Monodic Temporal Resolution

Anatoly Degtyarev[2], Michael Fisher[1], and Boris Konev[1⋆]

[1] Department of Computer Science, University of Liverpool, Liverpool L69 7ZF, U.K.
{M.Fisher, B.Konev}@csc.liv.ac.uk
[2] Department of Computer Science, King's College, Strand, London WC2R 2LS, U.K.
anatoli@dcs.kcl.ac.uk

Abstract. First-order temporal logic is a concise and powerful notation, with many potential applications in both Computer Science and Artificial Intelligence. While the full logic is highly complex, recent work on *monodic* first-order temporal logics have identified important enumerable and even decidable fragments. In this paper we present the first resolution-based calculus for monodic first-order temporal logic. Although the main focus of the paper is on establishing completeness results, we also consider implementation issues and define a basic loop-search algorithm that may be used to guide the temporal resolution system.

1 Introduction

Temporal Logic has achieved a significant role in Computer Science, in particular, within the formal specification and verification of concurrent and distributed systems [13, 11, 9]. However, even though *first-order* temporal logics have been studied over a number of years and have been recognised as a concise and powerful formalism, most of the temporal logics used remain essentially propositional. The reason for this is that it is easy to show that first-order temporal logic is, in general, incomplete (i.e. not recursively-enumerable [14]). In fact, until recently, it has been difficult to find *any* non-trivial fragment of first-order temporal logic that has reasonable properties. A breakthrough by Hodkinson *et. al.* [8] showed that *monodic* fragments of first-order temporal logic could be complete, even decidable. (In spite of this, the addition of equality or function symbols leads to the loss of recursive enumerability [15].)

The definition of the monodic fragment holds great promise for increasing the power of logic-based formal methods. However, there were, until now, no practical proof techniques for monodic fragments of first-order temporal logics. A general framework, which provides conditions to yield a tableau-based procedure for decidable monodic fragments, and a number of its instantiations, has been presented in [10]. In this paper, we provide a complete resolution calculus for monodic first-order temporal logic, based on our work on clausal temporal resolution over a number of years [5, 7, 1, 2].

Some technical proofs are omitted due to lack of space and can be found in the full version of the paper available as a technical report [3].

2 First-Order Temporal Logic

First-Order (linear time) Temporal Logic, FOTL, is an extension of classical first-order logic with operators that deal with a linear and discrete model of time (isomorphic to

⋆ On leave from Steklov Institute of Mathematics at St.Petersburg

$\mathbb{N}$, and the most commonly used model of time). The first-order temporal language is constructed in a standard way [6, 8] from: *predicate symbols* $P_0, P_1, \ldots$ each of which is of some fixed arity (null-ary predicate symbols are called *propositions*); *individual variables* $x_0, x_1, \ldots$; *individual constants* $c_0, c_1, \ldots$; *Boolean operators* $\wedge$, $\neg$, $\vee$, $\Rightarrow$, $\equiv$ **true** ('true'), **false** ('false'); *quantifiers* $\forall$ and $\exists$; together with *unary temporal operators*, such as[3] $\square$ ('always in the future'), $\lozenge$ ('sometime in the future'), and $\bigcirc$ ('at the next moment'). There are no function symbols and equality in our FOTL language. For a given formula, ϕ, $const(\phi)$ denotes the set of constants occurring in ϕ.

Formulae in FOTL are interpreted in *first-order temporal structures* of the form $\mathfrak{M} = \langle D, I \rangle$, where D is a non-empty set, the *domain* of $\mathfrak{M}$, and I is a function associating with every moment of time, $n \in \mathbb{N}$, an interpretation of predicate and constant symbols over D. We require that the interpretation of constants is *rigid*. Thus, for every constant c and all moments of time $i, j \geq 0$, we have $I_i(c) = I_j(c)$.

A *(variable) assignment* $\mathfrak{a}$ over D is a function from the set of individual variables to D. For every moment of time, n, there is a corresponding *first-order* structure $\mathfrak{M}_n = \langle D, I_n \rangle$, where $I_n = I(n)$. Intuitively, FOTL formulae are interpreted in sequences of *worlds*, $\mathfrak{M}_0, \mathfrak{M}_1, \ldots$ with truth values in different worlds being connected by means of temporal operators.

The *truth* relation $\mathfrak{M}_n \models^{\mathfrak{a}} \phi$ in a structure $\mathfrak{M}$, for an assignment $\mathfrak{a}$, is defined inductively in the usual way under the following understanding of temporal operators:

$$\mathfrak{M}_n \models^{\mathfrak{a}} \bigcirc\phi \text{ iff } \mathfrak{M}_{n+1} \models^{\mathfrak{a}} \phi;$$
$$\mathfrak{M}_n \models^{\mathfrak{a}} \lozenge\phi \text{ iff there exists } m \geq n \text{ such that } \mathfrak{M}_m \models^{\mathfrak{a}} \phi;$$
$$\mathfrak{M}_n \models^{\mathfrak{a}} \square\phi \text{ iff for all } m \geq n, \mathfrak{M}_m \models^{\mathfrak{a}} \phi.$$

$\mathfrak{M}$ is a *model* for a formula ϕ (or ϕ is *true* in $\mathfrak{M}$) if there exists an assignment $\mathfrak{a}$ such that $\mathfrak{M}_0 \models^{\mathfrak{a}} \phi$. A formula is *satisfiable* if it has a model. A formula is *valid* if it is true in any temporal structure under any assignment.

This logic is complex. It is known that even "small" fragments of FOTL, such as the *two-variable monadic* fragment (all predicates are unary), are not recursively enumerable [12, 8]. However, the set of valid *monodic* formulae is known to be finitely axiomatisable [15].

Definition 1. *An* FOTL*-formula ϕ is called* monodic *if any subformulae of the form $\mathcal{T}\psi$, where $\mathcal{T}$ is one of $\bigcirc$, $\square$, $\lozenge$, contains at most one free variable.*

3 Divided Separated Normal Form

Definition 2 (Temporal Step Clauses). *A* temporal step clause *is a formula either in the form $p \Rightarrow \bigcirc l$, where p is a proposition and l is a propositional literal, or $\forall x(P(x) \Rightarrow \bigcirc M(x))$, where P is a unary predicate and M is a unary literal. We call a clause of the first type an (original)* ground *step clause, and of the second type an (original)* non-ground *step clause.*

[3] W.r.t. satisfiability, binary temporal operators $\mathcal{U}$ ('until') and $\mathcal{W}$ ('week until') can be represented using these operators [6, 1].

Definition 3 (Monodic Temporal Problem). *A* monodic temporal problem in Divided Separated Normal Form (DSNF) *is a quadruple* $\langle \mathcal{U}, \mathcal{I}, \mathcal{S}, \mathcal{E} \rangle$, *where*

1. *the universal part,* $\mathcal{U}$, *is given by a set of arbitrary closed first-order formulae;*
2. *the initial part,* $\mathcal{I}$, *is, again, given by a set of arbitrary closed first-order formulae;*
3. *the step part,* $\mathcal{S}$, *is given by a set of original (ground and non-ground) temporal step clauses; and*
4. *the eventuality part,* $\mathcal{E}$, *is given by a set of eventuality clauses of the form* $\Diamond L(x)$ *(a* non-ground *eventuality clause) and* $\Diamond l$ *(a* ground eventuality *clause), where* l *is a propositional literal and* $L(x)$ *is a unary non-ground literal.*

The sets $\mathcal{U}$, $\mathcal{I}$, $\mathcal{S}$, *and* $\mathcal{S}$ *are finite.*

Note that, in a monodic temporal problem, we do not allow two different temporal step clauses with the same left-hand sides. A problem with the same left-hand sides can be easily transformed by renaming into one without. To each monodic temporal problem, we associate the formula $\mathcal{I} \wedge \Box\mathcal{U} \wedge \Box\forall x\mathcal{S} \wedge \Box\forall x\mathcal{E}$. Now, when we talk about particular properties of temporal problems (e.g., satisfiability, validity, logical consequences etc) we mean properties of the associated formula.

Following [6, 7], it was noted in [1] that any monodic FOTL formula can be reduced to a normal form where, in addition to the parts above, *conditional* eventuality clauses of the form $P(x) \Rightarrow \Diamond L(x)$ and $p \Rightarrow \Diamond l$ are allowed. The translation can be described as a number of steps.

1. Translate a given monodic formula to negation normal form.
2. Recursively rename innermost temporal subformulae, $\bigcirc\phi(x)$, $\Diamond\phi(x)$, $\Box\phi(x)$, by new unary predicates $P_i(x)$. Renaming introduces formulae defining $P_i(x)$ as follows:

$$(a)\ \Box\forall x(P_{i_1}(x) \Rightarrow \bigcirc\phi(x)); \quad (b)\ \Box\forall x(P_{i_2}(x) \Rightarrow \Diamond\phi(x));$$
$$(c)\ \Box\forall x(P_{i_3}(x) \Rightarrow \Box\phi(x)).$$

   Formulae of the form (a) and (b) are in the normal form[4], formulae of the form (c) require extra reduction by removing the temporal operators using their fixed point definitions.
3. Use fixed point definitions.
   $\Box\forall x(P(x) \Rightarrow \Box\phi(x))$ is satisfiability equivalent to

$$\Box\forall x(P(x) \Rightarrow R(x)) \wedge \Box\forall x(R(x) \Rightarrow \bigcirc R(x)) \wedge \Box\forall x(R(x) \Rightarrow \phi(x)),$$

   where $R(x)$ is a new unary predicate.

