COMP558
Network Games

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Local Connection Game
- Model
- Characterizing Solutions and Price of Stability
- Price of Anarchy

Global Connection Game
- Model
- Price of Anarchy
- Price of Stability

Facility Location
- Model
- Existence of pure NE and PoS
- Utility games and PoA
Scenario

- Consider users constructing a shared network
- Each user has its own interest and is driven by:
  - Minimising the price he pays for creating/using the network
  - Receiving a high quality of service
- We wish to model the networks generated by such selfish behaviour of the users and compare them to the optimal networks
Objectives

- How to evaluate the overall quality of a network?
  - social cost = sum of players’ costs
- What are stable networks?
  - we use Nash equilibrium as solution concept
  - we refer to networks corresponding to Nash equilibrium as being stable
- Our main goal: bounding the efficiency loss resulting from selfishness
  - Price of Anarchy
  - Price of Stability
Local vs. Global Connection Game

- Local connection game:
  - users buy edges
  - bought edge can be used by all users
  - users wish to minimise distance to all other nodes
  - while minimizing the number of edges they buy
  - Resembles formation of P2P networks

- Global connection game:
  - users want to connect two nodes $s_i, t_i$ in the network
  - users share the cost of used edges
  - users minimise their cost
  - Resembles use of a large scale shared network
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Local Connection Game

Model

- **n players**: nodes in a graph $G$ on which the network will be build
- Strategy $S_u$ of player $u \in V$ is a set of undirected edges that $u$ will build (all are incident to $u$)
- For a strategy vector $S$, the union of all edges in players’ strategies form a network $G(S)$
- $\alpha$ ... cost for building an edge
- $dist_S(u, v)$ ... distance (number of edges on shortest path) between $u$ and $v$ in $G(S)$
- Cost of player $u \in V$:

$$C_u(S) = \sum_{v \in V} dist_S(u, v) + \alpha \cdot n_u,$$

where $n_u$ is number of edges bought by player $u$
Local Connection Game

A network $G = (V, E)$ is stable for a value $\alpha$, if there is a NE $S$ that forms $G$.

Social Cost of a Network $G = (V, E)$ ($=$ sum of players’ costs)

$$SC(G) = \sum_{u \neq v} \text{dist}(u, v) + \alpha \cdot |E|$$

Observations

- Since the graph is undirected an edge $(u, v)$ is available to both $u$ and $v$
- At Nash equilibrium at most one of the nodes $u, v$ pays for the edge $(u, v)$
- At Nash equilibrium we must have a connected graph, since $\text{dist}(u, v) = \infty$, if $u$ and $v$ are not connected
Characterizing Solutions and Price of Stability

Lemma 4.1 (Lem. 19.1)
If $\alpha \geq 2$ then any star is an optimal solution, and if $\alpha \leq 2$ then the complete graph is an optimal solution.

Lemma 4.2 (Lem. 19.2)
If $\alpha \geq 1$ then any star is a Nash equilibrium, and if $\alpha \leq 1$ then the complete graph is a Nash equilibrium.

Remark: There are also other Nash equilibria.

Theorem 4.3 (Thm. 19.3)
If $\alpha \geq 2$ or $\alpha \leq 1$, the price of stability is 1. For $1 < \alpha < 2$, the price of stability is at most $4/3$. 
Price of Anarchy

To prove upper bound on the Price of Anarchy we
1. bound the diameter of a stable (NE) graph
2. use diameter to bound cost

Lemma 4.4 (Lem. 19.4)

If a graph $G$ at Nash equilibrium has diameter $d$, then $\text{PoA}(G) = O(d)$.
**Price of Anarchy**

**Theorem 4.5**

The diameter of a stable graph $G$ is at most $2\sqrt{\alpha}$, and hence $\text{PoA}(G) = O(\sqrt{\alpha})$.

**Theorem 4.6**

The price of anarchy is $O(1)$ whenever $\alpha = O(\sqrt{n})$. More generally, $\text{PoA} = O(1 + \frac{\alpha}{\sqrt{n}})$. 

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Figure 19.2. Consider how the distance decreases to at most $d$ for nodes $u$ and $v$ that are at maximum distance $d'$. Adding the edge $(u, w)$ will improve the distance to the nodes on the second half of the shortest path.

We again use Lemma 19.4, so all we have to do is improve our bound on the price of anarchy that was given by Lin (2003) and independently also by Albers et al., 2006 for $B$. The main observation is that by adding the edge $(v, u)$, the distance decreases to at most $\alpha$. Combining these, we get the bound $\text{PoA} = O(\sqrt{\alpha})$.

This implies that $\text{PoA} = O(\sqrt{n})$. Considering how the distance to all nodes in $A_w$ after the node $w$ leaves the set $B_u$, we must have that $\alpha \geq \frac{|B_u|}{2} \geq \frac{n(d' - 1)}{4} \geq n(d' - 1)^2/\alpha$. Therefore, node $u$ is at distance $\alpha \geq (d - 1)\sqrt{n}/2$. This yields $\alpha \geq (d - 1)n(d' - 1)/4 \geq n(d' - 1)^2/\alpha$.
Topic 4: Network Formation Games

- Local Connection Game
  - Model
  - Characterizing Solutions and Price of Stability
  - Price of Anarchy

- Global Connection Game
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- Facility Location
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Global Connection Game

