1. (45 marks) Let $S$ be the signature consisting of the unary predicates $\text{researcher}, \text{teacher}, \text{module}$, the binary predicate symbol $\text{teaches}$ and the individual constants $\text{John}$ and $\text{Comp1}$. Translate the following sentences into first-order predicate logic:

(a) Comp1 is a module.
(b) Every researcher is a teacher.
(c) No researcher is a teacher.
(d) John teaches Comp1.
(e) Every teacher teaches a module.
(f) John teaches a module.
(g) Everybody who teaches a module is a teacher.
(h) Some researcher are not teachers.

2. (45 marks) Let $S = \{R, Q, w\}$ be a signature, where $R$ is a binary predicate symbol, $Q$ a unary predicate symbol, and $w$ an individual constant. Let

$$\mathcal{F} = (D^F, R^F, Q^F, w^F)$$

be defined by

- $D^F = \{a, b, c, d, e\}$;
- $R^F = \{(a, b), (b, c), (c, d), (d, e), (e, e)\}$;
- $Q^F = \{a, e\}$;
- $w^F = a$.

(a) Does $\mathcal{F} \models Q(w)$ hold? Explain your answer.
(b) Does $\mathcal{F} \models \forall x. Q(x)$ hold? Explain your answer.
(c) Does $\mathcal{F} \models \forall x. R(x, x)$ hold? Explain your answer.
(d) Does $F \models \exists x. R(x, x)$ hold? Explain your answer.

(e) Does $F \models \forall x. \exists y. R(x, y)$ hold? Explain your answer.

(f) Does $F \models \exists x. \forall y. R(x, y)$ hold? Explain your answer.

(g) Does $F \models \forall x. \exists y. R(y, x)$ hold? Explain your answer.

(h) Does $F \models \forall x. (\neg \exists y. R(y, x) \rightarrow Q(x))$ hold? Explain your answer.

3. **(10 marks)** Discuss the difference between the truth (or satisfaction) relation $F \models G$ between a structure $F$ and a sentence $G$ and the logical consequence relation $X \models G$ between a set $X$ of sentences and a sentence $G$.

**Solution for 1.**

(a) Comp1 is a module: `module(Comp1)`.

(b) Every researcher is a teacher: $\forall x. (\text{researcher}(x) \rightarrow \text{teacher}(x))$.

(c) No researcher is a teacher: $\forall x. (\text{researcher}(x) \rightarrow \neg \text{teacher}(x))$.

(d) John teaches Comp1: `$\text{teaches}(\text{John}, \text{Comp1})$`.

(e) Every teacher teaches a module:

$$\forall x. (\text{teacher}(x) \rightarrow \exists y. (\text{teaches}(x, y) \land \text{module}(y)))$$

(f) John teaches a module: $\exists x. (\text{teaches}(\text{John}, x) \land \text{module}(x))$.

(g) Everybody who teaches a module is a teacher:

$$\forall x. (\exists y. (\text{teaches}(x, y) \land \text{module}(y)) \rightarrow \text{teacher}(x))$$

(h) Some researchers are not teachers: $\exists x. (\text{researcher}(x) \land \neg \text{teacher}(x))$.

**Solution for 2.**

(a) $F \models Q(w)$ holds because $w^F = a \in Q^F$. 
(b) \( F \models \forall x.Q(x) \) does not hold. To show this take an assignment \( a \) with \( a(x) = b \). Then \( b \notin Q^F \) and so \((F, a) \not\models Q(x)\). Hence \( F \not\models \forall x.Q(x) \).

(c) \( F \models \forall x.R(x, x) \) does not hold. To show this take an assignment \( a \) with \( a(x) = b \). Then \((b, b) \notin R^F \) and so \((F, a) \not\models R(x, x)\). Hence \( F \not\models \forall x.R(x, x) \).

(d) \( F \models \exists x.R(x, x) \) holds. To show this let \( a(x) = e \). Then \((e, e) \in R^F \) and so \((F, a) \models R(x, x)\). Hence \( F \models \exists x.R(x, x) \).

(e) \( F \models \forall x.\exists y.R(x, y) \) holds. To show this let \( a(x) \) be an element of \( D^F \). Then there exists \( a(y) \) such that \((a(x), a(y)) \in R^F \). Intuitively, the sentence says that every element has an \( R \)-successor — which is the case.

(f) \( F \models \exists x.\forall y.R(x, y) \) does not hold since for every domain element \( a(x) \) there exists another domain element \( a(y) \) such that \((a(x), a(y)) \notin R^F \). For \( a \) one can choose \( c \), for \( b \) one can choose \( d \), for \( c \) one can choose \( e \), for \( d \) one can choose \( a \), and for \( e \) one can choose \( a \) as well (in each case there are many other options).

(g) \( F \models \forall x.\exists y.R(y, x) \) does not hold. To show that take an assignment \( a \) with \( a(x) = a \). Then there is no \( g \in D^F \) such that \((g, a(x)) \in R^F \). Intuitively, this means that \( a \) has no \( R \)-predecessor and so the sentence is false.

(h) \( F \models \forall x.(-\exists y.R(y, x) \rightarrow Q(x)) \) holds since \( a \) is the only element of \( D^F \) without a \( R \)-predecessor and \( a \) is a member of \( Q^F \).

**Solution for 3.** See lecture notes.