In [2], a reduction from conditional problems to unconditional ones for the propositional case is given. For the first-order case, satisfiability of $\Phi \cup \{\Box\forall x(P(x) \Rightarrow \Diamond L(x))\}$ is equivalent to satisfiability of

$$\Phi \cup \left\{ \begin{array}{l} \Box\forall x((P(x) \wedge \neg L(x)) \Rightarrow \mathit{waitforL}(x)), \\ \Box\forall x((\mathit{waitforL}(x) \wedge \bigcirc\neg L(x)) \Rightarrow \bigcirc\mathit{waitforL}(x)), \\ \Box\forall x\Diamond\neg\mathit{waitforL}(x) \end{array} \right\},$$

[4] Possibly, after (first-order) renaming the complex expression $\phi(x)$; the formulae introduced by renaming are put in the universal part.

where $\mathit{waitforL}(x)$ is a new unary predicate symbol. (The second clause is translated into a step and a universal clauses.)

Theorem 1 (Transformation). *Every monodic first-order temporal formula can be reduced, in a satisfiability equivalence preserving way, to a monodic temporal problem with at most a linear increase in the size of the problem.*

4 Temporal Resolution for Monodic Non-Ground Case

As in the propositional case [5, 2], our calculus works with *merged step clauses*, but here the notion of merged step clauses is much more complex. This is, of course, because of the first-order nature of the problem and the fact that skolemisation is not allowed under temporal operators. First, we provide some required definitions.

Definition 4 (Derived Step Clauses). *Let P be a monodic temporal problem, and let $P_{i_1}(x) \Rightarrow \bigcirc M_{i_1}(x), \ldots, P_{i_k}(x) \Rightarrow \bigcirc M_{i_k}(x)$ be a subset of the set of its original non-ground step clauses. Then*

$$\begin{gathered}\forall x(P_{i_1}(x) \vee \cdots \vee P_{i_k}(x)) \Rightarrow \bigcirc \forall x(M_{i_1}(x) \vee \cdots \vee M_{i_k}(x)),\\ \exists x(P_{i_1}(x) \wedge \cdots \wedge P_{i_k}(x)) \Rightarrow \bigcirc \exists x(M_{i_1}(x) \wedge \cdots \wedge M_{i_k}(x)),\\ P_{i_j}(c) \Rightarrow \bigcirc M_{i_j}(c)\end{gathered}$$

are derived *step clauses, where c is a constant occurring in P and $j = 1 \ldots k$.*

Definition 5 (Merged Derived Step Clauses). *Let $\{\Phi_1 \Rightarrow \bigcirc \Psi_1, \ldots, \Phi_n \Rightarrow \bigcirc \Psi_n\}$ be a set of derived step clauses or original* ground *step clauses. Then $\bigwedge_{i=1}^{n} \Phi_i \Rightarrow \bigcirc \bigwedge_{i=1}^{n} \Psi_i$ is called a* merged derived step clause. *Note that the left-hand and right-hand sides of any merged derived step clause are closed formulae.*

Definition 6 (Full Merged Step Clauses). *Let $\mathcal{A} \Rightarrow \bigcirc \mathcal{B}$ be a merged derived step clause, $P_1(x) \Rightarrow \bigcirc M_1(x), \ldots, P_k(x) \Rightarrow \bigcirc M_k(x)$ be original step clauses, and $A(x) \stackrel{\text{def}}{=} \bigwedge_{i=1}^{k} P_i(x)$, $B(x) \stackrel{\text{def}}{=} \bigwedge_{i=1}^{k} M_i(x)$. Then $\forall x(\mathcal{A} \wedge A(x) \Rightarrow \bigcirc(\mathcal{B} \wedge B(x)))$ is called a* full merged step clause. *(In the case $k = 0$, the conjunctions $A(x)$, $B(x)$ are empty, i.e., their truth value is* **true***, and the merged step clause is just a merged derived step clause.)*

Definition 7 (Constant Flooding). *Let P be a monodic temporal problem, $\mathsf{P}^c = \mathsf{P} \cup \{\Diamond L(c) \mid \Diamond L(x) \in \mathcal{E}, c \in \mathit{const}(\mathsf{P})\}$ is the* constant flooded form[5] *of P. Evidently, P^c is satisfiability equivalent to P.*

Inference Rules. In what follows, $\mathcal{A} \Rightarrow \bigcirc \mathcal{B}$ and $\mathcal{A}_i \Rightarrow \bigcirc \mathcal{B}_i$ denote merged derived step clauses, $\forall x(\mathcal{A} \wedge A(x) \Rightarrow \bigcirc(\mathcal{B} \wedge B(x)))$ and $\forall x(\mathcal{A}_i \wedge A_i(x) \Rightarrow \bigcirc(\mathcal{B}_i \wedge B_i(x)))$ denote full merged step clauses, and $\mathcal{U}$ denotes the (current) universal part of the problem.

[5] Strictly speaking, P^c is not in DSNF: we have to rename ground eventualities by propositions. Rather than 'flooding', we could have introduced special inference rules to deal with constants.

– *Step resolution rule w.r.t. $\mathcal{U}$*: $\dfrac{\mathcal{A} \Rightarrow \bigcirc\mathcal{B}}{\neg\mathcal{A}}$ $(\bigcirc^{\mathcal{U}}_{res})$, where $\mathcal{U} \cup \{\mathcal{B}\} \models \bot$.
– *Initial termination rule w.r.t. $\mathcal{U}$:* The contradiction $\bot$ is derived and the derivation is (successfully) terminated if $\mathcal{U} \cup I \models \bot$.
– *Eventuality resolution rule w.r.t. $\mathcal{U}$:*

$$\frac{\begin{array}{c} \forall x(\mathcal{A}_1 \wedge A_1(x) \Rightarrow \bigcirc(\mathcal{B}_1 \wedge B_1(x))) \\ \dots \\ \forall x(\mathcal{A}_n \wedge A_n(x) \Rightarrow \bigcirc(\mathcal{B}_n \wedge B_n(x))) \end{array} \qquad \Diamond L(x)}{\forall x \bigwedge_{i=1}^{n} (\neg\mathcal{A}_i \vee \neg A_i(x))} \; (\Diamond^{\mathcal{U}}_{res}),$$

where $\forall x(\mathcal{A}_i \wedge A_i(x) \Rightarrow \bigcirc\mathcal{B}_i \wedge B_i(x))$ are full merged step clauses such that for all $i \in \{1,\dots,n\}$, the *loop* side conditions $\forall x(\mathcal{U} \wedge \mathcal{B}_i \wedge B_i(x) \Rightarrow \neg L(x))$ and $\forall x(\mathcal{U} \wedge \mathcal{B}_i \wedge B_i(x) \Rightarrow \bigvee_{j=1}^{n}(\mathcal{A}_j \wedge A_j(x))$ are both valid.
The set of merged step clauses, satisfying the loop side conditions, is called a *loop in* $\Diamond L(x)$ and the formula $\bigvee_{j=1}^{n}(\mathcal{A}_j(x) \wedge A_j(x))$ is called a *loop formula*.
– *Eventuality termination rule w.r.t. $\mathcal{U}$:* The contradiction $\bot$ is derived and the derivation is (successfully) terminated if $\mathcal{U} \models \forall x \neg L(x)$, where $\Diamond L(x) \in \mathcal{E}$.
– *Ground eventuality resolution w.r.t. $\mathcal{U}$* and *Ground eventuality termination w.r.t. $\mathcal{U}$:* These rules repeat the eventuality resolution and eventuality termination rules with the only difference that *ground eventualities* and merged *derived step clauses* are used instead of non-ground eventualities and full merged step clauses.

A *derivation* is a sequence of universal parts, $\mathcal{U} = \mathcal{U}_0 \subseteq \mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq \dots$, extended little by little by the conclusions of the inference rules. Successful termination means that the given problem is unsatisfiable. The I, $\mathcal{S}$ and $\mathcal{E}$ parts of the temporal problem are not changed in a derivation.

