COMP558
Network Games

Martin Gairing

University of Liverpool, Computer Science Dept

2nd Semester 2013/14
Topic 3: Selfish routing / congestion games

- Non-atomic congestion games: Wardrop Model
  - Model
  - Optimal Flows & Wardrop Equilibria
  - Existence of Wardrop Equilibria
  - Price of Anarchy

- (Atomic) congestion games
  - Model
  - Existence of Nash equilibria
  - Computation of Nash equilibria
  - Price of Anarchy
  - Price of Stability
Wardrop Model

We are given a

- directed graph $G = (V, E)$
- $k$ commodities, for each $i \in [k]$
  - $s_i, t_i$ .. source-sink pair
  - $r_i$ .. flow demand to route from $s_i$ to $t_i$
  - normalise: $r = \sum_{i \in [k]} r_i = 1$
  - $P_i$ .. set of paths between $s_i$ and $t_i$
- $P = \bigcup_{i \in [k]} P_i$
- $c_e : [0, 1] \rightarrow \mathbb{R}$ .. latency (cost) function of edge $e \in E$
  - continuous, non-decreasing, non-negative
- The triple $(G, r, c)$ is an instance of the routing problem
Wardrop Model

Flow and Latency:

- **Flow vector** \((f_P)_{P\in\mathcal{P}}\)
  - \(f_e = \sum_{P \ni e} f_P\)
- A flow is feasible if it satisfies flow demands and flow conservation
- \(c_e(f) = c_e(f_e)\) .. edge latency / cost
- \(c_P(f) = \sum_{e \in P} c_e(f)\) .. path latency

**Total latency (the total cost)**

\[
C(f) = \sum_{P \in \mathcal{P}} f_P \cdot c_P(f)
= \sum_{e \in E} f_e \cdot c_e(f)
\]
Wardrop Model

- So far, we basically just introduced a flow model.
- In the Wardrop model we assume:
  - flow is controlled by an infinite number of agents
  - each agent is responsible for an infinitesimal fraction of the flow
  - agents strive to minimise their own latency

**Definition (Wardrop equilibrium)**

A feasible flow $f$ is a Wardrop equilibrium if for every commodity $i \in [k]$ and every pair $P_1, P_2 \in \mathcal{P}_i$ with $f_{P_1} > 0$, we have

$$c_{P_1}(f) \leq c_{P_2}(f).$$

This implies:
- All used paths of the same commodity have the same latency.
Examples

Pigou's Example

\[ \ell_1(x) = 1 \]

\[ \ell_2(x) = x \]

Braess' Paradox

\[ x \]

\[ 1 \]

\[ 0 \]

\[ 1 \]

\[ x \]
Characterizing Optimum Flows

Optimum flow: minimise the value of $C(f)$ among all feasible flows.

Formulation as a convex program

$$\min_{\vec{f}} \sum_{e \in E} f_e \cdot c_e(f)$$

subject to

$$\sum_{P \ni e} f_P = r_i \quad \forall i \in [k]$$

$$f_e = \sum_{P \ni e} f_P \quad \forall e \in E$$

$$f_P \geq 0 \quad \forall P \in \mathcal{P}$$

Remarks:

- The linear constraints of this program define a convex polyhedron.
- We assume that the terms $f_e \cdot c_e(f)$ are convex.
- Number of variables is exponential in network size. (Can be reduced!)
Marginal Cost

- Suppose we are shifting flow from path $P_1$ to $P_2$.
- Contribution of edges on $P_1$ to total latency decreases, contribution of $P_2$ increases.
- If decrease on $P_1$ is more than the increase on $P_2$ then total latency decreases.
- The flow was not optimal.

**Definition: Marginal Cost**

\[
\begin{align*}
  c_e^*(x) &= (x \cdot c_e(x))' = c_e(x) + x \cdot c_e'(x) \\
  c_P^*(x) &= \sum_{e \in P} c_e^*(x)
\end{align*}
\]

**Observation**

In an optimal flow, for each commodity $i \in [k]$, the marginal cost of all used paths is the same.
Optimum via marginal costs

Assume \( x \cdot c_e(x) \) is convex and continuously differentiable for all \( e \in E \).

Lemma 3.1 (Characterisation of optimal flows) Prop.18.9
A feasible flow \( f \) is optimal if and only if for all commodities \( i \in [k] \), and paths \( P_1, P_2 \in \mathcal{P}_i \) with \( f_{P_1} > 0 \), we have \( c_{P_1}^*(f) \leq c_{P_2}^*(f) \).

