Logic in Computer Science (COMP118)

Class Test 1

1. **(35 marks)** Consider the formula

$$P = (\neg p \wedge (p \vee (\neg q \wedge \neg p))).$$

   Check whether P is satisfiable using (a) truth tables and (b) the tableau algorithm. When using the tableau algorithm, show how the rules $\Longrightarrow_{\wedge}$, $\Longrightarrow_{\vee}$, $\Longrightarrow_{\neg}$, $\Longrightarrow_{\neg\wedge}$ and $\Longrightarrow_{\neg\vee}$ from the lecture notes are applied.

2. **(30 marks)** Transform the following formulas into CNF (conjunctive normal form):

   - $\neg\neg((p \wedge q) \vee r)$;
   - $(\neg p \wedge \neg\neg r)$;
   - $\neg(p \wedge (\neg p \vee r))$.

   In your answer, state which laws (de Morgan, distributive law, complement law, etc) are applied.

3. **(20 marks)** Let $S = \{P, R, a, b, c\}$ be a signature in which P is a unary predicate symbol, R is a binary predicate symbol, and a, b, c are individual constant symbols. Let the S-structure

$$\mathcal{F} = (D^{\mathcal{F}}, P^{\mathcal{F}}, R^{\mathcal{F}}, a^{\mathcal{F}}, b^{\mathcal{F}}, c^{\mathcal{F}})$$

   be defined by

   - $D^{\mathcal{F}} = \{0, 1, 2, 3\}$;
   - $P^{\mathcal{F}} = \{0, 1\}$;
   - $R^{\mathcal{F}} = \{(0, 2), (2, 3), (1, 1)\}$;
   - $a^{\mathcal{F}} = 0$;
   - $b^{\mathcal{F}} = 1$;
   - $c^{\mathcal{F}} = 2$.

Which of the following statements are true? Explain your answers.

(a) $\mathcal{F} \models P(a)$;

(b) $\mathcal{F} \models R(a, b)$;

(c) $\mathcal{F} \models \neg R(a, a)$;

(d) $\mathcal{F} \models P(c) \lor R(a, c)$;

(e) $\mathcal{F} \models P(b) \land R(b, c)$.

4. **(15 marks)** Consider the signature $S = \{P, R, a, b, c\}$ from the previous question. Let $F = R(a, b) \land R(b, c) \land \neg P(c)$ be a ground sentence. Show that F is satisfiable. Do this by defining an S-structure

$$\mathcal{G} = (D^{\mathcal{G}}, P^{\mathcal{G}}, R^{\mathcal{G}}, a^{\mathcal{G}}, b^{\mathcal{G}}, c^{\mathcal{G}})$$

such that $\mathcal{G} \models F$. (It remains to define $D^{\mathcal{G}}$, $P^{\mathcal{G}}$, $R^{\mathcal{G}}$, $a^{\mathcal{G}}$, $b^{\mathcal{G}}$, and $c^{\mathcal{G}}$.)

Solution for 1.

p	q	$\neg p$	$\neg q$	$(\neg q \land \neg p)$	$p \lor (\neg q \land \neg p)$	P
1	1	0	0	0	1	0
1	0	0	1	0	1	0
0	1	1	0	0	0	0
0	0	1	1	1	1	1

It follows that $I(P) = 0$ for the interpretation I with $I(p) = 0$ and $I(q) = 0$. Hence P is satisfiable.

Using the tableau algorithm, we set $S_0 = \{P\}$.

- An application of $\Longrightarrow_\land$ to S_0 gives

$$S_1 = S_0 \cup \{\neg p, (p \lor (\neg q \land \neg p))\}$$

- One possible application of $\Longrightarrow_\lor$ to S_1 gives

$$S_2 = S_1 \cup \{p\}$$

S_2 contains the clash $\{p, \neg p\}$.

- The other possible application of $\Longrightarrow_\vee$ to S_1 gives

$$S_3 = S_1 \cup \{(\neg q \wedge \neg p)\}$$

- An application of $\Longrightarrow_\wedge$ to S_3 gives

$$S_4 = S_3 \cup \{\neg q, \neg p\}$$

  S_4 is complete and contains no clash. It follows that P is satisfiable.

Solution for 2.

- By complement law,

$$\neg\neg((p \wedge q) \vee r) \equiv ((p \wedge q) \vee r)$$

  By distributive law,

$$((p \wedge q) \vee r) \equiv ((p \vee r) \wedge (q \vee r))$$

  This is in CNF.

- By complement law,

$$(\neg p \wedge \neg\neg r) \equiv (\neg p \wedge r)$$

  which is in CNF.

- By de Morgan law,

$$\neg(p \wedge (\neg p \vee r)) \equiv (\neg p \vee \neg(\neg p \vee r))$$

  By de Morgan law:

$$(\neg p \vee \neg(\neg p \vee r)) \equiv (\neg p \vee (\neg\neg p \wedge \neg r))$$

  By complement law

$$(\neg p \vee (\neg\neg p \wedge \neg r)) \equiv (\neg p \vee (p \wedge \neg r))$$

  By distributive law:

$$(\neg p \vee (p \wedge \neg r)) \equiv ((\neg p \vee p) \wedge (\neg p \vee \neg r))$$

  which is in CNF.

Solution for 3.

1. $\mathcal{F} \models P(a)$ holds since $a^{\mathcal{F}} = 0 \in P^{\mathcal{F}}$
2. $\mathcal{F} \models R(a,b)$ does not hold since $(a^{\mathcal{F}}, b^{\mathcal{F}}) = (0,1) \notin R^{\mathcal{F}}$;
3. $\mathcal{F} \models \neg R(a,a)$ holds since $(a^{\mathcal{F}}, a^{\mathcal{F}}) = (0,0) \notin R^{\mathcal{F}}$;
4. $\mathcal{F} \models P(c) \vee R(a,c)$ holds since $(a^{\mathcal{F}}, c^{\mathcal{F}}) = (0,2) \in R^{\mathcal{F}}$;
5. $\mathcal{F} \models P(b) \wedge R(b,c)$ does not hold since $(b^{\mathcal{F}}, c^{\mathcal{F}}) = (1,2) \notin R^{\mathcal{F}}$.

Solution for 4.

There are many such S-structures. One option is:

- $D^{\mathcal{G}} = \{0,1,2\}$;
- $P^{\mathcal{F}} = \emptyset$;
- $R^{\mathcal{F}} = \{(0,1),(1,2)\}$;
- $a^{\mathcal{F}} = 0$;
- $b^{\mathcal{F}} = 1$;
- $c^{\mathcal{F}} = 2$.