Example 1. Let us consider an unsatisfiable temporal problem given by

$$I = \{\, i1.\ \exists x Q(x) \,\},\ \mathcal{U} = \left\{ \begin{array}{ll} u1. & \Box\exists x(P_1(x) \wedge P_2(x)) \\ u2. & \Box\forall x(Q(x) \wedge \exists y(\neg P_1(y) \wedge \neg P_2(y)) \Rightarrow L(x)) \end{array} \right\},$$

$$\mathcal{E} = \{\, e1.\ \Box\forall x \Diamond\neg L(x) \,\},\ \mathcal{S} = \left\{ \begin{array}{ll} s1. & \Box\forall x(P_1(x) \Rightarrow \bigcirc\neg P_1(x)) \\ s2. & \Box\forall x(P_2(x) \Rightarrow \bigcirc\neg P_2(x)) \\ s3. & \Box\forall x(Q(x) \Rightarrow \bigcirc Q(x)) \end{array} \right\}$$

and apply temporal resolution to this. First, we produce the following derived step clause from $s1$ and $s2$: $g1.\ \exists y(P_1(y) \wedge P_2(y)) \Rightarrow \bigcirc\exists y(\neg P_1(y) \wedge \neg P_2(y))$.
Then merge $g1$ and $s3$ to give

$$m1.\quad \Box\forall x(\exists y(P_1(y) \wedge P_2(y)) \wedge Q(x) \Rightarrow \bigcirc(\exists y(\neg P_1(y) \wedge \neg P_2(y)) \wedge Q(x))).$$

It can be immediately checked that the loop side conditions are valid for $m1$, i.e.,

$$\begin{array}{ll} \exists y(\neg P_1(y) \wedge \neg P_2(y)) \wedge Q(x) \Rightarrow L(x) & \text{(see } u2\text{)}, \\ \exists y(\neg P_1(y) \wedge \neg P_2(y)) \wedge Q(x) \Rightarrow \exists y(P_1(y) \wedge P_2(y)) \wedge Q(x) & \text{(see } u1\text{)}. \end{array}$$

We apply the eventuality resolution rule to $e1$ and $m1$ and derive a new universal clause

$$nu1.\quad \Box\forall x(\neg(\exists y(P_1(y) \wedge P_2(y))) \vee \neg Q(x))$$

which contradicts clauses $u1$ and $i1$ (the initial termination rule is applied).

Theorem 2 (Soundness and Completeness of Temporal Resolution). *The rules of temporal resolution preserve satisfiability. If a monodic constant flooded temporal problem* P *is unsatisfiable, then there exists a successfully terminating derivation from it.*

Proof From consideration of the models, it straightforwardly follows that the temporal resolution rules preserve satisfiability. Consider, for example, the step resolution rule. Let $\mathcal{A} \Rightarrow \bigcirc\mathcal{B}$ be a merged derived rule and assume that $\mathfrak{M}_0 \models^{\mathfrak{a}} \square(\mathcal{A} \Rightarrow \bigcirc\mathcal{B})$, but for some $i \geq 0$, $\mathfrak{M}_i \not\models^{\mathfrak{a}} \neg\mathcal{A}$. Then $\mathfrak{M}_{i+1} \models^{\mathfrak{a}} \mathcal{B}$ in contradiction with the rule side condition.

The proof of completeness is difficult, and Section 5 is entirely devoted to this issue. ❑

5 Completeness of Temporal Resolution

In order to prove completeness of the temporal resolution method, we introduce additional concepts (some of which were already defined in [1]). Let $\mathsf{P} = \langle \mathcal{U}, I, \mathcal{S}, \mathcal{E} \rangle$ be a monodic temporal problem. Let $\{P_1, \ldots, P_N\}$ and $\{p_1, \ldots, p_n\}$, $N, n \geq 0$, be the sets of all (monadic) predicate symbols and all propositional symbols, respectively, occurring in $\mathcal{S} \cup \mathcal{E}$. Let Δ be the set of all mappings from $\{1, \ldots, N\}$ to $\{0,1\}$, and Θ be the set of all mappings from $\{1, \ldots, n\}$ to $\{0,1\}$. An element $\delta \in \Delta$ ($\theta \in \Theta$) is represented by the sequence $[\delta(1), \ldots, \delta(N)] \in \{0,1\}^N$ ($[\theta(1), \ldots, \theta(n)] \in \{0,1\}^n$). Let us call elements of Δ and Θ predicate and propositional *colours*, respectively. Let Γ be a subset of Δ, θ be an element of Θ, and ρ be a map from the set of constants of P to Γ. A triple (Γ, θ, ρ) is called a *colour scheme*, and ρ is called a *constant distribution*. If a predicate $P_i(x)$ from $\mathcal{S} \cup \mathcal{E}$ "occurs" in a predicate colour γ (i.e., $\gamma(i) = 1$), we also write $P_i(x) \in \gamma$; and if $\gamma(i) = 0$, we also write $P(x) \notin \gamma$ or $\neg P(x) \in \gamma$. The same convention is used for propositional colours and constant distributions.

For every colour scheme $\mathcal{C} = \langle \Gamma, \theta, \rho \rangle$ let us construct the formulae $\mathcal{F}_{\mathcal{C}}$, $\mathcal{A}_{\mathcal{C}}$, $\mathcal{B}_{\mathcal{C}}$ in the following way. For every $\gamma \in \Gamma$ and for every θ, introduce the conjunctions:

$$F_\gamma(x) = \bigwedge_{i \leq N,\, \gamma(i)=1} P_i(x) \;\wedge \bigwedge_{i \leq N,\, \gamma(i)=0} \neg P_i(x), \qquad F_\theta = \bigwedge_{i \leq n,\, \theta(i)=1} p_i \;\wedge \bigwedge_{i \leq n,\, \theta(i)=0} \neg p_i.$$

Let us define two sets of indexes

$J_\gamma = \{i,\ 1 \leq i \leq N \mid \gamma(i) = 1 \text{ and } P_i(x) \Rightarrow \bigcirc M_i(x) \text{ belongs to } \mathcal{S} \text{ for some } M_i\}$ and

$J_\theta = \{j,\ 1 \leq i \leq n \mid \theta(j) = 1 \text{ and } p_j \Rightarrow \bigcirc m_i \text{ belongs to } \mathcal{S} \text{ for some } m_i\}$.

(Recall that there are no two different step clauses with the same left-hand side.) Let $A_\gamma(x) = \bigwedge_{i \in J_\gamma} P_i(x)$, $\quad B_\gamma(x) = \bigwedge_{i \in J_\gamma} M_i(x)$, $\quad A_\theta = \bigwedge_{i \in J_\theta} p_i$, $\quad B_\theta = \bigwedge_{i \in J_\theta} m_i$.

Now $\mathcal{F}_{\mathcal{C}}$, $\mathcal{A}_{\mathcal{C}}$, $\mathcal{B}_{\mathcal{C}}$ are of the following forms:

$$\begin{aligned}
\mathcal{F}_{\mathcal{C}} &= \bigwedge_{\gamma \in \Gamma} \exists x F_\gamma(x) \wedge F_\theta \wedge \bigwedge_{c \in C} F_{\rho(c)}(c) \wedge \forall x \bigvee_{\gamma \in \Gamma} F_\gamma(x),\\
\mathcal{A}_{\mathcal{C}} &= \bigwedge_{\gamma \in \Gamma} \exists x A_\gamma(x) \wedge A_\theta \wedge \bigwedge_{c \in C} A_{\rho(c)}(c) \wedge \forall x \bigvee_{\gamma \in \Gamma} A_\gamma(x),\\
\mathcal{B}_{\mathcal{C}} &= \bigwedge_{\gamma \in \Gamma} \exists x B_\gamma(x) \wedge B_\theta \wedge \bigwedge_{c \in C} B_{\rho(c)}(c) \wedge \forall x \bigvee_{\gamma \in \Gamma} B_\gamma(x).
\end{aligned}$$

We can consider the formula $\mathcal{F}_{\mathcal{C}}$ as a "categorical" formula specification of the quotient structure given by a colour scheme. In turn, the formula $\mathcal{A}_{\mathcal{C}}$ represents the part of this

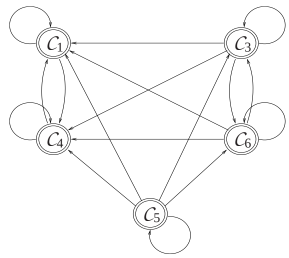

Fig. 1. Behaviour graph for the problem from Example 2.

specification which is "responsible" just for "transferring" requirements from the current world (quotient structure) to its immediate successors, and $\mathcal{B}_{\mathcal{C}}$ represents the result of transfering.

Definition 8 (Canonical merged derived step clauses). *Let* P *be a first-order temporal problem,* $\mathcal{C}$ *be a colour scheme for* P. *Then the clause* $(\mathcal{A}_{\mathcal{C}} \Rightarrow \bigcirc \mathcal{B}_{\mathcal{C}})$, *is called a* canonical merged derived step clause *for* P. *If all the sets* J_γ, *for all* $\gamma \in \Gamma$, *and* J_θ *are empty, the clause* $(\mathcal{A}_{\mathcal{C}} \Rightarrow \bigcirc \mathcal{B}_{\mathcal{C}})$ degenerates *to* $(\mathbf{true} \Rightarrow \bigcirc \mathbf{true})$. *If a conjunction* $A_\gamma(x)$, $\gamma \in \Gamma$, *is empty, that is its truth value is* **true**, *then the formula* $\forall x \bigvee_{\gamma \in \Gamma} A_\gamma(x)$ *(or* $\forall x \bigvee_{\gamma \in \Gamma} B_\gamma(x)$*) disappears from* $\mathcal{A}_{\mathcal{C}}$ *(or from* $\mathcal{B}_{\mathcal{C}}$ *respectively). In the propositional case, the clause* $(\mathcal{A}_{\mathcal{C}} \Rightarrow \bigcirc \mathcal{B}_{\mathcal{C}})$ *reduces to* $(A_\theta \Rightarrow \bigcirc B_\theta)$.

