Lemma 4.21

**First let \( \alpha \geq 1 \):**

We show that a star is a NE.

- Let center \( u \) pay for all edges.
- If \( u \) stops paying for an edge \((u,v)\)
  - then \( \text{dist}(u,v) = \infty \)

  \( \Rightarrow \) \( u \) will not do this (stop paying for \((u,v)\))

- Every leaf node \( v \) can only add edges
  - if \( u \) buys \( k \in [1, n-2] \) edges this adds edge cost of \( \alpha \cdot k \)
  - and only save \( k \) in distance cost

  \( \Rightarrow \) cost of \( v \) would not decrease since \( \alpha \geq 1 \).

**Now let \( \alpha \leq 1 \):**

We show that the complete graph is a NE.

Consider the complete graph where nodes pay arbitrarily
for incident edges. Only 1 player pays for each edge.

If a player stops paying for \( k \) edges this saves \( \alpha \cdot k \) in edge costs but adds at least \( k \) in distance cost.

\( \Rightarrow \) cost would not decrease, since \( \alpha \leq 1 \).
Theorem 4.3:

Recall from Lemma 4.1:

If $G$ is a star or complete graph

then $SC(G) = (\alpha - 2) \cdot m + 2n(n-1)$ \hspace{1cm} (\ast)

$\alpha \geq 2$ or $\alpha \leq 1$ follows immediately from the two lemmas.

So let $1 < \alpha < 2$:

$\Rightarrow$ star $G_1$, is a NE \hspace{1cm} (m_1 = n-1 \text{ edges})

$\Rightarrow$ complete graph $G_2$ is an optimum \hspace{1cm} (m_2 = \frac{n}{2} \cdot (n-1) \text{ edges})

\[
\frac{SC(G_1)}{SC(G_2)} = \frac{(\alpha-2)m_1 + 2n(n-1)}{(\alpha-2)m_2 + 2n(n-1)}
\]

\[
= \frac{(\alpha-2)(n-1) + 2n(n-1)}{(\alpha-2)\frac{n}{2}(n-1) + 2n(n-1)}
\]

\[
= \frac{\alpha - 2 + 2n}{\frac{\alpha}{2} \cdot \frac{n}{2} + 2n}
\]

\[
= \frac{\alpha + 2(n-1)}{\frac{\alpha}{2} \cdot n + n}
\]

(decreasing in $\alpha$ for $n \geq 3$)

\[
1 \leq \frac{2n-1}{\frac{\alpha}{2} \cdot n} = \frac{4n-2}{\frac{\alpha}{3} \cdot n} \leq \frac{4}{3}
\]