Observe the similarity in the characterisation of optimal flows and Wardrop equilibria.

Theorem 3.2 (Wardrop equilibrium vs. Optimum). Cor.18.10
A feasible flow \( f \) is optimal for \((G, r, c)\) if and only if \( f \) is a Wardrop equilibrium for \((G, r, c^*)\).
Existence of Wardrop Equilibria

Theorem 3.3 (Existence and essential uniqueness) Th.18.8

(a) Every instance \((G, r, c)\) admits a Wardrop equilibrium.
(b) If \(f\) and \(\tilde{f}\) are Wardrop equilibria, then \(c_e(f) = c_e(\tilde{f})\) for every \(e \in E\).

Proofs are based on the following potential function [ Beckmann, McGuire, Winston, 1956] :

\[
H(f) = \sum_{e \in E} h_e(f_e)
\]

with
\[
h_e(x) = \int_0^x c_e(u) \, du
\]
Price of Anarchy

Let $f$ be a Wardrop equilibrium and $f^*$ be an optimum for $(G, r, c)$.

Definition: Price of Anarchy

$$\rho(G, r, c) = \frac{C(f)}{C(f^*)}$$

Theorem 3.4 (Upper bound via potential function)

Suppose for every $e \in E$ and all $x \geq 0$,

$$x \cdot c_e(x) \leq \alpha \cdot \int_0^x c_e(u) \, du.$$ 

Then, $\rho(G, r, c) \leq \alpha$. 
Corollary 3.5 (polynomial latency functions)  Cor.18.17
Suppose latency functions are of the form $c_e(x) = \sum_{i=0}^{d} a_{e,i} \cdot x^i$. Then, $\rho(G, r, c) \leq d + 1$.

Theorem 3.6 (linear latency functions)
Suppose latency functions are linear with non-negative slope and offset. Then, $\rho(G, r, c) \leq \frac{4}{3}$.

Theorem 3.7 (bicriteria bound)
If $f$ is a Wardrop equilibrium for $(G, r, c)$ and $f^*$ a feasible flow for $(G, 2r, c)$, then $C(f) \leq C(f^*)$. 

Price of Anarchy
Topic 3: Selfish routing / congestion games

- Non-atomic congestion games: Wardrop Model
  - Model
  - Optimal Flows & Wardrop Equilibria
  - Existence of Wardrop Equilibria
  - Price of Anarchy

- (Atomic) congestion games
  - Model
  - Existence of Nash equilibria
  - Computation of Nash equilibria
  - Price of Anarchy
  - Price of Stability
(Weighted) congestion games

\[ \Gamma = ([k], (w_i)_{i \in [k]}, E, (S_i)_{i \in [k]}, (c_e)_{e \in E}) \]

- \([k]\) ... set of \(k\) players
- \(w_i\) ... weight of player \(i \in [k]\)
- \(E\) ... set of resources
  e.g. edges in a graph
- \(S_i \subseteq 2^E\) ... set of strategies of player \(i\)
  e.g. set of paths from \(o_i\) to \(d_i\)
- \(c_e\) ... latency function of resource \(e\)
Subclasses of (weighted) congestion games

- unweighted congestion games (or simply congestion games):
\[ w_i = 1 \quad \text{for all player } i \in [k] \]

- symmetric games:
\[ S_i = S_j \quad \text{for all player } i, j \in [k] \]

- network congestion games

- singleton congestion games
Load and Private Cost

**Strategy profile**

\[ s = (s_1, \ldots, s_n) \in S_1 \times \ldots \times S_n \]

**Traffic on resource** \( e \in E \)

\[ x_e(s) = \sum_{i \in [k]: e \in s_i} w_i \]

**Private cost of player** \( i \in [k] \)

\[ C_i(s) = w_i \cdot \sum_{e \in s_i} c_e(x_e(s)) \]

\[ C_1(s) = 3 \cdot (c_a(8) + c_d(3)) \]
\[ C_2(s) = 5 \cdot (c_a(8) + c_c(5) + c_e(7)) \]
\[ C_3(s) = 2 \cdot (c_b(2) + c_e(7)) \]
Nash Equilibrium

A strategy profile $s$ is a Nash equilibrium if and only if all players $i \in [k]$ are satisfied, that is,

$$C_i(s) \leq C_i(s_{-i}, s'_i)$$

for all $i \in [k]$ and $s'_i \in S_i$.