Definition 9 (Canonical merged step clause). *Let* $\mathcal{C}$ *be a colour scheme,* $\mathcal{A}_{\mathcal{C}} \Rightarrow \bigcirc \mathcal{B}_{\mathcal{C}}$ *be a canonical merged derived step clause, and* $\gamma \in \mathcal{C}$. $\forall x(\mathcal{A}_{\mathcal{C}} \wedge A_\gamma(x) \Rightarrow \bigcirc(\mathcal{B}_{\mathcal{C}} \wedge B_\gamma(x)))$ *is called a* canonical merged step clause. *If the set* J_γ *is empty, the truth value of the conjunctions* $A_\gamma(x)$, $B_\gamma(x)$ *is* **true**, *and the canonical merged step clause is just a canonical merged derived step clause.* $\gamma \in \mathcal{C}$ *abbreviates here* $\gamma \in \Gamma$, *where* $\mathcal{C} = (\Gamma, \theta, \rho)$.

Now, given a temporal problem $\mathsf{P} = \langle \mathcal{U}, I, \mathcal{S}, \mathcal{E} \rangle$ we define a finite directed graph G as follows. Every vertex of G is a colour scheme $\mathcal{C}$ for P such that $\mathcal{U} \cup \mathcal{F}_{\mathcal{C}}$ is satisfiable. For each vertex $\mathcal{C} = (\Gamma, \theta, \rho)$, there is an edge in G to $\mathcal{C}' = (\Gamma', \theta', \rho')$, if $\mathcal{U} \wedge \mathcal{F}_{\mathcal{C}'} \wedge \mathcal{B}_{\mathcal{C}}$ is satisfiable. They are the only edges originating from $\mathcal{C}$. A vertex $\mathcal{C}$ is designated as an *initial* vertex of G if $I \wedge \mathcal{U} \wedge \mathcal{F}_{\mathcal{C}}$ is satisfiable. The *behaviour graph* H of P is the subgraph of G induced by the set of all vertices reachable from the initial vertices.

Example 2. Consider a monodic temporal problem, P, given by

$$I = \emptyset,\ \mathcal{U} = \{l \Rightarrow \exists x P(x)\},\ \mathcal{S} = \{P(x) \Rightarrow \bigcirc P(x)\},\ \mathcal{E} = \{\Diamond \neg P(x), \Diamond l\}.$$

For this problem, there exist two predicate colours, $\gamma_1 = [1]$ and $\gamma_2 = [0]$; two propositional colours $\theta_1 = [1]$ and $\theta_2 = [0]$; and six colour schemes, $\mathcal{C}_1 = (\{\gamma_1\}, \theta_1)$, $\mathcal{C}_2 = (\{\gamma_2\}, \theta_1)$, $\mathcal{C}_3 = (\{\gamma_1, \gamma_2\}, \theta_1)$. $\mathcal{C}_4 = (\{\gamma_1\}, \theta_2)$, $\mathcal{C}_5 = (\{\gamma_2\}, \theta_2)$, $\mathcal{C}_6 = (\{\gamma_1, \gamma_2\}, \theta_2)$.

$$\begin{array}{lll}
\mathcal{F}_{\mathcal{C}_1} = \exists x P(x) \wedge \forall x P(x) \wedge l & \mathcal{A}_{\mathcal{C}_1} = \exists x P(x) \wedge \forall x P(x) & \mathcal{B}_{\mathcal{C}_1} = \exists x P(x) \wedge \forall x P(x) \\
\mathcal{F}_{\mathcal{C}_2} = \exists x \neg P(x) \wedge \forall x \neg P(x) \wedge l & \mathcal{A}_{\mathcal{C}_2} = \mathbf{true} & \mathcal{B}_{\mathcal{C}_2} = \mathbf{true} \\
\mathcal{F}_{\mathcal{C}_3} = \exists x P(x) \wedge \exists x \neg P(x) \wedge l & \mathcal{A}_{\mathcal{C}_3} = \exists x P(x) & \mathcal{B}_{\mathcal{C}_3} = \exists x P(x) \\
\mathcal{F}_{\mathcal{C}_4} = \exists x P(x) \wedge \forall x P(x) \wedge \neg l & \mathcal{A}_{\mathcal{C}_4} = \exists x P(x) \wedge \forall x P(x) & \mathcal{B}_{\mathcal{C}_4} = \exists x P(x) \wedge \forall x P(x)
\end{array}$$

$$\mathcal{F}_{\mathcal{C}_5} = \exists x \neg P(x) \wedge \forall x \neg P(x) \wedge \neg l \quad \mathcal{A}_{\mathcal{C}_5} = \mathbf{true} \quad \mathcal{B}_{\mathcal{C}_5} = \mathbf{true}$$
$$\mathcal{F}_{\mathcal{C}_6} = \exists x P(x) \wedge \exists x \neg P(x) \wedge \neg l \quad \mathcal{A}_{\mathcal{C}_6} = \exists x P(x) \quad \mathcal{B}_{\mathcal{C}_6} = \exists x P(x)$$

Note that $\mathcal{F}_{\mathcal{C}_2} \wedge \mathcal{U} \models \perp$. The behaviour graph for P, given in Fig. 1, consists of five vertices; all of them are initial.

Definition 10 (Path; Path Segment). *A* path, π, *through a behaviour graph, H, is a function from $\mathbb{N}$ to the vertices of the graph such that for any $i \geq 0$ there is an edge $\langle \pi(i), \pi(i+1) \rangle$ in H. In the similar way, we define a* path segment *as a function from $[m,n]$, $m < n$, to the vertices of H with the same property.*

Lemma 1. *Let $\mathsf{P}_1 = \langle \mathcal{U}_1, \mathcal{I}, \mathcal{S}, \mathcal{E} \rangle$ and $\mathsf{P}_2 = \langle \mathcal{U}_2, \mathcal{I}, \mathcal{S}, \mathcal{E} \rangle$ be two problems such that $\mathcal{U}_1 \subseteq \mathcal{U}_2$. Then the behaviour graph of P_2 is a subgraph of the behaviour graph of P_1.*

Definition 11 (Suitability). *For $\mathcal{C} = (\Gamma, \theta, \rho)$ and $\mathcal{C}' = (\Gamma', \theta', \rho')$, let $(\mathcal{C}, \mathcal{C}')$ be an ordered pair of colour schemes for a temporal problem P. An ordered pair of predicate colours (γ, γ') where $\gamma \in \Gamma$, $\gamma' \in \Gamma'$ is called* suitable *if the formula $\mathcal{U} \wedge \exists x(F_{\gamma'}(x) \wedge B_{\gamma}(x))$ is satisfiable. Similarly, an ordered pair of propositional colours (θ, θ') is suitable if $\mathcal{U} \wedge F_{\theta'} \wedge B_{\theta}$ is satisfiable, and an ordered pair of constant distributions (ρ, ρ') is suitable if, for every $c \in C$, the pair $(\rho(c), \rho'(c))$ is suitable.*

Note that the satisfiability of $\exists x(F_{\gamma'}(x) \wedge B_{\gamma}(x))$ implies $\models \forall x(F_{\gamma'}(x) \Rightarrow B_{\gamma}(x))$ as the conjunction $F_{\gamma'}(x)$ contains a valuation at x of *all* predicates occurring in $B_{\gamma}(x)$.

Lemma 2. *Let H be the behaviour graph for the problem $\mathsf{P} = \langle \mathcal{U}, \mathcal{I}, \mathcal{S}, \mathcal{E} \rangle$ with an edge from a vertex $\mathcal{C} = (\Gamma, \theta, \rho)$ to a vertex $\mathcal{C}' = (\Gamma', \theta', \rho')$. Then for every $\gamma \in \Gamma$ there exists a $\gamma' \in \Gamma'$ such that the pair (γ, γ') is suitable; for every $\gamma' \in \Gamma'$ there exists a $\gamma \in \Gamma$ such that the pair (γ, γ') is suitable; the pair of propositional colours (θ, θ') is suitable; the pair of constant distributions (ρ, ρ') is suitable.*

Definition 12 (Run/E-Run). *Let π be a path through a behaviour graph H of a temporal problem P, and $\pi(i) = (\Gamma_i, \theta_i, \rho_i)$. By a* run *in π we mean a function $r(n)$ from $\mathbb{N}$ to $\bigcup_{i \in \mathbb{N}} \Gamma_i$ such that for every $n \in \mathbb{N}$, $r(n) \in \Gamma_n$ and the pair $(r(n), r(n+1))$ is suitable. In the similar way, we define a* run segment *as a function from $[m,n]$, $m < n$, to $\bigcup_{i \in \mathbb{N}} \Gamma_i$ with the same property. A run r is called an* e-run *if $\forall i \geq 0 \forall \Diamond L(x) \in \mathcal{E} \exists j > i (L(x) \in r(j))$*[6].

