We show that \( \text{dist}(u,v) \leq 2\sqrt{x} \) for any pair of nodes \( u,v \).

Suppose \( \text{dist}(u,v) = 2k \) for some integer \( k \).

(case of odd distance is similar)

\[
\begin{array}{cccccccc}
& u & v_1 & v_2 & v_3 & \cdots & v_{2k} & v & \\
& \quad & \quad & \quad & \quad & \cdots & \quad & \quad & \\
\end{array}
\]

There is a path between \( u \) and \( v \).

Because \( \Delta \) is stable, \( u \) cannot improve by buying \( (u, v_{k+1}) \).

- Adding \( (u, v_{k+1}) \) improves the distance of \( u \) to all nodes \( v_{k+1}, \ldots, v_{2k} \) by \( k \).
- So the improvement in total distance is at least \( k^2 \).
- If \( x < k^2 \) then \( u \) would benefit from buying \( (u, v_{k+1}) \).

\[
\Rightarrow \quad x \geq k^2 \quad \Rightarrow \quad k \leq \sqrt{x}
\]

- Thus \( \text{dist}(u,v) = 2k \leq 2\sqrt{x} \).

\( \text{L} \) which proves the first part of the theorem.

- Applying Lemma 4.4 yields the upper bound on \( \text{PoA} \).