Remarks

- For simplicity we restrict to pure Nash equilibria.
- Many results hold also for mixed Nash equilibria.
  - Players randomize over their pure strategies
  - Guaranteed to exist [NASH, 1951]
Price of Anarchy

Social Cost

\[
SC(s) = \sum_{i \in [k]} C_i(s) \\
= \sum_{e \in E} x_e(s) \cdot c_e(x_e(s))
\]

Let \( G \) be a class of games.

Price of Anarchy

\[
PoA(G) = \sup_{\Gamma \in G, \ s \text{ is NE in } \Gamma} \frac{SC(s)}{OPT}
\]
Topic 3: Selfish routing / congestion games

- Non-atomic congestion games: Wardrop Model
  - Model
  - Optimal Flows & Wardrop Equilibria
  - Existence of Wardrop Equilibria
  - Price of Anarchy

- (Atomic) congestion games
  - Model
  - Existence of Nash equilibria
  - Computation of Nash equilibria
  - Price of Anarchy
  - Price of Stability
Existence of pure NE: positive result

**Theorem 3.8**

Every unweighted congestion game possesses a pure Nash equilibrium.

Define $\Phi : (S_1 \times \ldots \times S_n) \rightarrow \mathbb{N}$ by

$$\Phi(s) = \sum_{e \in E} \sum_{j=1}^{x_e(s)} c_e(j).$$

Consider two strategy profiles $s = (s_1, \ldots, s_k)$ and $s' = (s_i', s_{-i})$:

$$\Phi(s) - \Phi(s') = \sum_{e \in s_i - s_i'} c_e(x_e(s)) - \sum_{e \in s_i' - s_i} c_e(x_e(s'))$$

$$= C_i(s) - C_i(s').$$

Therefore: $\Phi(s)$ minimal $\Rightarrow$ $s$ is Nash equilibrium.
Existence of pure NE: negative result

Theorem 3.9 Ex.18.7

There is a weighted network congestion game that does not admit a pure Nash equilibrium.

Consider the following instance:

- 2 players
- \( w_1 = 1 \)
- \( w_2 = 2 \)

Existence of pure NE in weighted games

**Theorem 3.10** [Fotakis, Kontogiannis, Spirakis, 2004]

Every **weighted** congestion game with **linear** latency functions possesses a pure Nash equilibrium.

Proof is based on the following potential function:

\[ \tilde{\Phi}(s) = \sum_{i \in [k]} w_i \cdot \sum_{e \in s_i} (c_e(x_e(s)) + c_e(w_i)) \]

\[ = \sum_{e \in E} x_e(s) \cdot c_e(x_e(s)) + \sum_{i \in [k]} w_i \cdot \sum_{e \in s_i} c_e(w_i). \]

If \( s = (s_1, \ldots, s_k) \) and \( s' = (s'_j, s_{-j}) \) for some \( j \in [k] \) and \( s'_j \in S_j \), then

\[ \tilde{\Phi}(s) - \tilde{\Phi}(s') = 2 \cdot (C_j(s) - C_j(s')). \]
Topic 3: Selfish routing / congestion games

- Non-atomic congestion games: Wardrop Model
  - Model
  - Optimal Flows & Wardrop Equilibria
  - Existence of Wardrop Equilibria
  - Price of Anarchy

- (Atomic) congestion games
  - Model
  - Existence of Nash equilibria
  - Computation of Nash equilibria
  - Price of Anarchy
  - Price of Stability
We will focus on network congestion games.

Symmetric case uses reduction to Min Cost Flow.

For asymmetric case we study the simplified reduction from MaxCut in [Ackermann, Rölling, Vocking, 2006].
Computation of NE: Min Cost Flow

- Min Cost Flow Instance:
  - $G(V, E)$ .. directed graph
  - $c_{ij}$ .. cost of edge $(i, j) \in E$
  - $u_{ij}$ .. capacity of edge $(i, j) \in E$
  - $b(i)$ .. supply/demand of node $i \in V$

- Min Cost Flow Problem:

\[
\begin{align*}
\min & \quad \sum_{(i,j) \in E} c_{ij} x_{ij} \\
\text{s.t.} & \quad \sum_{\{j: (i,j) \in E\}} x_{ij} - \sum_{\{j: (j,i) \in E\}} x_{ji} = b(i) \quad \text{for all } i \in V \\
& \quad x_{ij} \in [0, u_{ij}] \quad \text{for all } (i, j) \in E
\end{align*}
\]

- Can be solved in polynomial time.
  - Integer solution for integer inputs.
Theorem 3.11
For symmetric network congestion games we can compute a pure NE in polynomial time.

