Lemma 4.1

Let $G$ be an optimum solution using $m = 1$ E1 edges.

$SC(G) = \sum_{u \neq v} \text{dist}(u,v) + \alpha \cdot m$

$$= \alpha \cdot m + \sum_{(u,v) \in E} \text{dist}(u,v) + \sum_{(u,v) \in E} \text{dist}(u,v)$$

$$\geq (n(n+1)/2 - 2m) \cdot 2 = 2m$$

$$\geq \alpha \cdot m + 2n(n-1) - 2m$$

($x$) $= (\alpha - 2) m + 2n(n-1)$

(For star and complete graph "\(\geq\)" can be replaced by "\(=\)", since \(\text{dist}(u,v) \leq 2 \forall (u,v) \in V^2\))

So both, the star and complete graph match this lower bound.

($x$) is minimised

1) for $\alpha \geq 2$: if $n$ is as small as possible => for star

2) for $\alpha \leq 2$: if $n$ is as large as possible => for complete graph

Remark:

If $G$ is star or complete graph then

$SC(G) = (\alpha - 2) m + 2n(n-1)$