2(c_1, c_1) \land \neg P_1(c_1))$.
- Define an $S$-structure $\mathcal{F}$ such that $\mathcal{F} \models (P_2(c_1, c_2) \land P_2(c_2, c_1))$.

4. Database instances correspond to finite structures. Let $S = \{\text{manager of}, \text{player of}\}$ be a signature in which $\text{manager of}$ and $\text{player of}$ are binary predicate symbols. Consider the following table for $\text{manager of}$:

<table>
<thead>
<tr>
<th>Rodgers</th>
<th>LFC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Moyes</td>
<td></td>
</tr>
<tr>
<td>Ferguson</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Everton</th>
<th>ManU</th>
</tr>
</thead>
</table>

and the following table for $\text{player of}$:

<table>
<thead>
<tr>
<th>Reina</th>
<th>LFC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rooney</td>
<td></td>
</tr>
<tr>
<td>Cahill</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>LFC</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>ManU</td>
<td></td>
</tr>
<tr>
<td>Everton</td>
<td></td>
</tr>
</tbody>
</table>

Represent both tables using a single $S$-structure $\mathcal{F}$. 
5. Similar to propositional logic, many functions \( F \) which have as domain the set of ground sentences can be defined by structural induction. To do so, one specifies

- \( F(P(c_1, \ldots, c_n)) \), for all atomic sentences \( P(c_1, \ldots, c_n) \);
- \( F(G_1 \land G_2) \), given the values \( F(G_1) \) and \( F(G_2) \);
- \( F(G_1 \lor G_2) \), given the values \( F(G_1) \) and \( F(G_2) \);
- \( F(\neg G) \), given the value \( F(G) \).

(a) Define the set \( \text{sub}(G) \) of subsentences of a ground sentence \( G \) by structural induction.

(b) Define the set \( \text{Const}(G) \) of constant symbols that occur in a ground sentence \( G \) by structural induction.

(c) Let \( c \) be a constant symbol. Define the number of occurrences \( n_c(G) \) of \( c \) in \( G \) by structural induction.

6. (Advanced) Encoding ground sentences into propositional logic

(a) Define, for a ground sentence \( G \), the propositional formula \( G^{pro} \) by structural induction (introduced on slide 25).

(b) With the definitions from slide 26: show by structural induction for all ground sentences \( E \) over \( S \):

\[
I_F(E^{pro}) = 1 \iff F \models E.
\]

(c) With the definitions from slide 27: show by structural induction for all ground sentences \( E \) over \( S \):

\[
I(E^{pro}) = 1 \iff F_I \models E.
\]

Solution for 1.

1. \( D^F = \{a, b, c\} \), \( P^F = \{a\} \), \( R^F = \{(a, b)\} \), \( c^F = \emptyset \) does not define an \( F \)-structure since \( c^F \) is not an element of \( D^F \).
2. \(D^F = \{a, b, c\}, P^F = \emptyset, R^F = \{(a, b), (a, a)\}, c^F = a\) defines an \(F\)-structure.

3. \(D^F = \{a, b, c\}, P^F = \{a\}, R^F = \emptyset, c^F = b\) defines an \(F\)-structure.

4. \(D^F = \{a, b, c\}, P^F = \{a\}, R^F = \{(a, b), (c, d)\}, c^F = a\) does not define an \(F\)-structure since \(R^F\) is not a subset of \(D^F \times D^F\).

5. \(D^F = \{a, b, c\}, P^F = \{a\}, R^F = \{(a, b), (a, a)\}, c^F = \{a\}\) does not define an \(F\)-structure since \(c^F\) is not an element of \(D^F\).

6. \(D^F = \{a, b, c\}, P^F = \{a\}, R^F = \{a\}, c^F = a\) does not define an \(F\)-structure since \(R^F\) is not a subset of \(D^F \times D^F\).

**Solution for 2.**

1. \(F \models P(c_1)\) holds since \(c_1^F = a \in P^F\);

2. \(F \models P(c_3)\) does not hold since \(c_3^F = c \notin P^F\);

3. \(F \models R(c_1, c_3)\) does not hold since \((c_1^F, c_3^F) = (a, c) \notin R^F\);

4. \(F \models \neg R(c_1, c_3)\) holds since \(F \nmodels R(c_1, c_3)\);

5. \(F \models R(c_4, c_3)\) does not hold since \((c_4^F, c_3^F) = (d, c) \notin R^F\);

6. \(F \models R(c_1, c_2) \lor P(c_3)\) holds since \((c_1^F, c_2^F) = (a, b) \in R^F\);

**Solution for 3.**

There are many \(S\)-structures satisfying the conditions. We always give one.