- replace each edge \( e \in E \) by \( k \) parallel edges \( e_1, \ldots, e_k \).
  - edge \( e_j \) has capacity 1 and cost \( c_e(j) \)

\[
\Phi(s) = \sum_{e \in E} \sum_{j=1}^{x_e(s)} c_e(j).
\]

Min cost flow minimizes potential: \( \Phi(s) = \sum_{e \in E} \sum_{j=1}^{x_e(s)} c_e(j) \).
Asymmetric games and PLS Complexity

- For asymmetric network congestion games this reduction is not possible. Why?

## Complexity class PLS

- The class **PLS** (polynomial time local search) consists of local search problems for which local optimality can be verified in polynomial time.

- Solving a PLS problem means finding a solution, which is known to exist.

- Many local search problems were shown to be complete for this class, including graph partitioning, weighted satisfiability and traveling salesman problems.
A PLS problem: TSP with 2-OPT neighborhood

▶ Feasible solutions:
Tours for the traveling salesman.

▶ 2-OPT neighborhood of a tour:
All tours that can obtained by deleting two edges \( \{a, b\}, \{c, d\} \)
and replacing them by \( \{a, d\}, \{c, b\} \).

```
a, b, c, d
\Rightarrow
a, b, c, d
```

old tour  new tour
Definition: PLS-Problem [Johnson, Papadimitriou, Yannakakis, 1988]

A PLS problem $L$ is specified by:

- a set $\mathcal{I}_L$ of problem instances,
- for each instance $I \in \mathcal{I}_L$ there is a set $F(I)$ of feasible solutions,
- objective function $c : F(I) \mapsto \mathbb{N}$ (cost of feasible solution)
- neighborhood $N(S, I) \subseteq F(I)$ for each feasible solution $S \in F(I)$.

Three polynomial time algorithms $A$, $B$, $C$ have to exist:

- $A$ computes for every $I \in \mathcal{I}_L$ a feasible solution $S \in F(I)$
- $B$ computes the objective function $c(S)$ for every $S \in F(I)$ and $I \in \mathcal{I}_L$
- $C$ determines for every $S \in F(I)$ and $I \in \mathcal{I}_L$ whether $S$ is locally optimal and, if not, finds a better solution $S' \in N(S, I)$ in the neighborhood of $S$. 
**Definition: PLS-Reduction**

PLS-Problem $L$ is **PLS-reducible** to the PLS-Problem $K$ if there are polynomial-time computable functions $f$ and $g$, such that...

1. $f$ maps a given instance $I \in \mathcal{I}_L$ to an instance $f(I) \in \mathcal{I}_K$,
2. $g$ calculates, for a given feasible solution $S \in F_K(f(I))$, a feasible solution $S' = g(S, I) \in F_L(I)$,
3. if $S$ is a local optimum for instance $f(I) \in \mathcal{I}_K$, then $g(S, I)$ is local optimum for $I \in \mathcal{I}_L$.
PLS-Problem: pure NE for Congestion Games

- **Feasible solutions:**
  all pure strategy profiles \( s = (s_1, \ldots, s_n) \in (S_1 \times \ldots \times S_n) \).

- **Neighborhood of** \( s \):
  all profiles \( s' = (s_1, \ldots, s_{i-1}, s'_i, s_{i+1}, \ldots, s_n) \in (S_1 \times \ldots \times S_n) \).

- **Cost function:**
  potential function \( \Phi(s) = \sum_{e \in E} \sum_{j=1}^{x_e(s)} c_e(j) \).

⇒ Feasible solution is local optimum iff it is a Nash equilibrium!

We show that computing a pure NE in network congestion games is PLS complete.

- By reduction from the PLS-complete (local) MaxCut problem.
- Use intermediate problem Quadratic Threshold Game.
MaxCut to Quadratic Threshold Game

**MaxCut**

- **Input:** A complete graph $G = (V, E)$ with edge weights $(w_e)_{e \in E}$
- **Feasible solutions:** Set of cuts $U, W$ such that $U$ and $W$ are disjoint and $U \cup W = V$.
- **Neighborhood:** Single node moves between $U$ and $W$
- **Objective:** maximise sum of weights crossing the cut.

