On dynamic topological and metric logics

B. Konev[1], R. Kontchakov[2], F. Wolter[1], and M. Zakharyaschev[2]

[1]Department of Computer Science, University of Liverpool,
Liverpool L69 7ZF, U.K.
{b.konev, frank}@csc.liv.ac.uk,

[2]Department of Computer Science, King's College London,
Strand, London WC2R 2LS, U.K.
{romanvk, mz}@dcs.kcl.ac.uk

Abstract

The first result of this paper shows that some dynamic topological logics interpreted in various topological spaces with homeomorphisms are not recursively enumerable (and so are not recursively axiomatisable). This gives a 'negative' solution to a conjecture of Kremer and Mints [11]. Second, we prove the non-elementary decidability of the dynamic metric logic with distance operators of the form 'somewhere in the ball of radius a,' for $a \in \mathbb{Q}^+$, interpreted in arbitrary metric spaces with distance preserving automorphisms.

1 Introduction

Dynamic topological logics were first introduced in 1997 (see, e.g., [9, 10, 12, 2, 11]) as a logical formalism for describing the behaviour of *dynamical systems*, e.g., in order to specify liveness and safety properties of hybrid systems [5]. Dynamical systems [4, 8] are usually represented by some 'mathematical' space W (modelling possible system states) and a function f on W (modelling the evolution of the system), with one of the main research problems being the study of iterations of f, in particular, the orbits $O(w) = \{w, f(w), f^2(w), \dots\}$ of states $w \in W$.

A natural logical formalism for speaking about such iterations is a variant of temporal logic. For example, given a subset V of W, we can introduce the standard temporal operators $\bigcirc$ ('at the next moment'), $\Box_F$ ('always in the future'), and its dual $\Diamond_F$ ('eventually') by taking

$$\bigcirc V = f^{-1}(V), \qquad \Box_F V = \bigcap_{0<n<\omega} f^{-n}(V) \qquad \text{and} \qquad \Diamond_F V = \bigcup_{0<n<\omega} f^{-n}(V).$$

Using this language we can describe in a succinct and transparent way properties like

- starting from a state in some region P, one will never leave a region Q: $P \to \Box_F Q$;
- starting from a state in a region P, one will eventually reach a state in Q: $P \to \Diamond_F Q$;
- w 'visits' P ever and ever again: $w \in \Box_F \Diamond_F P$.

To speak about the structure of the underlying space W—important examples are (subspaces of) the Euclidean spaces $\mathbb{R}^n$, general topological spaces, metric spaces, and measure spaces—as well as the type of the intended functions f, one may require different non-temporal operators. So far, research has mainly been focused on topological spaces with continuous mappings. The corresponding logical constructors are those of modal logic **S4** which can be regarded also as the topological closure and interior operators—we denote them by **C** and **I**, respectively. For example, a property similar to Poincaré's recurrence theorem corresponds in this language to the validity of the formula $\mathbf{C}(\mathbf{I}p \to \bigcirc\Diamond_F \mathbf{I}p)$ in spaces based on the unit disc with measure preserving continuous mappings.

Metric operators were suggested in [15] in order to formulate safety properties. For example, using the operator $\exists^{\leq a}$, where a is a positive rational number, the formula $P \to \Box_F \neg\exists^{\leq a} Q$ states that, having started from a point in P, one can never reach the a-neighbourhood of some 'unsafe' area Q.

The resulting combinations of temporal and topological/metric logics are of a clear 'two-dimensional character,' which makes it very difficult to analyse their computational properties (see, e.g., [6]). Perhaps this is the main reason why in the field of dynamic topological systems no (un)decidability or axiomatisability results have been obtained yet for the full language containing both $\bigcirc$ and the infinitary $\Box_F$.

This note provides answers to some of the open problems. First, we show that some dynamic topological logics introduced in [11] and interpreted in various topological spaces with homeomorphisms are not recursively enumerable (and so are not recursively axiomatisable). This result gives negative solutions to Conjectures 2.7 (ii) and 2.7 (iv) from [11]. Second, we prove the non-elementary decidability of the dynamic metric logic with distance operators of the form $\exists^{\leq a}$ from [13] interpreted in arbitrary metric spaces with distance preserving automorphisms.

Although numerous problems remain open, the obtained results clearly indicate that the logics for dynamic systems are very sensitive to the available operators (say, topological vs metric) as well as the constraints imposed on the spaces $\langle W, f\rangle$ (e.g., the proof of the undecidability result mentioned above does not go through for continuous functions, while the decidability proof only works for *arbitrary* metric spaces, but not for, say, compact ones).

2 Definitions

Syntax. The language $\mathcal{DTL}$ of *dynamic topological logic* (or *dynamic topo-logic*, for short) [2, 11] is constructed from a countably infinite set of propositional variables using the Booleans $\wedge$ and $\neg$, the modal operators **I** and **C** (for topological interior and closure), and the temporal operators $\bigcirc$ (for 'next'), $\Box_F$ and $\Diamond_F$ (for 'always' and 'eventually'). By $\mathcal{DTL}_\bigcirc$ we denote the fragment of $\mathcal{DTL}$ which does not use $\Box_F$ and $\Diamond_F$. We write $\Box_F^+\varphi$ for $\varphi \wedge \Box_F\varphi$ and dually $\Diamond_F^+\varphi = \varphi \vee \Diamond_F\varphi$, for every $\mathcal{DTL}$-formula φ.