Let π be a path, the set of all runs in π is denoted by $\mathcal{R}(\pi)$, and the set of all e-runs in π is denoted by $\mathcal{R}_e(\pi)$. If π is clear, we may omit it.

Example 3. $\pi = \mathcal{C}_3, \mathcal{C}_6, \mathcal{C}_3, \mathcal{C}_6, \ldots$ is a path through the behaviour graph given in Fig. 1. $r_1 = \gamma_1, \gamma_1, \ldots$ and $r_2 = \gamma_1, \gamma_2, \gamma_1, \gamma_2, \ldots$ are both runs in π. r_2 is an e-run, but r_1 is not.

Theorem 3 (Existence of a model). *Let $\mathsf{P} = \langle \mathcal{U}, \mathcal{I}, \mathcal{S}, \mathcal{E} \rangle$ be a temporal problem. Let H be the behaviour graph of P, let $\mathcal{C}$ and $\mathcal{C}'$ be vertices of H such that $\mathcal{C} = (\Gamma, \theta, \rho)$ and $\mathcal{C}' = (\Gamma', \theta', \rho')$. If both the set of initial vertices of H is non-empty and the following conditions hold*[7]

$$\forall \gamma \in \Gamma \, \forall \mathcal{C} \forall \Diamond L(x) \in \mathcal{E} \exists \gamma' \in \Gamma' \exists \mathcal{C}' \, \left((\mathcal{C}, \gamma) \to^{+} \left(\mathcal{C}', \gamma' \right) \wedge L(x) \in \gamma' \right), \tag{1}$$

[6] To make the presentation compact, we abuse the notation by allowing the use of logical symbols at meta-level.

[7] Here $(\mathcal{C}, \gamma) \to^{+} (\mathcal{C}', \gamma')$ denotes that there exists a path segment π from $\mathcal{C}$ to $\mathcal{C}'$ such that γ and γ' belong to a run segment r in π, i.e., $\pi(m) = \mathcal{C}$, $\pi(n) = \mathcal{C}'$, $r(m) = \gamma \in \Gamma$, and $r(n) = \gamma' \in \Gamma'$ for some $m < n$; $\mathcal{C} \to^{+} \mathcal{C}'$ denotes that there exists a path segment from $\mathcal{C}$ to $\mathcal{C}'$.

$$\forall c \in const(\mathsf{P})\ \forall \mathcal{C} \forall \Diamond L(x) \in \mathcal{E}\ \ \exists \mathcal{C}'\ \left(\mathcal{C} \to^{+} \mathcal{C}' \wedge L(x) \in \rho'(c)\right), \tag{2}$$

$$\forall \mathcal{C} \forall \Diamond l \in \mathcal{E}\ \ \exists \mathcal{C}'\ \left(\mathcal{C} \to^{+} \mathcal{C}' \wedge l \in \theta'\right), \tag{3}$$

then P *has a model.*

Note 1. For *constant flooded* problems condition 3 of Theorem 3 implies condition 2.

This theorem generalises its ground eventuality counterpart in [1] (Lemma 5) and its proof, therefore, is omitted and given in full in [3]. This generalisation is made possible by the following intricate, but essential, lemma.

Lemma 3. *Under the conditions of Theorem 3, there exists a path* π *through* H *where:*

(a) $\pi(0)$ *is an initial vertex of* H*;*
(b) for every colour scheme $\mathcal{C} = \pi(i)$, $i \geq 0$, *and every ground eventuality literal* $\Diamond l \in \mathcal{E}$ *there exists a colour scheme* $\mathcal{C}' = \pi(j)$, $j > i$, *such that* $l \in \theta'$*;*
(c) for every colour scheme $\mathcal{C} = \pi(i)$, $i \geq 0$ *and every predicate colour* γ *from the colour scheme there exists an e-run* $r \in \mathcal{R}_e(\pi)$ *such that* $r(i) = \gamma$*; and*
(d) for every constant $c \in \mathcal{L}$, *the function* $r_c(n)$ *defined by* $r_c(n) = \rho_n(c)$, *where* ρ_n *is the constant distribution from* $\pi(n)$, *is an e-run in* π.

Proof [of Lemma 3] Let $\Diamond L_1(x), \ldots, \Diamond L_k(x)$ be all non-ground eventuality literals from $\mathcal{E}$; $\Diamond l_1, \ldots, \Diamond l_p$ be all ground eventuality literals from $\mathcal{E}$; and $c_1, \ldots, c_q$ be all constants of P. Let $\mathcal{C}_0$ be an initial vertex of H. We construct the path π as follows. Let $\{\gamma_1, \ldots, \gamma_{s_0}\}$ be all predicate colours from $\Gamma_{\mathcal{C}_0}$. By condition (1) there exists a vertex $\mathcal{C}_0^{(\gamma_1,L_1)}$ and a predicate colour $\gamma_1^{(1)} \in \Gamma_{\mathcal{C}_0^{(\gamma_1,L_1)}}$ such that $(\mathcal{C}_0, \gamma_1) \to^{+} (\mathcal{C}_0^{(\gamma_1,L_1)}, \gamma_1^{(1)})$ and $L_1(x) \in \gamma_1^{(1)}$. In the same way, there exists a vertex $\mathcal{C}_0^{(\gamma_1,L_2)}$ and a predicate colour $\gamma_1^{(2)} \in \Gamma_{\mathcal{C}_0^{(\gamma_1,L_2)}}$ such that $(\mathcal{C}_0^{(\gamma_1,L_1)}, \gamma_1^{(1)}) \to^{+} (\mathcal{C}_0^{(\gamma_1,L_2)}, \gamma_1^{(2)})$ and $L_2(x) \in \gamma_1^{(2)}$. And so on. Finally, there exists a vertex $\mathcal{C}_0^{(\gamma_1,L_k)}$ and a predicate colour $\gamma_1^{(k)} \in \Gamma_{\mathcal{C}_0^{(\gamma_1,L_k)}}$ such that $(\mathcal{C}_0^{(\gamma_1,L_{k-1})}, \gamma_1^{(k-1)}) \to^{+} (\mathcal{C}_0^{(\gamma_1,L_k)}, \gamma_1^{(k)})$ and $L_k(x) \in \gamma_1^{(k)}$. Clearly, γ_1, $\ldots, \gamma_1^{(1)}, \ldots, \gamma_1^{(2)}, \ldots, \gamma_1^{(k)}$ forms a segment of a run and every non-ground eventuality is satisfied along this segment.

Now, let $\gamma_2^{(0)}$ be any successor of γ_2 in $\Gamma_{\mathcal{C}_0^{(\gamma_1,L_k)}}$. As above, there exists a sequence of vertices $\mathcal{C}_0^{(\gamma_2,L_1)}, \ldots, \mathcal{C}_0^{(\gamma_2,L_k)}$ and a sequence of predicate colours $\gamma_2^{(1)} \in \Gamma_{\mathcal{C}_0^{(\gamma_2,L_1)}}, \ldots,$ $\gamma_2^{(k)} \in \Gamma_{\mathcal{C}_0^{(\gamma_2,L_k)}}$ such that $\gamma_2, \ldots, \gamma_2^{(0)}, \ldots, \gamma_2^{(1)}, \ldots, \gamma_2^{(k)}$ forms a segment of a run and every non-ground eventuality is satisfied along this segment. And so on. At a certain point we construct a segment of a path from $\mathcal{C}_0$ to a vertex $\mathcal{C}_0^{(\gamma_{s_0},L_k)}$ such that for every $\gamma \in \mathcal{C}_0$ there exists $\gamma' \in \mathcal{C}_0^{(\gamma_{s_0},L_k)}$ such that all eventualities are satisfied on the run-segment from γ to γ'.

In a similar way we can construct a vertex $\mathcal{C}_0^{(c_1,L_1)}$ such that $\mathcal{C}_0^{(\gamma_{s_0},L_k)} \to^{+} \mathcal{C}_0^{(c_1,L_1)}$ and $L_1(x) \in \rho_{\mathcal{C}_0^{(c_1,L_1)}}(c_1)$. And so on. Then we can construct a vertex $\mathcal{C}_0^{(l_1)}$ such that $\mathcal{C}_0^{(c_q,L_k)} \to^{+} \mathcal{C}_0^{(l_1)}$ and $l_1 \in \theta_{\mathcal{C}_0^{(l_1)}}$. And so on.

Finally, we construct a vertex $\mathcal{C}_0' = \mathcal{C}_0^{(l_p)}$ such that $\mathcal{C}_0 \to^+ \mathcal{C}_0'$ and on this path segment all conditions of the theorem hold for $\mathcal{C} = \mathcal{C}_0$. Let us denote this path segment as λ_0, and let $\mathcal{C}_1$ be any successor of $\mathcal{C}_0'$.