---

[Schäffer and Yannakakis, 1991]

Computing a local MaxCut is PLS-complete.
MaxCut to Quadratic Threshold Game

Quadratic Threshold Game

- Special class of congestion games, where each player has only 2 strategies.
- Two classes of resources $R_{in}$ and $R_{out}$
  - $R_{out} = \{ r_i \mid i \in [k] \}$; where $r_i$ has constant latency $T_i$ (the threshold)
  - $R_{in} = \{ r_{ij} \mid \{i, j\} \in [k]^2, i \neq j \}$; one for each unordered pair
- Each player $i \in [k]$ has 2 strategies:
  - $S_{i}^{out} = \{ r_i \}$
  - $S_{i}^{in} = \{ r_{ij} \mid j \in [k], j \neq i \}$

Theorem 3.12

Computing a pure NE for quadratic threshold games is PLS-complete.
Quadratic Threshold Game to Network Congestion Game

Theorem 3.13
Computing a pure NE in network congestion games is PLS complete.

Proof by reduction from Quadratic Threshold Games.
Topic 3: Selfish routing / congestion games

- Non-atomic congestion games: Wardrop Model
  - Model
  - Optimal Flows & Wardrop Equilibria
  - Existence of Wardrop Equilibria
  - Price of Anarchy

- (Atomic) congestion games
  - Model
  - Existence of Nash equilibria
  - Computation of Nash equilibria
  - Price of Anarchy
  - Price of Stability
Price of Anarchy: Example

Nash Equilibrium

\[ SC = 14 + 14 = 28 \]

Price of Anarchy = \( \frac{28}{24} = \frac{7}{6} \)

If multiple equilibria, look at worst one

OPT

\[ SC = 14 + 10 = 24 \]
Price of Anarchy: State of the Art

(1) **Analytical simple classes** of cost functions ⇒ **exact formula for PoA**.
   - linear
     - [Christodoulou, Koutsoupias, STOC’05]
     - [Awerbuch, Azar, Epstein, STOC’05]
   - bounded degree polynomials
     - [Aland et al., STACS’06]

(2) For **every set** of allowable cost functions ⇒ **recipe** for computing PoA.
   - non-atomic (Wardrop model)
     - [Roughgarden, Tardos, JACM’00]
   - unweighted
     - [Roughgarden, STOC’09]
   - weighted
     - [Bhawalkar, Gairing, Roughgarden, ESA’10]

(3) Understanding of **game complexity** required for worst-case PoA to be realized.
   - Ideally independent of cost functions.
   - e.g. symmetric strategy sets, singleton strategy sets
Abstract Setup

- $n$ players, each picks a strategy $s_i$
- player $i$ incurs cost $C_i(s)$
- Important Assumption: objective function is $SC(s) = \sum_i C_i(s)$

Definition: [Roughgarden, STOC’09]
A game is $(\lambda, \mu) – smooth$ if for every pair $s, s^*$ of outcomes:

$$\sum_i C_i(s_{-i}, s_i^*) \leq \lambda \cdot SC(s^*) + \mu \cdot SC(s).$$

($\lambda > 0, \mu < 1$)
Smoothness $\implies$ PoA bound

**Theorem 3.14**

If a game $G$ is $(\lambda, \mu)$ – *smooth*, then

$$\text{PoA}(G) \leq \frac{\lambda}{1 - \mu}.$$
Back to congestion games

- $\mathcal{C}$ ... arbitrary class of cost functions

Consider the set:

- $\mathcal{A}(\mathcal{C}) = \{(\lambda, \mu) : x^* \cdot c(x + x^*) \leq \lambda \cdot x^* \cdot c(x^*) + \mu \cdot x \cdot c(x)\}$

where

- $0 \leq \mu < 1$ and $\lambda > 0$
- constraints range over all $c \in \mathcal{C}$ and $x \geq 0$ and $x^* > 0$.

### Proposition 3.15

For a class of functions $\mathcal{C}$, if $(\lambda, \mu) \in \mathcal{A}(\mathcal{C})$ then for every weighted congestion game with cost functions in $\mathcal{C}$ we have: $\text{PoA} \leq \frac{\lambda}{1-\mu}$.