Semantics. In this paper, by a *dynamic topological structure* (or DTS, for short) we understand a pair of the form $\mathfrak{F} = \langle \mathfrak{T}, f\rangle$, where $\mathfrak{T} = \langle T, \mathbb{I}\rangle$ is a topological space with an interior operator $\mathbb{I}$ (satisfying the standard Kuratowski axioms) and f is a homeomorphism[1] (i.e., a bijective continuous and open mapping) on $\mathfrak{T}$. A *dynamic topological model* (or DTM) is a pair $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$, where $\mathfrak{F}$ is a DTS and $\mathfrak{V}$, a *valuation*, associates with each propositional variable p a subset $\mathfrak{V}(p)$ of T. The *truth-relation* $(\mathfrak{M}, w) \models \varphi$, for a $\mathcal{DTL}$-formula φ, is defined as follows:

$$
\begin{array}{lll}
(\mathfrak{M}, w) \models p & \text{iff} & w \in \mathfrak{V}(p), \\
(\mathfrak{M}, w) \models \mathbf{I}\varphi & \text{iff} & w \in \mathbb{I}\{v \in T \mid (\mathfrak{M}, v) \models \varphi\},
\end{array}
$$

[1] In a more general setting, f can be a continuous mapping.

$$
\begin{array}{lll}
(\mathfrak{M},w) \models \mathbf{C}\varphi & \text{iff} & w \in \mathbb{C}\{v \in T \mid (\mathfrak{M},v) \models \varphi\}, \\
(\mathfrak{M},w) \models \bigcirc\varphi & \text{iff} & (\mathfrak{M}, f(w)) \models \varphi, \\
(\mathfrak{M},w) \models \Box_F\varphi & \text{iff} & (\mathfrak{M}, f^n(w)) \models \varphi \text{ for every } n > 0, \\
(\mathfrak{M},w) \models \Diamond_F\varphi & \text{iff} & (\mathfrak{M}, f^n(w)) \models \varphi \text{ for some } n > 0.
\end{array}
$$

Here $f^n(w) = \overbrace{f \dots f}^{n}(w)$. If $(\mathfrak{M},w) \models \varphi$ for some $w \in T$, then we say that φ is *satisfied* in $\mathfrak{M}$. A $\mathcal{DTL}$-formula φ is *satisfiable* in a DTS $\mathfrak{F}$ if φ is satisfied in a DTM based on $\mathfrak{F}$.

Given a class $\mathcal{K}$ of dynamic topological structures, we denote by $\mathrm{Log}\,\mathcal{K}$ (respectively, $\mathrm{Log}_{\bigcirc}\mathcal{K}$) the *logic* of $\mathcal{K}$ *in the language* $\mathcal{DTL}$ (or $\mathcal{DTL}_{\bigcirc}$), i.e., the set of all $\mathcal{DTL}$-formulas (respectively, $\mathcal{DTL}_{\bigcirc}$-formulas) φ such that $(\mathfrak{M},w) \models \varphi$ holds for every model $\mathfrak{M}$ based on a structure in $\mathcal{K}$ and every point w in $\mathfrak{M}$.

We remind the reader that every quasi-order $\mathfrak{G} = \langle W,R\rangle$ (R is a reflexive and transitive relation on W) gives rise to a topological space $\mathfrak{T}_{\mathfrak{G}} = \langle W, \mathbb{I}_{\mathfrak{G}}\rangle$, where, for every $X \subseteq W$,

$$
\mathbb{I}_{\mathfrak{G}}X = \{x \in X \mid \forall y \in W\ (xRy \to y \in X)\}.
$$

Such spaces are known as *Aleksandrov spaces*. Alternatively they can be defined as topological spaces where arbitrary (not only finite) intersections of open sets are open; for details see [1, 3]. Clearly, for $\mathfrak{M} = \langle\langle \mathfrak{T}_{\mathfrak{G}}, f\rangle, \mathfrak{V}\rangle$ we have

$$
\begin{array}{lll}
(\mathfrak{M},w) \models \mathbf{I}\varphi & \text{iff} & (\mathfrak{M},v) \models \varphi \text{ for every } v \in W \text{ with } wRv, \\
(\mathfrak{M},w) \models \mathbf{C}\varphi & \text{iff} & \text{there is } v \in W \text{ such that } wRv \text{ and } (\mathfrak{M},v) \models \varphi.
\end{array}
$$

It should be also clear that a function $f\colon W \to W$ is a continuous mapping on $\mathfrak{T}_{\mathfrak{G}}$ if, for all $w,v \in W$,

$$
wRv \quad \text{implies} \quad f(w)Rf(v).
$$

The function f is a *homeomorphism* on $\mathfrak{T}_{\mathfrak{G}}$ if f is bijective and the converse implication holds as well.

Let $\mathbb{R}^n$ denote the standard Euclidean space of dimension n and $\mathbb{R}$ is the real line. For $n \geq 2$, a *unit ball* is a DTS $\mathfrak{B}^n = \langle B^n, f\rangle$, where B^n is a ball in $\mathbb{R}^n$ of radius 1, and f is a *measure preserving* homeomorphism on B^n.

The results of the theorem below were explicitly proved in or easily follow from [2, 12, 11].

Theorem 1. *The four dynamic topo-logics listed below coincide, have the finite model property, are finitely axiomatisable, and so decidable:*

1. $\mathrm{Log}_{\bigcirc}\{\langle \mathfrak{T}, f\rangle \mid \langle \mathfrak{T}, f\rangle$ *a DTS*$\}$,
2. $\mathrm{Log}_{\bigcirc}\{\langle \mathbb{R}^n, f\rangle \mid \langle \mathbb{R}^n, f\rangle$ *a DTS*, $n \geq 1\}$,
3. $\mathrm{Log}_{\bigcirc}\{\langle \mathfrak{T}, f\rangle \mid \langle \mathfrak{T}, f\rangle$ *a DTS*, $\mathfrak{T}$ *an Aleksandrov space*$\}$,
4. $\mathrm{Log}_{\bigcirc}\{\mathfrak{B}^n \mid \mathfrak{B}^n$ *a unit ball*, $n \geq 2\}$.

Later on we will use the fact that $\mathrm{Log}_{\bigcirc}\{\langle \mathbb{R}, x \mapsto x+1\rangle\}$ coincides with all of the logics above as well (see [11]).

We show now that the computational behaviour of dynamic topo-logics becomes completely different if we allow the use of the operators $\Box_F$ and $\Diamond_F$.