Model

- directed $G = (V, E)$ with non-negative edge cost $c_e$
- $k$ players, each player $i \in [k]$ has a source $s_i$ and sink node $t_i$
- A strategy for a player $i$ is a path $P_i$ from $s_i$ to $t_i$ in $G$
- Given each player's strategy we define the constructed network to be $\bigcup_i P_i$
- Players who use edge $e$ divide the cost $c_e$ according to some cost sharing mechanism.
- We will consider the equal-division mechanism:

$$
\text{cost}_i(S) = \sum_{e \in P_i} \frac{c_e}{k_e}
$$

- $S = (P_1, \ldots, P_k)$
- $k_e$ ... number of players whose path contains $e$
Existence of Stable Networks

- Does every global connection game have a pure Nash equilibrium?
  - Yes.
- Why?
  - It is a special congestion game!
- Rosenthal’s potential function

\[
\Phi(S) = \sum_{e \in E, k_e > 0} \sum_{j=1}^{k_e} \frac{c_e}{j} \\
= \sum_{e \in E} c_e \cdot H_{k_e}
\]

\((H_k\) is k-th harmonic number)
Price of Anarchy

Social Cost (total cost of used edges)

\[ SC(S) = \sum_{i \in [k]} cost_i(S) \]

Example

- optimal network has cost 1
- best NE: all players use the left edge
  \[ \Rightarrow \text{PoS} = 1 \]
- worst NE: all players use the right edge
  \[ \Rightarrow \text{PoA} = k \]

Theorem 4.7

For any global connection game with \( k \) players, \( \text{PoA} \leq k \).
Price of Stability: a lower bound \((\varepsilon > 0\) arbitrary small)\)

There are two equilibria with costs \(1\) and \(1 + \varepsilon\) respectively. Since the latter is also optimal, Is it stable?

- optimal network has a cost of \(1 + \varepsilon\)
- cost of unique stable network: \(\sum_{j=1}^{k} \frac{1}{j} = H_k\)
Price of Stability: an upper bound

Lemma 4.8 (Lem. 19.8)
For any strategy profile $S = (P_1, \ldots, P_k)$ we have

$$SC(S) \leq \Phi(S) \leq H_k \cdot SC(S)$$

Lemma 4.8 and Lemma 3.18 directly imply:

Theorem 4.9 (Thm. 19.10)
The price of stability in the global connection game with $k$ players is at most $H_k$.

Remark: $H_k = \Theta(\log k)$
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Facility Location Game

Model

- $k$ service providers and a set of clients $[m]$
- each provider $i \in [k]$ has a set of possible locations $A_i$ where he can locate his facility; denote $A = \bigcup_{i \in [k]} A_i$
- $c_{js_i}$ .. cost of serving customer $j \in [m]$ from location $s_i \in A_i$
- $\pi_j$ .. value of client $j \in [m]$ for being served
Facility Location Game

Model cont.

- If provider \( i \) serves client \( j \) from location \( s_i \) at a price \( p \)
  - \( \pi_j - p \) .. benefit for client \( j \)
  - \( p - c_{js_i} \) .. profit for provider \( i \)
  - \( \Rightarrow \) social value: \( \pi_j - c_{js_i} \)
    (independent of price)

- To simplify notation assume \( \pi_j \geq c_{js_i} \) for all \( j, i \) and \( s_i \in A_i \).
  - This does not change social value.

- Given \( s = (s_1, \ldots, s_k) \), each client \( j \)
  - is assigned to provider \( i \) with lowest cost \( c_{js_i} \),
  - pays price \( p_{ij} = \min_{i' \neq i} c_{js_i'} \)

- total social value of \( s = (s_1, \ldots, s_k) \):
  \[
  V(s) = \sum_{j \in [m]} \left( \pi_j - \min_{i \in [k]} c_{js_i} \right)
  \]
Facility Location Game is Potential Game

**Theorem 4.10** (Thm. 19.16)

$V(s)$ is a potential function for the facility location game.

**Corollary 4.11**

1. Each facility location game admits a pure NE.
2. Every optimum is also a NE and thus $PoS = 1$.

Up next:

- bound price of anarchy
- we do this for the more general class of utility games
Utility Games

- each player $i \in [k]$ has a set of available strategies $A_i$ (e.g. locations)
- $A = \bigcup_{i \in [k]} A_i$
- Social welfare function $V(B)$ defined for all $B \subseteq A$
- $\alpha_i(s)$ .. welfare of player $i$ in $s = (s_1, \ldots, s_k)$

Utility games satisfy the following properties:

(i) $V(B)$ is submodular: for any sets $B \subseteq B' \subseteq A$ and any element $e \in A$, we have $V(B \cup \{e\}) - V(B) \geq V(B' \cup \{e\}) - V(B')$.

(ii) The total value for players is less than or equal to the total social value: $\sum_{i \in [k]} \alpha_i(s) \leq V(s)$.

(iii) The value for player $i$ is at least his added value for the society: $\alpha_i(s) \geq V(s) - V(s - s_i)$.

Utility game is basic, if (iii) is satisfied with equality, and monotone if for all $B \subseteq B' \subseteq A$, $V(B) \leq V(B')$. 
PoA in Utility Games

To view facility location game as a utility game define for each
\( B \subseteq A = \bigcup_{i \in [k]} A_i \):

\[
V(B) = \sum_{j \in [m]} (\pi_j - \min_{e \in B} c_{je})
\]

**Theorem 4.12**
The facility location game is a monotone basic utility game.

**Theorem 4.13**
For all monotone utility games \( G \), we have \( PoA(G) \leq 2 \).