By analogy, we can construct a vertex $\mathcal{C}_1'$ and a path segment λ_1 from $\mathcal{C}_1$ to $\mathcal{C}_1'$ such that all conditions of the theorem hold for $\mathcal{C} = \mathcal{C}_1$. An so forth. Eventually, we construct a sequence $\mathcal{C}_0$, $\mathcal{C}_1$,..., $\mathcal{C}_j$ such that there exists n, $0 \le n < j$ and $\mathcal{C}_n = \mathcal{C}_j$ because there are only finitely many different colour schemes. Let $\pi_1 = \lambda_0, \ldots, \lambda_{n-1}, \pi_2 = \lambda_n, \ldots \lambda_{j-1}$. Now, we define our path π as $\pi_1(\pi_2)^*$. Properties (a) and (b) evidently hold on π.

Let $\mathcal{C} = \pi(i)$ and $\gamma \in \Gamma_{\mathcal{C}}$. Clearly, there exist $\gamma' \in \mathcal{C}_0$ and $\gamma'' \in \mathcal{C}_n$ such that $(\mathcal{C}_0, \gamma') \to^+ (\mathcal{C}, \gamma)$ and $(\mathcal{C}, \gamma) \to^+ (\mathcal{C}_n, \gamma'')$. Since for every $\gamma'' \in \mathcal{C}_n$ there exists $\gamma''' \in \mathcal{C}_n^{(\gamma_{s_n}, L_k)}$ such that all eventualities are satisfied on the run-segment from γ'' to γ''' and there exists $\gamma^{(4)} \in \mathcal{C}_n$, $(\mathcal{C}_n^{(\gamma_{s_n}, L_k)}, \gamma''') \to^+ (\mathcal{C}_n, \gamma^{(4)})$, then there is an e-run, r, such that $r(i) = \gamma$, i.e., property (c) holds.

Note that, for every constant c of P the sequence $r_c(n)$ is a run in π. By construction, for every $\Diamond L(x) \in \mathcal{E}$ there is a vertex $\mathcal{C}_n^{(c,L)}$ in π_2 such that $L(x) \in \rho_{\mathcal{C}_n^{(c,L)}}(c)$. Therefore, $r_c(n)$ is an e-run in π and property (d) holds. ❑

Proof [**Theorem 2: completeness of temporal resolution**] The proof proceeds by induction on the number of vertices in the behaviour graph H for $\mathsf{P} = \langle \mathcal{U}, I, \mathcal{S}, \mathcal{E} \rangle$, which is finite. If H is empty then the set $\mathcal{U} \cup I$ is unsatisfiable. In this case the derivation is successfully terminated by the initial termination rule.

Now suppose H is not empty. Let $\mathcal{C}$ be a vertex of H which has no successors. In this case the set $\mathcal{U} \cup \mathcal{B}_{\mathcal{C}}$ is unsatisfiable. Indeed, suppose $\mathcal{U} \cup \{\mathcal{B}_{\mathcal{C}}\}$ is true in a model $\langle D', I' \rangle$. Then we can define a colour scheme $\mathcal{C}'$ such that $\langle D', I' \rangle \models \mathcal{F}_{\mathcal{C}'}$. (Indeed, for every $a \in D'$ let $\gamma_{(a)}$ be a map from $\{1, \ldots, N\}$ to $\{0, 1\}$ such that $\gamma_{(a)}(i) = 1$ if, and only if, $\mathfrak{M} \models P_i(a)$ for every $1 \le i \le N$. Similarly, let θ be a map from $\{1, \ldots, n\}$ to $\{0, 1\}$ such that $\theta(j) = 1$ if, and only if, $\mathfrak{M} \models p_i$ for every $1 \le j \le n$. Define Γ as $\{\gamma_{(a)} \mid a \in D'\}$, and $\rho(c)$ as $\gamma_{(c^{I'})}$.) As $\mathcal{B}_{\mathcal{C}} \wedge \mathcal{F}_{\mathcal{C}'}$ is satisfiable, there exists an edge from the vertex $\mathcal{C}$ to the vertex $\mathcal{C}'$ in the contradiction with the choice of C as having no successor.

The conclusion of the step resolution rule, $\neg \mathcal{A}_{\mathcal{C}}$, is added to the set $\mathcal{U}$; this implies removing the vertex $\mathcal{C}$ from the behaviour graph because the set $\{\mathcal{F}_{\mathcal{C}}, \neg \mathcal{A}_{\mathcal{C}}\}$ is not satisfiable.
Next, we check the possibility where H is not empty and every vertex H has a successor. Ought to Note 1, we consider two cases of violation of the conditions of Theorem 3.

First condition of Theorem 3 does not hold. The negation of (1) gives the following:

$$\exists \mathcal{C}\, \exists \gamma \in \Gamma\, \exists \Diamond L(x) \in \mathcal{E}\, \forall \gamma' \in \Gamma'\, \forall \mathcal{C}'\, ((\mathcal{C}, \gamma) \to^+ (\mathcal{C}', \gamma') \Rightarrow L(x) \notin \gamma'). \tag{4}$$

Let $\mathcal{C}_0$, γ_0, and $\Diamond L_0(x)$ be the vertex, colour and eventuality, respectively, determined by the existential quantifiers of (4). Let $\mathfrak{I}$ and $\mathfrak{I}_i, i \in \mathfrak{I}$ be finite nonempty sets of indexes such that $\{\mathcal{C}_i \mid i \in \mathfrak{I}\}$ is the set of *all* successors of $\mathcal{C}_0$ (possibly including $\mathcal{C}_0$ itself) and $\{\gamma_{i,j} \in \Gamma_i \mid i \in \mathfrak{I},\ j \in \mathfrak{I}_i,\ \gamma_0 \to^+ \gamma_{i,j}\}$ is the set of *all* predicate colours such that there exists a run going through γ_0 and the colour. (To unify notation, if $0 \notin \mathfrak{I}$, we define $\mathfrak{I}_0$ as $\{0\}$, and $\gamma_{0,0}$ as γ_0; and if $0 \in \mathfrak{I}$, we add the index of γ_0 to $\mathfrak{I}_0$. Therefore, $\mathfrak{I}_0$ is always defined and without loss of generality we may assume that $\gamma_{0,0} = \gamma_0$.)

Let $\mathcal{C}_{i_1},\ldots,\mathcal{C}_{i_k}$ be the set of all immediate successors of $\mathcal{C}_0$. To simplify the proof, we will represent canonical merged derived step clauses $\mathcal{A}_{\mathcal{C}_i} \Rightarrow \bigcirc\mathcal{B}_{\mathcal{C}_i}$ ($\mathcal{A}_{\mathcal{C}_{i_l}} \Rightarrow \bigcirc\mathcal{B}_{\mathcal{C}_{i_l}}$) simply as $\mathcal{A}_i \Rightarrow \bigcirc\mathcal{B}_i$ ($\mathcal{A}_{i_l} \Rightarrow \bigcirc\mathcal{B}_{i_l}$), and formulae $\mathcal{F}_{\mathcal{C}_i}$ ($\mathcal{F}_{\mathcal{C}_{i_l}}$) simply as $\mathcal{F}_i$ ($\mathcal{F}_{i_l}$).

Consider two cases depending on whether the canonical merged derived step clause $\mathcal{A}_0 \Rightarrow \bigcirc\mathcal{B}_0$ (or any of $\mathcal{A}_i \Rightarrow \bigcirc\mathcal{B}_i$, $i \in \mathfrak{I}$) degenerates or not.

Let $\mathcal{A}_0 = \mathcal{B}_0 = \mathbf{true}$. It follows that $\mathcal{U} \models \forall x \neg L_0(x)$. Indeed, suppose $\mathcal{U} \cup \{\exists x L_0(x)\}$ has a model, $\langle D', I'\rangle$. Then we can construct a colour scheme $\mathcal{C}'$ such that $\langle D', I'\rangle \models \mathcal{F}_{\mathcal{C}'}$. Since $\mathcal{C}_{i_1},\ldots,\mathcal{C}_{i_k}$ is the set of all immediate successors of $\mathcal{C}_0$ and $\mathcal{B}_0 = \mathbf{true}$, it holds that there exists $j, 1 \le j \le k$, such that $\mathcal{C}_{i_j} = \mathcal{C}'$. Since $B_{\gamma_0}(x) = \mathbf{true}$, every pair (γ_0, γ'), where $\gamma' \in \Gamma'$, is suitable; hence $\neg L_0(x) \in \gamma'$ for every $\gamma' \in \Gamma'$, and $\mathcal{F}_{\mathcal{C}'} \models \forall x \neg L_0(x)$ leading to a contradiction. Therefore, $\mathcal{U} \models \forall x \neg L_0(x)$ and the eventuality termination rule can be applied. The same holds if any one of $\mathcal{A}_i \Rightarrow \bigcirc\mathcal{B}_i$ degenerates.

Let none of the $\mathcal{A}_i \Rightarrow \bigcirc\mathcal{B}_i$ degenerate. We are going to prove that the eventuality resolution rule can be applied. First, we have to check the side conditions for such an application.