3 Undecidability and non-axiomatisability

Theorem 2. *No logic from the list below is recursively enumerable:*

1. $\mathsf{Log}\,\{\langle\mathfrak{T}, f\rangle \mid \langle\mathfrak{T}, f\rangle$ *a DTS*$\}$,
2. $\mathsf{Log}\,\{\langle\mathbb{R}^n, f\rangle \mid \langle\mathbb{R}^n, f\rangle$ *a DTS*, $n \geq 1\}$,
3. $\mathsf{Log}\,\{\langle\mathfrak{T}, f\rangle \mid \langle\mathfrak{T}, f\rangle$ *a DTS*, $\mathfrak{T}$ *an Aleksandrov space*$\}$,
4. $\mathsf{Log}\,\{\mathfrak{B}^n \mid \mathfrak{B}^n$ *a unit ball*, $n \geq 2\}$.

Remark 3. Before proceeding to the proof, we note that all logics mentioned in this theorem are different. As was shown in [17], the formula $\mathbf{I}\Diamond_F(p \wedge \mathbf{CI}\neg p)$ is not satisfiable in any DTS of the form $\langle\mathbb{R}^n, f\rangle$, while it is clearly satisfiable. According to [11], the formula $\mathbf{C}(\mathbf{I}p \to \bigcirc\Diamond_F\mathbf{I}p)$ is valid in all unit balls, but refuted in a DTS based on both an Aleksandrov space and $\langle\mathbb{R}^n, x \mapsto x+1\rangle$. Finally, the formula $\Diamond_F\mathbf{C}p \leftrightarrow \mathbf{C}\Diamond_F p$ is valid in DTSs based on Aleksandrov spaces, but refuted both in $\langle\mathbb{R}^n, f\rangle$ and in all unit balls.

We prove Theorem 2 by reduction of *Post's correspondence problem* or PCP, for short [16]. (Cases (2) and (4) will only be proved for $n = 1$ and $n = 2$, respectively.) The idea of the proof is taken from [6]. Let A be a finite alphabet and P a finite set of pairs $\langle\mathbf{v}_1, \mathbf{u}_1\rangle, \dots, \langle\mathbf{v}_k, \mathbf{u}_k\rangle$ of nonempty finite words

$$\mathbf{v}_i = \left\langle b^i_1, \dots, b^i_{l_i}\right\rangle, \qquad \mathbf{u}_i = \left\langle c^i_1, \dots, c^i_{r_i}\right\rangle \qquad (i = 1, \dots, k)$$

over A. For a sequence of indices $i_1, \dots, i_N$ ranging over $1, \dots, k$, let

$$m_j = l_{i_1} + \dots + l_{i_j} \qquad \text{and} \qquad n_j = r_{i_1} + \dots + r_{i_j},$$

for $1 \leq j \leq N$. The following problem is undecidable (for a proof see, e.g., [7]): *given a set P of pairs of words as above, decide whether there exist an $N \geq 1$ and a sequence $i_1, \dots, i_N$ of indices such that*

$$\mathbf{v}_{i_1} * \dots * \mathbf{v}_{i_N} = \mathbf{u}_{i_1} * \dots * \mathbf{u}_{i_N}, \tag{1}$$

where $*$ is the concatenation operation. If the condition above holds for a PCP instance P then we say that *P has a solution*. Later, we will use a consequence of this result, namely, that the set of PCP instances without solutions *is not recursively enumerable*.

The reduction formula $\phi_{A,P}$ is constructed using the propositional variables r, s, *left* and *right*, $left_a$ and $right_a$, for every $a \in A$, as well as $pair_i$, for every pair $\langle\mathbf{v}_i, \mathbf{u}_i\rangle$ in P, $1 \leq i \leq k$.

The variable s is used to introduce a new operator $\mathbf{S}$ (which can be interpreted as a 'strict diamond' in Kripke quasi-ordered frames). Namely, for every $\mathcal{DTL}$-formula ψ, we put

$$\mathbf{S}\psi = \big(s \to \mathbf{C}(\neg s \wedge \mathbf{C}\psi)\big) \wedge \big(\neg s \to \mathbf{C}(s \wedge \mathbf{C}\psi)\big).$$

Denote by $\mathbf{S}^m$ a string of m operators $\mathbf{S}$ (so that $\mathbf{S}^0\psi = \psi$). The variable r is used to 'relativise' $\Box_F$ in the following ways: $\Box_F^{<\mathbf{r}}\psi = \Box_F^+(\Diamond_F r \to \psi)$ and $\Box_F^{\leq\mathbf{r}}\psi = \Box_F^+(\Diamond_F^+ r \to \psi)$.

Now $\phi_{A,P}$ is defined as the conjunction

$$\phi_{A,P} = \varphi_{eq} \wedge \varphi_{pair} \wedge \varphi_{stripe} \wedge \varphi_{left} \wedge \varphi_{right},$$

where

$$\varphi_{eq} = \Diamond_F\big(r \wedge \bigwedge_{a \in A}\mathbf{I}(left_a \leftrightarrow right_a)\big),$$

$$\varphi_{pair} = \Box_F^{<\mathbf{r}}\Big(\bigvee_{1 \leq i \leq k} pair_i \wedge \bigwedge_{1 \leq i < j \leq k}\neg(pair_i \wedge pair_j)\Big),$$

$$\varphi_{stripe} = \Box_F^{<\mathbf{r}}\mathbf{I}(s \leftrightarrow \bigcirc s),$$