For unweighted congestion games: redefine $\mathcal{A}(\mathcal{C})$:

- $\mathcal{A}(\mathcal{C}) = \{(\lambda, \mu) : x^* \cdot c(x + 1) \leq \lambda \cdot x^* \cdot c(x^*) + \mu \cdot x \cdot c(x)\}$
- and restrict $x, x^*$ to be integer
The set $\mathcal{A}(C) \implies$ upper bound

Best possible upper bound on PoA:

$\zeta(C) = \inf \left\{ \frac{\lambda}{1-\mu} : (\lambda, \mu) \in \mathcal{A}(C) \right\}$
Questions

Questions:

▶ Is this upper bound tight?
  ▶ Yes, for unweighted. [ROUGHGARDEN, 2009]
  ▶ Yes, for weighted with mild assumption on $C$. [BHAWALKAR, GAIRING, ROUGHGARDEN, 2010]

Closure under scaling and dilation:

If $c(x) \in C$ and $r \in \mathbb{R}^+$ then

▶ $r \cdot c(x) \in C$
▶ $c(r \cdot x) \in C$

▶ $\zeta(C)$ for linear/polynomial cost functions?
PoA for linear/polynomial

- Polynomial latency functions: \( C_d = \left\{ c \mid c(x) = \sum_{i=0}^{d} a_i \cdot x^i \right\} \)
- \( \Phi_d \) is solution to \((\Phi_d + 1)^d = \Phi_d^{d+1}\).
- \( k = \lfloor \Phi_d \rfloor \)

**Theorem 3.16**

If all latency functions are from \( C_d \), then for

(a) weighted congestion games: \( \text{PoA} = \Phi_d^{d+1} \)

(b) unweighted congestion games: \( \text{PoA} = \frac{(k+1)^{2d+1} - k^{d+1}(k+2)^d}{(k+1)^{d+1} - (k+2)^d + (k+1)^d - k^{d+1}} \)

**Corollary 3.17**

For the linear case \( (d = 1) \) we have:

(a) weighted congestion games: \( \text{PoA} = \Phi^2 = \frac{3 + \sqrt{5}}{2} \approx 2.618 \)

(b) unweighted congestion games: \( \text{PoA} = 2.5 \)
Proof Lower Bound Thm 3.16(b) (unweighted)

Proof Sketch.

- \( n \geq \lfloor \Phi_d \rfloor + 2 \) player
- \( E = \{g_1, \ldots, g_n\} \cup \{h_1, \ldots, h_n\} \)
- \( c_{g^*}(x) = a \cdot x^d, \quad c_{h^*}(x) = x^d \)
- \( S_i = \{Q_i, P_i\} \) with
  - \( Q_i = \{g_i, h_i\} \)
  - \( P_i = \{g_{i+1}, \ldots, g_{i+k}, h_{i+1}, \ldots, h_{i+k+1}\} \)

Choose \( a > 0 \) such that \( P = (P_i)_{i \in [n]} \) NE with \( C_i(P) = C_i(P_{-i}, Q_i) \).

\[ d = 2, \quad n = 4, \quad k = \lfloor \Phi_d \rfloor = \lfloor 2.148 \rfloor \]
Topic 3: Selfish routing / congestion games

- Non-atomic congestion games: Wardrop Model
  - Model
  - Optimal Flows & Wardrop Equilibria
  - Existence of Wardrop Equilibria
  - Price of Anarchy

- (Atomic) congestion games
  - Model
  - Existence of Nash equilibria
  - Computation of Nash equilibria
  - Price of Anarchy
  - Price of Stability
Potential Games and Price of Stability

Potential Games:

- All games that admit a potential function $\Phi$, s.t. for all outcomes $s$, all player $i$, and all alternative strategies $s'_i$,

  $$C_i(s'_i, s_{-i}) - C_i(s) = \Phi(s'_i, s_{-i}) - \Phi(s).$$

- Every congestion game is a potential game. (cf. Thm. 3.10)
- For every potential game there exists a congestion game having the same potential function. [Monderer, Shapley, 1996]

Definition: Price of Stability

For a game $G$:

$$\text{PoS}(G) = \min_{s \text{ is NE}} \frac{\text{SC}(s)}{\text{OPT}}$$

For a class of games $\mathcal{G}$:

$$\text{PoS}(\mathcal{G}) = \sup_{G \in \mathcal{G}} \text{PoS}(G)$$
Potential Games and Price of Stability

Theorem 3.18 (Thm.19.13)
Suppose that we have a potential game with potential function $\Phi$, and assume that for any outcome $s$, we have

$$\frac{SC(s)}{A} \leq \Phi(s) \leq B \cdot SC(s)$$

for some constants $A, B \geq 0$. Then the price of stability is at most $A \cdot B$.

For congestion games with linear latency functions this yields $\text{PoS} \leq 2$.

Theorem 3.19
Let $G$ be the class of unweighted congestion games with linear latency functions. Then,

$$\text{PoS}(G) = 1 + \frac{\sqrt{3}}{3} \approx 1.577$$