1. $\forall x(\mathcal{U} \wedge \mathcal{B}_i \wedge B_{\gamma_{i,j}}(x) \Rightarrow \neg L_0(x))$ for all $i \in \mathfrak{I} \cup \{0\}$, $j \in \mathfrak{I}_i$.
   Consider the case when $i = j = 0$ (for other indexes the arguments are similar).
   We show that $\forall x(\mathcal{U} \wedge \mathcal{B}_0 \wedge B_{\gamma_0}(x) \Rightarrow \bigvee\limits_{l \in \{1,\ldots,k\},\, \gamma' \in \Gamma_{i_l},\, \gamma \to \gamma'} F_{\gamma'}(x))$ is valid (it follows, in particular, that $\forall x(\mathcal{U} \wedge \mathcal{B}_0 \wedge B_{\gamma_0}(x) \Rightarrow \neg L_0(x))$ is valid). Suppose $\langle D', I'\rangle$ is a model for $\exists x(\mathcal{U} \wedge \mathcal{B}_0 \wedge B_{\gamma_0}(x) \wedge \bigwedge\limits_{l \in \{1,\ldots,k\},\, \gamma' \in \Gamma_{i_l},\, \gamma \to \gamma'} \neg F_{\gamma'}(x))$. Then there exists a colour scheme $\mathcal{C}'$ such that $\langle D', I'\rangle \models \mathcal{F}_{\mathcal{C}'}$. Since $\langle D', I'\rangle \models \mathcal{B}_0 \wedge \mathcal{F}_{\mathcal{C}'}$, we conclude that $\mathcal{C}'$ is among $\mathcal{C}_{i_1},\ldots,\mathcal{C}_{i_k}$. Note that $\langle D', I'\rangle \models \mathcal{F}_{\mathcal{C}'}$ follows. In particular $\langle D', I'\rangle \models \forall x \bigvee\limits_{\gamma'' \in \Gamma'} F_{\gamma''}(x)$ and, hence, $\langle D', I'\rangle \models \forall x(B_{\gamma_0}(x) \Rightarrow \bigvee\limits_{\gamma'' \in \Gamma'} F_{\gamma''}(x))$. Together with the fact that $\langle D', I'\rangle \models \exists x(B_{\gamma_0}(x) \wedge F_{\gamma''}(x))$ implies $\gamma_0 \to \gamma''$, we have $\langle D', I'\rangle \models \forall x(B_{\gamma_0}(x) \Rightarrow \bigvee\limits_{\gamma'' \in \Gamma',\, \gamma_0 \to \gamma''} F_{\gamma''}(x))$. This contradicts the choice of the structure $\langle D', I'\rangle$.
2. $\forall x(\mathcal{U} \wedge \mathcal{B}_i \wedge B_{\gamma_{i,j}}(x) \Rightarrow \bigvee\limits_{k \in \mathfrak{I} \cup \{0\},\, l \in \mathfrak{I}_k} (\mathcal{A}_k \wedge A_{\gamma_{k,l}}(x)))$ for all $i \in \mathfrak{I} \cup \{0\}$, $j \in \mathfrak{I}_i$.
   Again, consider the case $i = j = 0$. Suppose $\mathcal{U} \wedge \mathcal{B}_0 \wedge \exists x(B_{\gamma_0}(x) \wedge \bigwedge\limits_{k \in \mathfrak{I} \cup \{0\},\, l \in \mathfrak{I}_k} (\neg(\mathcal{A}_k \wedge A_{\gamma_{k,l}}(x))))$ is satisfied in a structure $\langle D', I'\rangle$. Let $\mathcal{C}'$ be a colour scheme such that $\langle D', I'\rangle \models \mathcal{F}_{\mathcal{C}'}$. By arguments similar to the ones given above, there is a vertex $\mathcal{C}_{i_l}$, $1 \le l \le k$, which is an immediate successor of $\mathcal{C}_0$, such that $\mathcal{C}_{i_l} = \mathcal{C}'$, and hence $\langle D', I'\rangle \models \mathcal{A}'$. It suffices to note that $\langle D', I'\rangle \models \forall x(B_{\gamma_0}(x) \Rightarrow \bigvee\limits_{\gamma'' \in \Gamma',\, \gamma_0 \to \gamma''} A_{\gamma''}(x))$. (As in the case 1 above, $\langle D', I'\rangle \models \forall x(B_{\gamma_0}(x) \Rightarrow \bigvee\limits_{\gamma'' \in \Gamma',\, \gamma_0 \to \gamma''} F_{\gamma''}(x))$, and for all $\gamma'' \in \Gamma'$, the formula $\forall x(F_{\gamma''}(x) \Rightarrow A_{\gamma''}(x))$ is valid.)

After applying the eventuality resolution rule we add to $\mathcal{U}$ its conclusion: $\forall x \bigwedge\limits_{i \in \mathfrak{I} \cup \{0\},\, j \in \mathfrak{I}_i} (\neg\mathcal{A}_i \vee \neg A_{\gamma_{i,j}}(x))$. Then, the vertex $\mathcal{C}_0$ will be removed from the behaviour graph (recall that $\mathcal{F}_0 \models \mathcal{A}_0 \wedge \exists x A_{\gamma_0}(x)$).

Third condition of Theorem 3 does not hold. This case was already considered in [1]. We sketch here the proof. The negation of (2) gives the following:

$$\exists \mathcal{C}\ \exists \Diamond l \in \mathcal{E}\ \forall \mathcal{C}'\ (\mathcal{C} \to^{+} \mathcal{C}' \Rightarrow l \notin \theta') \tag{5}$$

Let $\mathcal{C}_0$, and l_0 be the vertex and eventuality determined by the existential quantifiers of (5). Let $\mathfrak{I}$ be a finite nonempty set of indexes, $\{\mathcal{C}_i \mid i \in \mathfrak{I}\}$ be the set of all successors of $\mathcal{C}_0$ (possibly including $\mathcal{C}_0$ itself). As in the previous case, one can show that

- If any of $\mathcal{A}_i \Rightarrow \bigcirc \mathcal{B}_i$ (where $i \in \mathfrak{I}$) degenerates then $\mathcal{U} \models \neg l$, and the ground eventuality termination rule can be applied.
- If none of the canonical merged derived step clauses degenerate then the following conditions hold
  - for all $i \in \mathfrak{I} \cup \{0\}$ $\quad \mathcal{U} \cup \mathcal{B}_i \models l_0$
  - for all $i \in \mathfrak{I} \cup \{0\}$ $\quad \mathcal{U} \cup \mathcal{B}_i \models \bigvee_{j \in \mathfrak{I} \cup \{0\}} \mathcal{A}_j$

  and so the ground eventuality resolution rule can be applied. ❑

Example 4 (example 2 contd.). We illustrate the proof of Theorem 2 on the temporal problem introduced in Example 2. The behaviour graph of the problem is not empty; every vertex has a successor. It is not hard to see that the first condition of Theorem 3 does not hold, and, following the proof, we can choose as $\mathcal{C}_0$, γ_0, and L_0, for example, $\mathcal{C}_1$, γ_1, and $\neg P(x)$, respectively. The set of all (and all immediate) successors of $\mathcal{C}_1$ is $\{\mathcal{C}_1, \mathcal{C}_4\}$. Note that the canonical full merged step clauses corresponding to $\mathcal{C}_1$ and $\mathcal{C}_4$ are identical, and none of them degenerates. For $i \in \{1,4\}$, the loop side conditions,

$$\forall x((\underbrace{(l \Rightarrow \exists x P(x))}_{\mathcal{U}_i} \wedge \underbrace{(\exists x P(x) \wedge \forall x P(x))}_{\mathcal{B}_i} \wedge \underbrace{P(x)}_{B_{\gamma_1}(x)}) \Rightarrow P(x))$$

and

$$\forall x((\underbrace{(l \Rightarrow \exists x P(x))}_{\mathcal{U}_i} \wedge \underbrace{(\exists x P(x) \wedge \forall x P(x))}_{\mathcal{B}_i} \wedge \underbrace{P(x)}_{B_{\gamma_1}(x)}) \Rightarrow \bigvee_{j \in \{1,4\}} (\underbrace{\exists x P(x) \wedge \forall x P(x)}_{\mathcal{A}_j} \wedge \underbrace{P(x)}_{A_{\gamma_1}(x)}))$$

hold. Therefore, we can apply the eventuality resolution rule whose conclusion can be simplified to $\exists x \neg P(x)$. After the conclusion of the rule is added to $\mathcal{U}$, vetices $\mathcal{C}_1$ and $\mathcal{C}_4$ and edges leading to and from them are deleted from the behaviour graph.

For the temporal problem with the new universal part, again the first condition of Theorem 3 does not hold, for example, for $\mathcal{C}_0 = \mathcal{C}_3$, $\gamma_0 = \gamma_1$, and $L_0(x) = \neg P(x)$. (Note that γ_2 is never a successor of γ_1.) The set of all (and all immediate) successors of $\mathcal{C}_3$ is $\{\mathcal{C}_3, \mathcal{C}_6\}$. The canonical full merged step clauses corresponding to $\mathcal{C}_3$ and $\mathcal{C}_6$ are identical, and none of them degenerates. In a similar way, the loop side conditions hold and the conclusion of the eventuality resolution rule simplifies to $\forall x \neg P(x)$. This time, vertices $\mathcal{C}_3$ and $\mathcal{C}_6$ are deleted from the behaviour graph.