φ_{left} is the conjunction of (2)–(8), for $1 \le i \le k$,

$$\bigwedge_{\substack{a\neq b\\ a,b\in A}} \Box_F^{\le\mathbf{r}}\mathbf{I}\neg\big(left_a \wedge left_b\big) \;\wedge\; \Box_F^{\le\mathbf{r}}\mathbf{I}\big(left \leftrightarrow \bigvee_{a\in A} left_a\big), \tag{2}$$

$$\bigwedge_{a\in A} \Box_F^{<\mathbf{r}}\mathbf{I}(left_a \to \bigcirc left_a), \tag{3}$$

$$\mathbf{I}\neg left \;\wedge\; \Box_F^{\le\mathbf{r}}\mathbf{I}(\neg left \to \neg\mathbf{S}left), \tag{4}$$

$$\Box_F^{<\mathbf{r}}\big(pair_i \to \mathbf{I}(\neg left \to \bigcirc\neg\mathbf{S}^{l_i}left)\big), \tag{5}$$

$$\Box_F^{<\mathbf{r}}\big(pair_i \to \bigwedge_{j<l_i} \bigcirc\mathbf{I}\big((\mathbf{S}^j left \wedge \neg\mathbf{S}^{j+1}left) \to left_{b^i_{l_i-j}}\big)\big), \tag{6}$$

$$pair_i \to \bigcirc lw_i, \tag{7}$$

$$\Box_F^{<\mathbf{r}}\big(pair_i \to \mathbf{I}((left \wedge \neg\mathbf{S}left) \to \mathbf{S}\bigcirc lw_i)\big), \tag{8}$$

where

$$lw_i = left_{b^i_1} \wedge \mathbf{S}\big(left_{b^i_2} \wedge \mathbf{S}(left_{b^i_3} \wedge \cdots \wedge \mathbf{S}left_{b^i_{l_i}})\dots\big)$$

(remember that l_i is the length of the word $\mathbf{v}_i = \langle b^i_1,\dots,b^i_{l_i}\rangle$), and the conjunct φ_{right}—the ‘right counterpart’ of φ_{left}—is defined by replacing in φ_{left} all occurrences of *left* with *right*, $left_a$ with $right_a$, l_i with r_i, etc.

We also require $\mathcal{DTL}_\bigcirc$-formulas $\phi^n_{A,P}$, $n > 0$, which are defined similarly to $\phi_{A,P}$ by replacing

- φ_{eq} with $\bigcirc^n \bigwedge_{a\in A}\mathbf{I}(left_a \leftrightarrow right_a)$, and
- every occurrence of $\Box_F^{\le\mathbf{r}}\psi$ (or $\Box_F^{<\mathbf{r}}\psi$) with $\Box_F^{\le n}\psi$ (respectively, $\Box_F^{<n}\psi$), where

$$\Box_F^{\le n}\psi = \psi \wedge \bigcirc\psi \wedge \bigcirc\bigcirc\psi \wedge \cdots \wedge \bigcirc^n\psi \qquad \text{and} \qquad \Box_F^{<n}\psi = \psi \wedge \bigcirc\psi \wedge \bigcirc\bigcirc\psi \wedge \cdots \wedge \bigcirc^{n-1}\psi.$$

We denote the subformula of $\phi^n_{A,P}$ corresponding to a subformula φ of $\phi_{A,P}$ by φ^n.

Lemma 4. *If $\phi_{A,P}$ is satisfiable in $\langle\mathfrak{T}, f\rangle$ then there is $n > 0$ such that $\phi^n_{A,P}$ is satisfiable in $\langle\mathfrak{T}, f\rangle$.*

Proof. Suppose $(\mathfrak{M}, w) \models \phi_{A,P}$. Then $(\mathfrak{M}, w) \models \Diamond_F(r \wedge \bigwedge_{a\in A}\mathbf{I}(left_a \leftrightarrow right_a))$, that is, there exists $m > 0$ such that $(\mathfrak{M}, f^m(w)) \models r \wedge \bigwedge_{a\in A}\mathbf{I}(left_a \leftrightarrow right_a)$. Let n be the minimal such m. One can easily check that $(\mathfrak{M}, w) \models \phi^n_{A,P}$. $\square$

Lemma 5. *If P has a solution, then the following hold:*

(i) *$\phi_{A,P}$ is satisfiable in a DTS $\langle\mathfrak{T}, f\rangle$, where $\mathfrak{T}$ is an Aleksandrov space;*

(ii) *$\phi_{A,P}$ is satisfiable in a DTS;*

(iii) *$\phi_{A,P}$ is satisfiable in $\mathfrak{B}^2$;*

(iv) *$\phi_{A,P}$ is satisfiable in $\langle\mathbb{R}, f\rangle$, where $f : x \mapsto x+1$.*

Proof. Suppose that P has a solution

$$\mathbf{v}_{i_1} * \cdots * \mathbf{v}_{i_N} = \mathbf{u}_{i_1} * \cdots * \mathbf{u}_{i_N}. \tag{9}$$

Let $\mathbf{v}_{i_1} * \cdots * \mathbf{v}_{i_N} = \langle b_0, \ldots, b_{m_N - 1} \rangle$.

(i) Define a quasi-order $\mathfrak{G} = \langle W, R \rangle$ by taking $W = \{0, \ldots, 2m_N\} \times \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers, and $(x,y)R(x',y')$ iff $x \leq x'$ and $y = y'$. Define $f : W \to W$ by taking $f(x,y) = (x, y+1)$. Clearly, f is a homeomorphism on $\mathfrak{T}_{\mathfrak{G}}$. Finally, define $\mathfrak{V}$ by taking

$$\begin{aligned} \mathfrak{V}(s) &= \{(2n,z) \mid 0 \leq 2n \leq 2m_N,\ z \in \mathbb{Z}\}, & \mathfrak{V}(r) &= \{(0,N)\}, \\ \mathfrak{V}(pair_i) &= \{(0, j-1) \mid i = i_j, \text{ for some } j \leq N\}, & & \\ \mathfrak{V}(left_a) &= \{(2k, j) \mid 0 < j \leq N,\ k < m_j,\ b_k = a\}, & \mathfrak{V}(left) &= \textstyle\bigcup_{a \in A} \mathfrak{V}(left_a), \\ \mathfrak{V}(right_a) &= \{(2k, j) \mid 0 < j \leq N,\ k < n_j,\ b_k = a\}, & \mathfrak{V}(right) &= \textstyle\bigcup_{a \in A} \mathfrak{V}(right_a). \end{aligned}$$

Let $\mathfrak{M} = \langle \mathfrak{T}_{\mathfrak{G}}, \mathfrak{V} \rangle$. It is an easy exercise to show $(\mathfrak{M}, (0,0)) \models \phi_{A,P}$. We leave this to the reader.