- Let \(F = (D^F, P_1^F, P_2^F, c_1^F, c_2^F)\) be defined by setting
  - \(D^F = \{a, b, c, d\}\);
  - \(P_1^F = \{a, b\}\);
  - \(P_2^F = \{(a, b)\}\);
\[c_1^F = a;\]
\[c_2^F = c.\]

Then \( \mathcal{F} \models P_1(c_1) \) since \( c_1^F = a \in P_1^F \). \( \mathcal{F} \not\models P_1(c_2) \) since \( c_2^F = c \notin P_1^F \). Thus, \( \mathcal{F} \models (P_1(c_1) \land \neg P_1(c_2)) \), as required.

- Let \( \mathcal{F} = (D^F, P_1^F, P_2^F, c_1^F, c_2^F) \) be defined by setting
  - \( D^F = \{a, b, c, d\} \);
  - \( P_1^F = \{b\} \);
  - \( P_2^F = \{(a, a)\} \);
  - \( c_1^F = a;\)
  - \( c_2^F = c.\)

Then \( \mathcal{F} \models P_2(c_1, c_1) \) since \( c_1^F = a \) and \( (a, a) \in P_2^F \). \( \mathcal{F} \not\models P_1(c_1) \) since \( c_1^F = a \notin P_1^F \). Thus, \( \mathcal{F} \models (P_2(c_1, c_1) \land \neg P_1(c_1)) \), as required.

- Let \( \mathcal{F} = (D^F, P_1^F, P_2^F, c_1^F, c_2^F) \) be defined by setting
  - \( D^F = \{a, b, c, d\} \);
  - \( P_1^F = \emptyset \);
  - \( P_2^F = \{(c, d), (d, c)\} \);
  - \( c_1^F = c;\)
  - \( c_2^F = d.\)

Then \( \mathcal{F} \models P_2(c_1, c_2) \) since \( c_1^F = c \) and \( c_2^F = d \) and \( (c, d) \in P_2^F \). \( \mathcal{F} \models P_2(c_2, c_1) \) since \( (d, c) \in P_2^F \). Thus, \( \mathcal{F} \models (P_2(c_1, c_2) \land P_2(c_2, c_1)) \), as required.

Solution for 4

The \( S \)-structure \( \mathcal{F} \) corresponding to the database instance is defined by setting

\[ \mathcal{F} = (D^F, \text{manager_of}^F, \text{player_of}^F) \]

where

\[ \text{manager_of}^F = \{(\text{Rodgers, LFC}), (\text{Moyes, Everton}), (\text{Ferguson, ManU})\} \]
and

\[ \text{player_of}^T = \{(\text{Reina, LFC}), (\text{Rooney, ManU}), (\text{Cahill, Everton})\} \]

Important difference to example from the lecture notes: here we do not have individual constants in the signature \( S \). This is relevant since there are no ground sentences over \( S \), but there are a lot of ground sentences of \( S \) extended by the individual constants for Rodgers, LFC, etc.

**Solution for 5.**

The set of subsentences \( \text{sub}(G) \) of a sentence \( G \) can be defined by structural induction as follows:

- \( \text{sub}(P(c_1, \ldots, c_n)) = \{P(c_1, \ldots, c_n)\} \), for every ground sentence \( P(c_1, \ldots, c_n) \),
- \( \text{sub}(G_1 \land G_2) = \{(G_1 \land G_2)\} \cup \text{sub}(G_1) \cup \text{sub}(G_2) \),
- \( \text{sub}(G_1 \lor G_2) = \{(G_1 \lor G_2)\} \cup \text{sub}(G_1) \cup \text{sub}(G_2) \),
- \( \text{sub}(\neg G) = \{\neg G\} \cup \text{sub}(G) \).

