1. (35 marks) Consider the formula

\[ P = (\neg p \land (p \lor (\neg q \land \neg p))). \]

Check whether \( P \) is satisfiable using (a) truth tables and (b) the tableau algorithm. When using the tableau algorithm, show how the rules \( \Rightarrow \land, \Rightarrow \lor, \Rightarrow \neg, \Rightarrow \neg \land \) and \( \Rightarrow \neg \lor \) from the lecture notes are applied.

2. (30 marks) Transform the following formulas into CNF (conjunctive normal form):

- \( \neg \neg ((p \land q) \lor r) \);
- \( (\neg p \land \neg \neg r) \);
- \( \neg (p \land (\neg p \lor r)) \).

In your answer, state which laws (de Morgan, distributive law, complement law, etc) are applied.

3. (20 marks) Let \( S = \{P, R, a, b, c\} \) be a signature in which \( P \) is a unary predicate symbol, \( R \) is a binary predicate symbol, and \( a, b, c \) are individual constant symbols. Let the \( S \)-structure

\[ \mathcal{F} = (D^\mathcal{F}, P^\mathcal{F}, R^\mathcal{F}, a^\mathcal{F}, b^\mathcal{F}, c^\mathcal{F}) \]

be defined by

- \( D^\mathcal{F} = \{0, 1, 2, 3\} \);
- \( P^\mathcal{F} = \{0, 1\} \);
- \( R^\mathcal{F} = \{(0,2), (2,3), (1,1)\} \);
- \( a^\mathcal{F} = 0 \);
- \( b^\mathcal{F} = 1 \);
- \( c^\mathcal{F} = 2 \).
Which of the following statements are true? Explain your answers.

(a) \( F \models P(a) \);
(b) \( F \models R(a, b) \);
(c) \( F \models \neg R(a, a) \);
(d) \( F \models P(c) \lor R(a, c) \);
(e) \( F \models P(b) \land R(b, c) \).

4. (15 marks) Consider the signature \( S = \{ P, R, a, b, c \} \) from the previous question. Let \( F = R(a, b) \land R(b, c) \land \neg P(c) \) be a ground sentence. Show that \( F \) is satisfiable. Do this by defining an \( S \)-structure

\[ G = (D^G, P^G, R^G, a^G, b^G, c^G) \]

such that \( G \models F \). (It remains to define \( D^G, P^G, R^G, a^G, b^G, \) and \( c^G \).)

Solution for 1.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( \neg p )</th>
<th>( \neg q )</th>
<th>( \neg q \land \neg p )</th>
<th>( p \lor (\neg q \land \neg p) )</th>
<th>( P )</th>
</tr>
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</tbody>
</table>

It follows that \( I(P) = 0 \) for the interpretation \( I \) with \( I(p) = 0 \) and \( I(q) = 0 \). Hence \( P \) is satisfiable.

Using the tableau algorithm, we set \( S_0 = \{ P \} \).

- An application of \( \implies \land \) to \( S_0 \) gives
  \[ S_1 = S_0 \cup \{ \neg p, (p \lor (\neg q \land \neg p)) \} \]

- One possible application of \( \implies \lor \) to \( S_1 \) gives
  \[ S_2 = S_1 \cup \{ p \} \]

\( S_2 \) contains the clash \( \{ p, \neg p \} \).
• The other possible application of \( \implies \lor \) to \( S_1 \) gives

\[
S_3 = S_1 \cup \{ (\neg q \land \neg p) \}
\]

• An application of \( \implies \land \) to \( S_3 \) gives

\[
S_4 = S_3 \cup \{ \neg q, \neg p \}
\]

\( S_4 \) is complete and contains no clash. It follows that \( P \) is satisfiable.

**Solution for 2.**

• By complement law,

\[
\neg\neg((p \land q) \lor r) \equiv ((p \land q) \lor r)
\]

By distributive law,

\[
((p \land q) \lor r) \equiv ((p \lor r) \land (q \lor r))
\]

This is in CNF.

• By complement law,

\[
(\neg p \land \neg\neg r) \equiv (\neg p \land r)
\]

which is in CNF.

• By de Morgan law,

\[
\neg(p \land (\neg p \lor r)) \equiv (\neg p \lor \neg(\neg p \lor r))
\]

By de Morgan law:

\[
(\neg p \lor \neg(\neg p \lor r)) \equiv (\neg p \lor (\neg\neg p \land \neg r))
\]

By complement law

\[
(\neg p \lor (\neg p \land \neg r)) \equiv (\neg p \lor (p \land \neg r))
\]

By distributive law:

\[
(\neg p \lor (p \land \neg r)) \equiv ((\neg p \lor p) \land (\neg p \lor \neg r))
\]

which is in CNF.
Solution for 3.

1. $\mathcal{F} \models P(a)$ holds since $a^F = 0 \notin P^F$
2. $\mathcal{F} \not\models R(a, b)$ does not hold since $(a^F, b^F) = (0, 1) \notin R^F$;
3. $\mathcal{F} \models \neg R(a, a)$ holds since $(a^F, a^F) = (0, 0) \notin R^F$;
4. $\mathcal{F} \models P(c) \lor R(a, c)$ holds since $(a^F, c^F) = (0, 2) \in R^F$;
5. $\mathcal{F} \not\models P(b) \land R(b, c)$ does not hold since $(b^F, c^F) = (1, 2) \notin R^F$.

Solution for 4.

There are many such $S$-structures. One option is:

- $D^G = \{0, 1, 2\}$;
- $P^F = \emptyset$;
- $R^F = \{(0, 1), (1, 2)\}$;
- $a^F = 0$;
- $b^F = 1$;
- $c^F = 2$. 