(ii) follows from (i).

(iii) We only consider the two-dimensional unit ball $\mathfrak{B}^2 = \langle B^2, g \rangle$, where g is the *rotation* of B^2 clockwise by some *rational* angle α such that $0 < \alpha < 2\pi/N + 1$ and N is given by (9) (for $n > 2$, the construction is similar: we rotate the ball around a fixed axis by the same angle α). Obviously, g is a measure preserving homeomorphism.

Let E be an open set, say, a smaller open ball contained in the sector $[-\alpha/2, \alpha/2]$ of B^2 and let $E_i = g^i(E)$, for $i < \omega$. Note that $E = E_0, E_1, \ldots, E_N$ are pairwise disjoint sets. Let $\mathfrak{H} = \langle \{0, \ldots, 2m_N\}, \leq \rangle$. By the main result of McKinsey and Tarski [14], there exists a continuous mapping h_i from E_i *onto* $\mathfrak{T}_{\mathfrak{H}}$. Moreover, one may assume that, for every $x \in E_i$, we have $h_i(x) = h_{i+1}(g(x))$ and that $h_i(e_i) = 0$, where e_i is the center of the ball E_i. Define a valuation $\mathfrak{V}$ on $\mathfrak{B}^2$ by taking

$$\begin{aligned} \mathfrak{V}(s) &= \textstyle\bigcup_{i=0}^{N} \{h_i^{-1}(2n) \mid 0 \leq 2n \leq 2m_N\}, & \mathfrak{V}(r) &= \{e_N\}, \\ \mathfrak{V}(pair_i) &= \{e_{j-1} \mid i = i_j, \text{ for some } j \leq N\}, & & \\ \mathfrak{V}(left_a) &= \textstyle\bigcup_{j=1}^{N} \{h_j^{-1}(2k) \mid k < m_j,\ b_k = a\}, & \mathfrak{V}(left) &= \textstyle\bigcup_{a \in A} \mathfrak{V}(left_a), \\ \mathfrak{V}(right_a) &= \textstyle\bigcup_{j=1}^{N} \{h_j^{-1}(2k) \mid k < n_j,\ b_k = a\}, & \mathfrak{V}(right) &= \textstyle\bigcup_{a \in A} \mathfrak{V}(right_a). \end{aligned}$$

As α is rational, we have $g^j(e_0) = e_N$ iff $j = N$. It is not hard to check now that $(\langle \mathfrak{B}, \mathfrak{V} \rangle, e_0) \models \phi_{A,P}$.

(iv) We know from (i) and Lemma 4 that there exists $n > 0$ such that $\phi^n_{A,P}$ is satisfiable in a DTS $\langle \mathfrak{T}, f \rangle$. Then, by the remark following Theorem 1 and since $\phi^n_{A,P}$ is a $\mathcal{DTL}_{\bigcirc}$-formula, $(\mathfrak{M}, w) \models \phi^n_{A,P}$, for some model $\mathfrak{M} = \langle \langle \mathbb{R}, f \rangle, \mathfrak{V} \rangle$ and $f : x \mapsto x + 1$. Define a new valuation $\mathfrak{V}'$ on $\mathbb{R}$ which coincides with $\mathfrak{V}$ except for only one case: now we set $\mathfrak{V}'(r) = \{f^n(w)\}$. (Note that r does not occur in $\phi^n_{A,P}$.) Let $\mathfrak{M}' = \langle \langle \mathbb{R}, f \rangle, \mathfrak{V}' \rangle$. Then clearly $(\mathfrak{M}', w) \models \phi_{A,P}$. $\square$

Lemma 6. *Suppose that there exists $n > 0$ such that $\phi^n_{A,P}$ is satisfiable in a DTS based on an Aleksandrov space. Then P has a solution.*

Proof. Suppose that $(\mathfrak{M}, w_1^0) \models \phi^N_{A,P}$ for some DTM $\mathfrak{M} = \langle \langle \mathfrak{T}_{\mathfrak{G}}, f \rangle, \mathfrak{V} \rangle$, where $\mathfrak{G} = \langle W, R \rangle$ is a quasi-order, f a homeomorphism on $\mathfrak{T}_{\mathfrak{G}}$ and $w_1^0 \in W$. For $j < \omega$, let

$$W_j = \{w \in W \mid f^j(w_1^0) R w\}.$$

As $(\mathfrak{M}, w_1^0) \models \bigcirc^N \bigwedge_{a \in A} \mathbf{I}(left_a \leftrightarrow right_a)$, we have

$$(\mathfrak{M}, f^N(w_1^0)) \models \bigwedge_{a \in A} \mathbf{I}(left_a \leftrightarrow right_a). \tag{10}$$

Since $(\mathfrak{M}, w_1^0) \models \varphi^N_{stripe}$, for each $w \in W_0$ and each $j \leq N$, we have $(\mathfrak{M}, w) \models s$ iff $(\mathfrak{M}, f^j(w)) \models s$.

Denote by S_j, $j \leq N$, the transitive binary relation on W_j defined by taking wS_jv iff there is $u \in W_j$ such that $wRuRv$ and $(\mathfrak{M}, w) \models s$ iff $(\mathfrak{M}, u) \not\models s$. Then we clearly have that, for every $j \leq N$ and every $w \in W_j$,

$$(\mathfrak{M}, w) \models \mathbf{S}\psi \quad \text{iff} \quad \text{there is } v \in W_j \text{ such that } wS_jv \text{ and } (\mathfrak{M}, v) \models \psi.$$

Note that, since f is a homeomorphism and in view of $(\mathfrak{M}, w_1^0) \models \varphi^N_{stripe}$, for all $w, v \in W_0$ and $i \leq N$, we have wS_0v iff $f^i(w)S_if^i(v)$.

Let $i_1, \ldots, i_N$ be the sequence of indices such that, for $1 \leq j \leq N$, we have $(\mathfrak{M}, f^{j-1}(w_1^0)) \models pair_{i_j}$ (φ^N_{pair} ensures that there is a unique sequence of this sort). We claim that (1) holds for this sequence.

