3-Coloring is in \( NP \)

- **Certificate:** for each node a color from \( \{1, 2, 3\} \)
- **Certifier:** Check if for each edge \((u, v)\), the color of \(u\) is different from that of \(v\)

**Hardness:** We will show \( 3\text{-SAT} \leq_P 3\text{-Coloring} \)
Start with 3-SAT formula $\phi$ with $n$ variables $x_1, \ldots, x_n$ and $m$ clauses $C_1, \ldots, C_m$. Create graph $G_\phi$ such that $G_\phi$ is 3-colorable iff $\phi$ is satisfiable

- need to establish truth assignment for $x_1, \ldots, x_n$ via colors for some nodes in $G_\phi$.
- create triangle with node True, False, Base
- for each variable $x_i$ two nodes $v_i$ and $\overline{v}_i$ connected in a triangle with common Base
- If graph is 3-colored, either $v_i$ or $\overline{v}_i$ gets the same color as True. Interpret this as a truth assignment to $v_i$
- For each clause $C_j = (a \lor b \lor c)$, create a small gadget graph
  - gadget graph connects to nodes corresponding to $a, b, c$
  - needs to implement OR
Property: if $a$, $b$, $c$ are colored False in a 3-coloring then output node of OR-gadget has to be colored False.

Property: if one of $a$, $b$, $c$ is colored True then OR-gadget can be 3-colored such that output node of OR-gadget is colored True.
• create triangle with node True, False, Base
• for each variable $x_i$ two nodes $v_i$ and $\bar{v}_i$ connected in a triangle with common Base
• for each clause $C_j = (a \vee b \vee c)$, add OR-gadget graph with input nodes $a, b, c$ and connect output node of gadget to both False and Base
Example

\[ \varphi = (u \lor \neg v \lor w) \land (v \lor x \lor \neg y) \]
Correctness of Reduction

\( \phi \) is satisfiable implies \( G_\phi \) is 3-colorable

- if \( x_i \) is assigned True, color \( v_i \) True and \( \bar{v}_i \) False
- for each clause \( C_j = (a \lor b \lor c) \) at least one of \( a, b, c \) is colored True. OR-gadget for \( C_j \) can be 3-colored such that output is True.

\( G_\phi \) is 3-colorable implies \( \phi \) is satisfiable

- if \( v_i \) is colored True then set \( x_i \) to be True, this is a legal truth assignment
- consider any clause \( C_j = (a \lor b \lor c) \). it cannot be that all \( a, b, c \) are False. If so, output of OR-gadget for \( C_j \) has to be colored False but output is connected to Base and False!