The set \( \text{Const}(G) \) can be defined by structural induction as follows:

- \( \text{Const}(P(c_1, \ldots, c_n)) = \{c_1, \ldots, c_n\} \), for every ground sentence \( P(c_1, \ldots, c_n) \),
- \( \text{Const}(G_1 \land G_2) = \text{Const}(G_1) \cup \text{Const}(G_2) \),
- \( \text{Const}(G_1 \lor G_2) = \text{Const}(G_1) \cup \text{Const}(G_2) \),
- \( \text{Const}(\neg G) = \text{Const}(G) \).

The set \( n_c(G) \) can be defined by structural induction as follows:

- \( n_c(P(c_1, \ldots, c_n)) = |\{i \mid c = c_i\}| \), for every ground sentence \( P(c_1, \ldots, c_n) \),
- \( n_c(G_1 \land G_2) = n_c(G_1) + n_c(G_2) \),
- \( n_C(G_1 \lor G_2) = n_C(G_1) + n_c(G_2) \),
• \( n_c(\neg G) = n_C(G) \).

**Solution for 6.**

For every atomic sentence \( P(c_1, \ldots, c_n) \) we fix an atomic formula \( p_{P(c_1,\ldots,c_n)} \). Now the inductive definition of \( G^{pro} \) is as follows:

- \( P(c_1, \ldots, c_n)^{pro} = p_{P(c_1,\ldots,c_n)} \) for all atomic sentences \( P(c_1, \ldots, c_n) \),
- \( (G_1 \land G_2)^{pro} = G_1^{pro} \land G_2^{pro} \),
- \( (G_1 \lor G_2)^{pro} = G_1^{pro} \lor G_2^{pro} \),
- \( (\neg G)^{pro} = \neg G^{pro} \).

We now show
\[
I_{\mathcal{F}}(E^{pro}) = 1 \iff \mathcal{F} \models E.
\]
by structural induction.

- If \( E \) is an atomic sentence, this is true by definition of \( I_{\mathcal{F}} \).
- If \( E = G_1 \land G_2 \), then
\[
I_{\mathcal{F}}(E^{pro}) = 1 \iff I_{\mathcal{F}}(G_1) = 1 \text{ and } I_{\mathcal{F}}(G_2) = 1
\]
\[
\iff \mathcal{F} \models G_1 \text{ and } \mathcal{F} \models G_2 \text{ (by induction hypothesis)}
\]
\[
\iff \mathcal{F} \models G_1 \land G_2
\]
\[
\iff \mathcal{F} \models E
\]
- If \( E = G_1 \lor G_2 \), then
\[
I_{\mathcal{F}}(E^{pro}) = 1 \iff I_{\mathcal{F}}(G_1) = 1 \text{ or } I_{\mathcal{F}}(G_2) = 1
\]
\[
\iff \mathcal{F} \models G_1 \text{ or } \mathcal{F} \models G_2 \text{ (by induction hypothesis)}
\]
\[
\iff \mathcal{F} \models G_1 \lor G_2
\]
\[
\iff \mathcal{F} \models E
\]

•
• If $E = \neg G$, then

$$I_F(E^{pro}) = 1 \iff I_F(G) = 0 \
\iff F \not\models G \text{ (by induction hypothesis)} \
\iff F \models \neg G \
\iff F \models E$$

We show

$$I(E^{pro}) = 1 \iff F_I \models E.$$  

by structural induction.

• If $E$ is an atomic sentence, this is true by definition of $F_I$.

• If $E = G_1 \land G_2$, then

$$I(E^{pro}) = 1 \iff I(G_1) = 1 \text{ and } I(G_2) = 1 \
\iff F_I \models G_1 \text{ and } F_I \models G_2 \text{ (by induction hypothesis)} \
\iff F_I \models G_1 \land G_2 \
\iff F_I \models E$$

• If $E = G_1 \lor G_2$, then

$$I(E^{pro}) = 1 \iff I(G_1) = 1 \text{ or } I(G_2) = 1 \
\iff F_I \models G_1 \text{ or } F_I \models G_2 \text{ (by induction hypothesis)} \
\iff F_I \models G_1 \lor G_2 \
\iff F_I \models E$$

• If $E = \neg G$, then

$$I(E^{pro}) = 1 \iff I(G) = 0 \
\iff F_I \not\models G \text{ (by induction hypothesis)} \
\iff F_I \models \neg G \
\iff F_I \models E$$