For every j with $1 \leq j \leq N$, let

$$W_j^L = \{w \in W_j \mid (\mathfrak{M}, w) \models left\}.$$

Call a sequence $\langle w_1, \ldots, w_l \rangle$ of (not necessarily distinct) points from W_j^L an *S_j-path in W_j^L of length l* if $w_1S_jw_2S_j \ldots S_jw_l$, and set

$$leftword_j(w_1, \ldots, w_l) = \langle a_1, \ldots, a_l \rangle,$$

where the a_i are the (uniquely determined by (2)) symbols from A such that $(\mathfrak{M}, w_i) \models left_{a_i}$.

We show now that there is a sequence $\pi_1, \ldots, \pi_N$ such that, for every j with $1 \leq j \leq N$, the following hold:

(a) $\pi_j = \langle w_1^j, \ldots, w_{m_j}^j \rangle$ is an S_j-path in W_j^L of length m_j, and there is no S_j-path in W_j^L of length $> m_j$;

(b) $f(w_1^0) = w_1^1$ and if $j > 1$ then $w_m^j = f(w_m^{j-1})$, for all m, $1 \leq m \leq m_{j-1}$;

(c) $leftword_j(w_1^j, \ldots, w_{m_j}^j) = \mathbf{v}_{i_1} * \ldots * \mathbf{v}_{i_j}$;

(d) for every S_j-path $\langle w_1, \ldots, w_{m_j} \rangle$ in W_j^L of length m_j, we have $leftword_j(w_1, \ldots, w_{m_j}) = \mathbf{v}_{i_1} * \ldots * \mathbf{v}_{i_j}$.

Indeed, by $(\mathfrak{M}, w_1^0) \models pair_{i_1}$, (7), (4) and (5), there exists an S_1-path π_1 in W_1^L such that (a)–(c) hold. Condition (d) follows from (6).

Now assume inductively that conditions (a)–(d) hold for some $j-1$ with $1 \leq j-1 < N$. Let $\pi_{j-1} = \langle w_1^{j-1}, \ldots, w_{m_{j-1}}^{j-1} \rangle$ be an S_{j-1}-path in W_{j-1}^L for which (a)–(d) hold. By (3), the sequence $\langle f(w_1^{j-1}), \ldots, f(w_{m_{j-1}}^{j-1}) \rangle$ is an S_j-path in W_j^L. Since $(\mathfrak{M}, w_{m_{j-1}}^{j-1}) \models left \wedge \neg \mathbf{S} left$ and $(\mathfrak{M}, w_1^{j-1}) \models pair_{i_j}$, (8) means that there exists a sequence $w_{m_{j-1}+1}^j, \ldots, w_{m_{j-1}+l_{i_j}}^j$ of points in W_j^L such that

$$\pi_j = \langle f(w_1^{j-1}), \ldots, f(w_{m_{j-1}}^{j-1}), w_{m_{j-1}+1}^j, \ldots, w_{m_{j-1}+l_{i_j}}^j \rangle$$

is an S_j-path in W_j^L of length $m_j = m_{j-1} + l_{i_j}$ such that $leftword_j(w_{n_{j-1}+1}^j, \ldots, w_{m_{j-1}+l_{i_j}}^j) = \mathbf{v}_{i_j}$. By (5) and the induction hypothesis, there is no S_j-path in W_j^L of length $> m_j$. Thus, (a) and (b) hold for π_j, (c) follows from (3), and (d) from (6) and the induction hypothesis.

Now define sets W_j^R in the same way as W_j^L, but with *left* replaced by *right*, introduce the notion of an S_j-path in W_j^R, and, for every sequence $w_1, \ldots, w_l$ of points from W_j^R, set

$$rightword_j(w_1, \ldots, w_l) = \langle a_1, \ldots, a_l \rangle,$$

where the a_i are the uniquely determined (by 'right analogue' of (2)) elements of A such that $(\mathfrak{M}, w_i) \models right_{a_i}$. In precisely the same way as above we show now that there is a sequence $\pi_1', \ldots, \pi_N'$ such that, for every j with $1 \leq j \leq N$, (a′) $\pi'_j = \langle w^j_1, \ldots, w^j_{n_j} \rangle$ is an S_j-path in W^R_j of length n_j, and there is no S_j-path in W^R_j of length $> n_j$;

(b′) $f(w^0_1) = w^1_1$ and if $j > 1$ then $w^j_n = f(w^{j-1}_n)$, for all n with $1 \leq n \leq n_{j-1}$;

(c′) $\mathit{rightword}_j(w^j_1, \ldots, w^j_{n_j}) = \mathbf{u}_{i_1} * \ldots * \mathbf{u}_{i_j}$;

(d′) for every S_j-path $\langle w_1, \ldots, w_{n_j} \rangle$ in W^R_j of length n_j, we have $\mathit{rightword}_j(w_1, \ldots, w_{n_j}) = \mathbf{u}_{i_1} * \ldots * \mathbf{u}_{i_j}$.

Now it is easy to see that (10) means that

$$\mathbf{v}_{i_1} * \ldots * \mathbf{v}_{i_N} = \mathit{leftword}_N(w^N_1, \ldots, w^N_{m_N}) = \mathit{rightword}_N(w^N_1, \ldots, w^N_{n_N}) = \mathbf{u}_{i_1} * \ldots * \mathbf{u}_{i_N},$$

as required. □

Theorem 2 now follows immediately. Just observe that we have proved that, for any of the classes $\mathcal{K}$ of DTSs listed in Theorem 2, $\phi_{A,P}$ is satisfiable in $\mathcal{K}$ iff P has a solution. Indeed, the direction from right to left is Lemma 5. The direction from left to right for DTSs based on Aleksandrov spaces follows from Lemmas 4 and 6. For the remaining classes this direction follows from Lemmas 4, 6, and Theorem 1, since the $\phi^n_{A,P}$ are $\mathcal{DTL}_{\bigcirc}$-formulas.