For the new problem, the third condition of Theorem 3 does not hold for $\mathcal{C}_0 = \mathcal{C}_5$, $l_0 = l$. As the canonical full merged step clause degenerates (and $\mathcal{U} \models \neg l$), the ground eventuality termination rule can be applied.

Note that if, in the beginning, instead of $\mathcal{C}_1$ we had selected $\mathcal{C}_3$ (or $\mathcal{C}_6$) as $\mathcal{C}_0$, vertices $\mathcal{C}_1$, $\mathcal{C}_3$, $\mathcal{C}_4$, and $\mathcal{C}_6$ would be deleted after the first application of the eventuality resolution rule.

Input A temporal problem P and an eventuality clause $\Diamond L(x) \in \mathcal{E}$.
Output A formula $H(x)$ with at most one free variable.
Method:
1. Let $H_0(x) = \mathbf{true}$; $N_0 = \emptyset$; $i = 0$.
2. Let $N_{i+1} = \{\forall x(\mathcal{A}_j^{(i+1)}(x) \Rightarrow \bigcirc \mathcal{B}_j^{(i+1)}(x))\}_{j=1}^k$ be the set of *all* full merged step clauses such that for every $j \in \{1 \dots k\}$, $\forall x(\mathcal{U} \wedge \mathcal{B}_j^{(i+1)}(x) \Rightarrow (\neg L(x) \wedge H_i(x)))$ holds. (The set N_{i+1} possibly includes the degenerate clause $\mathbf{true} \Rightarrow \bigcirc \mathbf{true}$ in the case $\mathcal{U} \models \forall x(\neg L(x) \wedge H_i(x))$.)
3. If $N_{i+1} = \emptyset$, return **false**; else let $H_{i+1}(x) = \bigvee_{j=1}^{k} (\mathcal{A}_j^{(i+1)}(x))$.
4. If $\forall x(H_i(x) \Rightarrow H_{i+1}(x))$ return $H_{i+1}(x)$.
5. $i = i + 1$; goto 2.

Fig. 2. Breadth First Search algorithm.

6 Loop Search Algorithm

The notion of a full merged step clause is quite involved and the search for appropriate merging of simpler clauses is computationally hard. Finding *sets* of such full merged clauses needed for the temporal resolution rule is even more difficult. In Fig. 2 we present a search algorithm that finds a *loop formula* (cf. page 5)—a disjunction of the left-hand sides of full merged step clauses that together with an eventuality literal form the premises for the temporal resolution rule. The algorithm is based on a Dixon's loop search algorithm for the propositional case [4]. For the sake of space, in what follows we consider non-ground eventualities only. The algorithm and the proof of its properties for the ground case can be obtained by considering merged derived step clauses instead of the general case and by deleting the parameter "x" and quantifiers. We are going to show now that the algorithm terminates (Lemma 5), its output is a loop formula (lemmas 6 and 7), and temporal resolution is complete if we consider only the loops generated by the algorithm (Theorem 4).

Lemma 4. *For the formulae $H_i(x)$, $i \geq 0$, constructed by the BFS algorithm, the following holds:* $\forall x(H_{i+1}(x) \Rightarrow H_i(x))$.

Lemma 5. *The BFS algorithm terminates.*

Proof There are only finitely many different $H_i(x)$. Therefore, either there exists k such that $H_k(x) \equiv \mathbf{false}$ and the algorithm terminates by step 3, or there exist $l, m : l < m$ such that $\forall x(H_l(x) \equiv H_m(x))$. In the latter case, by Lemma 4 we have $\forall x(H_{m-1}(x) \Rightarrow H_l(x))$, that is $\forall x(H_{m-1}(x) \Rightarrow H_m(x))$. By step 4, the algorithm terminates. ❑

Lemma 6. *Let $H(x)$ be a formula produced by the BFS algorithm. Then* $\forall x(\mathcal{U} \wedge H(x) \Rightarrow \bigcirc \Box \neg L(x))$.

Lemma 7. *Let P be a monodic temporal problem, $\mathcal{L}$ be a loop in $\Diamond L(x) \in \mathcal{E}$, and $\mathbf{L}(x)$ be its loop formula. Then for the formula $H(x)$, produced by the BFS algorithm on $\Diamond L(x)$, the following holds:* $\forall x(\mathbf{L}(x) \Rightarrow H(x))$.

The proof of the completeness theorem goes by showing that there exists an eventuality $\Diamond L(x) \in \mathcal{E}$ and a loop $\mathcal{L} = \{\forall x(\mathcal{A}_i(x) \Rightarrow \bigcirc \mathcal{B}_i(x))\}_{i=1}^{k}$ such that the application of the eventuality resolution rule to $\Diamond L(x)$ and $\mathcal{L}$ leads to the deletion of some vertices from the eventuality graph. A vertex $\mathcal{C}$ is deleted if the categorical formula, $\mathcal{F}_{\mathcal{C}}$, together with the universal part, $\mathcal{U}$, is satisfiable, but $\mathcal{F}_{\mathcal{C}} \wedge \forall x \neg \bigvee_{j=1}^{k} \mathcal{A}_j(x) \wedge \mathcal{U}$ is unsatisfiable.

Theorem 4. *Temporal resolution is complete if we restrict ourselves to loops found by the BFS algorithm.*

Note 2. The need to include *all* full merged step clauses satisfying some particular conditions into N_{i+1} might lead to quite extensive computations. Note however that due to the trivial fact that if $\forall x(A(x) \Rightarrow B(x))$ then $\forall x((A(x) \vee B(x)) \equiv B(x))$, we can restrict the choice to only those full merged step clauses whose left-hand sides do not imply the left-hand side of any other clause in N_{i+1} yielding a formula $H'_{i+1}(x)$ equivalent to the original formula $H_{i+1}(x)$.

Example 5. Let us consider an unsatisfiable monodic temporal problem, P, given by $I = \{\exists x A(x)\}$, $\mathcal{U} = \{\forall x(B(x) \Rightarrow A(x) \wedge \neg L(x))\}$, $\mathcal{S} = \{A(x) \Rightarrow \bigcirc B(x)\}$, $\mathcal{E} = \{\Diamond L(x)\}$ and apply the BFS algorithm to $\Diamond L(x)$.

The set of all full merged step clauses, N_1, whose right-hand sides imply $\neg L(x)$, is:

$$(\forall y A(y)) \Rightarrow \bigcirc(\forall y B(y)), \tag{6}$$
$$(A(x) \wedge \forall y A(y)) \Rightarrow \bigcirc(B(x) \wedge \forall y B(y)), \tag{7}$$
$$(A(x) \wedge \exists y A(y)) \Rightarrow \bigcirc(B(x) \wedge \exists y B(y)). \tag{8}$$

Note that $\forall x(\forall y A(y) \Rightarrow A(x) \wedge \forall y A(y))$ and $\forall x(A(x) \wedge \forall y A(y)) \Rightarrow A(x) \wedge \exists y A(y))$; therefore, clauses (6) and (7) can be deleted from N_1 yielding $N'_1 = \{(A(x) \wedge \exists y A(y)) \Rightarrow \bigcirc(B(x) \wedge \exists y B(y))\}$ and $H'_1(x) = (A(x) \wedge \exists y A(y))$.

The set of all full merged step clauses N_2 whose right-hand sides imply $L(x) \wedge H'_1(x)$ coincides with N_1 and the output of the algorithm is $H'_2(x) \equiv H'_1(x)$. The conclusion of the eventuality resolution rule, $\forall x \neg A(x) \vee \neg \exists y A(y)$, simplified to $\forall x \neg A(x)$, contradicts the initial part of the problem.

Note that all full merged step clauses from N_1 are loops in $\Diamond L(x)$, but both conclusions of the eventuality resolution rule, applied to the loops (6) and (7), can be simplified to $\exists x \neg A(x)$ which does not contradict the initial part.

7 Conclusion and future work

In this paper, we have introduced a resolution-based calculus for the monodic fragment of first-order temporal logic. We have shown that the calculus is sound and complete and considered some problems of implementation. We have suggested an algorithm that "guides" the search for loops in order to avoid unnecessary enumeration of all possibilities. We are going to refine also the step resolution rule in a way similar to the original temporal resolution method for the propositional case [5] that could serve as a basis for a practical implementation.

An alternative tableaux-based approach [10] also utilises similar "separation" ideas dividing the proof search procedure into *temporal* and *first-order* parts. Note that the method in [10] requires the first-order component to give a finite representation of *all possible* first-order models; whereas our method requires from it just an *yes/no* answer (to test side conditions of the rules of temporal resolution). Our procedure is a decision procedure when the side condition checks are decidable, and is a semi-decision procedure otherwise.

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