4 Dynamic metric logic

The language $\mathcal{DML}$ of *dynamic metric logic* is defined in the same way as $\mathcal{DTL}$ with the exception that the topological operators are replaced by *metric operators* $\exists^{\leq a}$, for $a \in \mathbb{Q}^+$, where $\mathbb{Q}^+$ denotes the set of positive rational numbers.

A *dynamic metric structure* (or DMS, for short) is a pair $\mathfrak{F} = \langle \langle W, d \rangle, f \rangle$, where $\langle W, d \rangle$ is a metric space, and $f \colon W \to W$ is a *metric automorphism*, i.e., a bijection such that $d(x,y) = d(f(x), f(y))$ for all $x, y \in W$. For example, the mapping $x \mapsto x+1$ on $\mathbb{R}$ and the rotation g on B^2 considered above are metric automorphisms on the respective spaces with the Euclidean metric.

A *dynamic metric model* (or DMM) is a pair $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$, where $\mathfrak{F}$ is a DMS and $\mathfrak{V}$ is a valuation defined in the same way as in the topological case. The truth-relation is also defined as for DTMs with the exception that the truth-conditions for the topological operators **I** and **C** are replaced by

- $(\mathfrak{M}, x) \models \exists^{\leq a}\varphi$ iff there exists $y \in W$ such that $d(x,y) \leq a$ and $(\mathfrak{M}, x) \models \varphi$.

In contrast to the topological case, now we have the following:

Theorem 7. *The set of $\mathcal{DML}$-formulas that are valid in all DMSs is decidable. However, the decision problem is not elementary.*

Roughly, the idea of the decidability proof is similar to that of Theorem 13.6 from [6]: first we represents DMMs in the form of *quasimodels* and then show that quasimodels can be encoded in monadic second-order logic. The main novelty of this proof is the rather involved notion of a quasimodel. We give the definition of quasimodels and formulate their main properties below. Details of the proofs are left to the reader; they will be available on the web shortly.

Given a formula φ, denote by γ_φ the maximal numerical parameter in φ and by $M[\varphi] \subseteq \mathbb{Q}^+$ the smallest set containing all parameters in φ and closed under (i) if $a, b \in M[\varphi]$ and $a + b \leq \gamma_\varphi$, then $a + b \in M[\varphi]$, and (ii) if $a, b \in M[\varphi]$ and $a - b > 0$, then $a - b \in M[\varphi]$. Clearly, $M[\varphi]$ is finite. The closure $cl\,\varphi$ of φ is the set

$$\{\psi, \neg\psi \mid \psi \in sub\,\varphi\} \cup \{\exists^{\leq a}\psi, \neg\exists^{\leq a}\psi \mid \psi \in sub\,\varphi \text{ and } a \in M[\varphi]\},$$

where $sub\,\varphi$ is the set of all subformulas of φ. A *type* t for φ is a subset of $cl\,\varphi$ such that

(t1) for every $\neg\psi \in cl\,\varphi$, $\quad \psi \in t$ iff $\neg\psi \notin t$;

(t2) for every $\psi_1 \wedge \psi_2 \in cl\,\varphi$, $\quad \psi_1 \wedge \psi_2 \in t$ iff $\psi_1 \in t$ and $\psi_2 \in t$;

(t3) for every $\exists^{\leq a}\psi \in cl\,\varphi$, $\quad \exists^{\leq a}\psi \in t$ iff $\exists^{\leq b}\psi \in t$ for all $b \in M[\varphi] \cap [a, +\infty)$.

The *metric depth* of φ is defined as follows:

$$\begin{aligned} mtd(p) &= 0, & mtd(\exists^{\leq a}\varphi) &= mtd(\varphi) + a, \\ mtd(\neg\varphi) &= mtd(\varphi), & mtd(\bigcirc\varphi) &= mtd(\varphi), \\ mtd(\varphi_1 \wedge \varphi_2) &= \max(mtd(\varphi_1), mtd(\varphi_2)), & mtd(\Box_F\varphi) &= mtd(\varphi). \end{aligned}$$

A *quasistate* q for φ is a triple $q = \langle\langle T_q, <_q\rangle, \delta_q, t_q\rangle$, where

- $\langle T_q, <_q\rangle$ is an intransitive tree;
- $\delta_q : \{(u,v) \in T_q \times T_q \mid u <_q v\} \to M[\varphi]$ is a labelling function satisfying the following condition: the *metric depth*

$$\max_{u_0 <_q u_1 <_q \cdots <_q u_n} \sum_{i=1}^{n} \delta_q(u_{i-1}, u_i)$$

  of the tree metric space $\left\langle T_q, \delta_q^*\right\rangle$ induced by $\langle\langle T_q, <_q\rangle, \delta_q\rangle$ is bounded by $mtd(\varphi)$;
- t_q is a labelling function such that $t_q(u)$ is a type for φ, for every $u \in T_q$, and

  (qs1) for every $u \in T_q$ and every $\exists^{\leq a}\psi$, we have $\exists^{\leq a}\psi \in t_q(u)$ iff there is $v \in T_q$ such that $\delta_q^*(u,v) \leq a$ and $\psi \in t_q(v)$;

  (qs2) for every $u \in T_q$, there are no isomorphic substructures induced by immediate successors v of u located at the same distance $\delta_q(u,v)$.

We say that a point $u \in T_q$ has *index* $\langle a_1, \ldots, a_n\rangle$ if there is a sequence $u_0 <_q u_1 <_q \cdots <_q u_n$ of points in T_q such that u_0 is the root of $\left\langle T_q, <_q\right\rangle$, $u_n = u$ and $a_i = \delta_q(u_{i-1}, u_i)$, for all i, $1 \leq i \leq n$.

Let $\mathbf{q}$ be a function associating with each $i \in \mathbb{N}$ a quasistate $\mathbf{q}(i) = \langle\langle T_i, <_i\rangle, \delta_i, t_i\rangle$ for φ. A *run of index* $\langle a_1, \ldots, a_n\rangle$ through $\mathbf{q}$ is a function r mapping each $i \in \mathbb{N}$ to a point $r(i) \in T_i$ of index $\langle a_1, \ldots, a_n\rangle$ such that, for every $i \in \mathbb{N}$

- and every $\bigcirc\psi \in cl\,\varphi$, $\qquad \bigcirc\psi \in t_i(r(i))$ iff $\psi \in t_{i+1}(r(i+1))$;
- and every $\Box_F\psi \in cl\,\varphi$, $\qquad \Box_F\psi \in t_i(r(i))$ iff $\psi \in t_j(r(j))$ for all $j > i$.

Given a set $\mathfrak{R}$ of runs, we denote by $\mathfrak{R}_{\langle a_1,\ldots,a_n\rangle}$ its subset of all runs of index $\langle a_1, \ldots, a_n\rangle$.

A *quasimodel* for φ is a pair $\langle \mathbf{q}, \mathfrak{R}\rangle$, where $\mathbf{q}(i) = \langle\langle T_i, <_i\rangle, \delta_i, t_i\rangle$ is a quasistate for φ for every $i \in \mathbb{N}$ such that

(qm2) $\varphi \in t_0(u_0)$, where u_0 is the root of $\langle T_0, <_0\rangle$,

and $\mathfrak{R}$ is a set of runs through $\mathbf{q}$ satisfying the following condition

(qm3) $\mathfrak{R}_{\langle\rangle} \neq \emptyset$ and, for all $r \in \mathfrak{R}_{\langle a_1,\ldots,a_n\rangle}$, $i \in \mathbb{N}$ and $u \in T_i$, if $r(i) <_i u$ and $\delta_i(r(i), u) = a_{n+1}$ then there is a run $r' \in \mathfrak{R}_{\langle a_1,\ldots,a_n,a_{n+1}\rangle}$ such that $r'(i) = u$ and $r(i) <_i r'(i)$ for all $i \in \mathbb{N}$.

Using the decidability of monadic second-order logic over $\langle\mathbb{N}, <\rangle$, one can now prove the decidability result by showing that (i) a $\mathcal{DML}$-formula φ is satisfiable in a DMS iff there exists a quasimodel for φ, and, (ii) for every $\mathcal{DML}$-formula φ, one can effectively construct a sentence $\varphi^\sharp$ of monadic second-order logic such that there exists a quasimodel for φ iff $\varphi^\sharp$ is satisfiable in $\langle\mathbb{N}, <\rangle$.

The non-elementarity result can be proved by a polynomial reduction of the satisfiability problem for the product modal logic **PTL** $\times$ **K** (which is non-elementary by Theorem 6.37 from [6]) to the satisfiability problem for $\mathcal{DML}$-formulas in DMSs.

Open problems. Interesting and challenging open problems are (i) the decidability of dynamic topological logics interpreted in various topological spaces with *continuous* mappings, and (ii) the decidability of dynamic metric logics interpreted in various *compact* metric spaces; for a justification and more details see, e.g., [11].

Acknowledgements. The work on this paper was partially supported by U.K. EPSRC grants no. GR/S61973/01, GR/S63175/01, GR/S61966/01 and GR/S63182/01, and the U.S. Civilian Research & Development Foundation for the Independent States of the Former Soviet Union (CRDF) award no. RM1-2409-ST-02.

References

[1] P. S. Alexandroff. Diskrete Räume. *Matematicheskii Sbornik*, 2 (44):501–518, 1937.

[2] S. Artemov, J. Davoren, and A. Nerode. Modal logics and topological semantics for hybrid systems. Technical Report MSI 97-05, Cornell University, 1997.

[3] N. Bourbaki. *General topology, Part 1*. Hermann, Paris and Addison-Wesley, 1966.

[4] J. R. Brown. *Ergodic Theory and Topological Dynamics*. Academic Press, 1976.

[5] J. Davoren and A. Nerode. Logics for hybrid systems. *Proceedings of the IEEE*, 88:985–1010, 2000.

[6] D. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Many-Dimensional Modal Logics: Theory and Applications*, volume 148 of *Studies in Logic*. Elsevier, 2003.

[7] J. Hopcroft, R. Motwani, and J. Ullman. *Introduction to Automata Theory, Languages, and Computation*. Addison–Wesley, 2001.

[8] A. Katok and B. Hasselblatt. *Introduction to Modern Theory of Dynamical Systems*, volume 54 of *Encyclopedia of mathematics and its applications*. Elsevier, 1995.

[9] P. Kremer. Temporal logic over **S4**: an axiomatizable fragment of dynamic topological logic. *Bulletin of Symbolic Logic*, 3:375–376, 1997.

[10] P. Kremer and G. Mints. Dynamic topological logic. *Bulletin of Symbolic Logic*, 3:371–372, 1997.

[11] P. Kremer and G. Mints. Dynamic topological logic. Manuscript, 2003.

[12] P. Kremer, G. Mints, and V. Rybakov. Axiomatizing the next-interior fragment of dynamic topological logic. *Bulletin of Symbolic Logic*, 3:376–377, 1997.

[13] O. Kutz, H. Sturm, N.-Y. Suzuki, F. Wolter, and M. Zakharyaschev. Logics of metric spaces. *ACM Transactions on Computational Logic*, 4:260–294, 2003.

[14] J.C.C. McKinsey and A. Tarski. The algebra of topology. *Annals of Mathematics*, 45:141–191, 1944.

[15] T. Moor and J. Davoren. Robust controller synthesis for hybrid systems using modal logics. In M.D. Di Benedetto and A. Sangiovanni-Vincentelli, editors, *Hybrid Systems: Computation and Control* (*HSCC'01*), volume 2034 of *LNCS*, pages 433–446. Springer, 2001.

[16] E. Post. A variant of a recursively unsolvable problem. *Bulletin of the AMS*, 52:264–268, 1946.

[17] S. Slavnov. Two counterexamples in the logic of dynamic topological systems. Technical Report TR–2003015, Cornell University, 2003.