Foundations of Computer Science Comp109

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Part 2. (Naive) Set Theory

Comp109 Foundations of Computer Science

Reading

- K. H. Rosen. *Discrete Mathematics and Its Applications* Chapter 2

Contents

- Notation for *sets*.
- Important sets.
- What is a *subset* of a set?
- When are two sets *equal*?
- *Operations* on sets.
- *Algebra* of sets.
- Bit strings.
- *Cardinality* of sets.
- Russell's paradox.

Notation

A *set* is a collection of objects, called the *elements* of the set. For example:

- $\{7, 5, 3\}$;
- $\{\text{Liverpool}, \text{Manchester}, \text{Leeds}\}$.

We have written down the elements of each set and contained them between the *braces* $\{\ \}$.

We write $a \in S$ to denote that the object a is an element of the set S:

$$7 \in \{7, 5, 3\}, \quad 4 \notin \{7, 5, 3\}.$$

Notes

- The order of elements does not matter
- Repeatitions do not count

Notation

For a large set, especially an infinite set, we cannot write down all the elements. We use a predicate P instead.

$$S = \{x \mid P(x)\}$$

denotes the set of objects x for which the predicate $P(x)$ is true.

Examples: Let $S = \{1, 3, 5, 7, \ldots\}$. Then

$$S = \{x \mid x \text{ is an odd positive integer}\}$$

and

$$S = \{2n - 1 \mid n \text{ is a positive integer }\}.$$

More examples

Find simpler descriptions of the following sets by listing their elements:

- $A = \{x \mid x \text{ is an integer and } x^2 + 4x = 12\}$;
- $B = \{x \mid x \text{ a day of the week not containing "}u\text{" }\}$;
- $C = \{n^2 \mid n \text{ is an integer }\}$.

Important sets (notation)

The empty set has no elements. It is written as $\emptyset$ or as $\{\}$.

We have seen some other examples of sets in Part 1.

- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ (the natural numbers)
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ (the integers)
- $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ (the positive integers)
- $\mathbb{Q} = \{x/y \mid x \in \mathbb{Z}, y \in \mathbb{Z}, y \neq 0\}$ (the rationals)
- $\mathbb{R}$: (real numbers)
  - $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ the set of real numbers between a and b (inclusive)

Detour: Sets in python

Sets are the 'most elementary' data structures (though they don't always map well into the underlying hardware).

Some modern programming languages feature sets.

- For example, in Python one writes

```
empty = set()
m = {'a', 'b', 'c'}
n = {1, 2}
print 'a' in m
```


Computer representation of sets

Only finite sets can be represented

- Number of elements not fixed: List (?) Java&Python do differently
- All elements of A are drawn from some ordered sequence $S = s_1, \ldots, s_n$: the characteristic vector of A is the sequence $(b_1, \ldots, b_n)$ where

$$b_i = \begin{cases} 1 & \text{if} \quad s_i \in A \\ 0 & \text{if} \quad s_i \notin A \end{cases}$$

Sequences of zeros and ones of length n are called *bit strings* of length n. AKA *bit vectors* AKA *bit arrays*

Example

Let $S = \{1, 2, 3, 4, 5\}$, $A = \{1, 3, 5\}$ and $B = \{3, 4\}$.

- The characteristic vector of A is $(1, 0, 1, 0, 1)$.
- The characteristic vector of B is $(0, 0, 1, 1, 0)$.
- The set characterised by $(1, 1, 1, 0, 1)$ is $\{1, 2, 3, 5\}$.
- The set characterised by $(1, 1, 1, 1, 1)$ is $\{1, 2, 3, 4, 5\}$.
- The set characterised by $(0, 0, 0, 0, 0)$ is ...

Subsets

Definition A set B is called a *subset* of a set A if every element of B is an element of A. This is denoted by $B \subseteq A$.

Examples:

$$\{3, 4, 5\} \subseteq \{1, 5, 4, 2, 1, 3\}, \ \{3, 3, 5\} \subseteq \{3, 5\}, \ \{5, 3\} \subseteq \{3, 5\}.$$

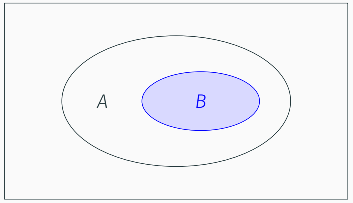


Figure 1: Venn diagram of $B \subseteq A$.

Detour: Subsets in Python

```
def isSubset(A, B):
    for x in A:
        if x not in B:
            return False
    return True
```

Testing the method:

```
print isSubset(n,m)
```

But then there is a built-in operation:

```
print n<m
```


Subsets and bit vectors

Let $S = \{1, 2, 3, 4, 5\}$, $A = \{1, 3, 5\}$ and $B = \{3, 4\}$.

- Is $A \subseteq B$?
- Is the set C, represented by $(1, 0, 0, 0, 1)$, a subset of the set D, represented by $(1, 1, 0, 0, 1)$?

Equality

Definition A set A is called *equal* to a set B if $A \subseteq B$ and $B \subseteq A$. This is denoted by $A = B$.

Examples:

$$\{1\} = \{1, 1, 1\},$$
$$\{1, 2\} = \{2, 1\},$$
$$\{5, 4, 4, 3, 5\} = \{3, 4, 5\}.$$

The union of two sets

Definition The union of two sets A and B is the set

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

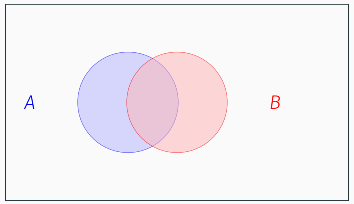


Figure 2: Venn diagram of $A \cup B$.

Example

Suppose

$$A = \{4, 7, 8\}$$

and

$$B = \{4, 9, 10\}.$$

Then

$$A \cup B = \{4, 7, 8, 9, 10\}.$$

Detour: Set union in Python

```python
def union(A, B):
    result = set()
    for x in A:
        result.add(x)
    for x in B:
        result.add(x)
    return result
```

Testing the method:

```python
print union(m, n)
```

But then there is a built-in operation:

```python
print m.union(n)
```


Union of sets represented by bit vectors

Let $S = \{1, 2, 3, 4, 5\}$, $A = \{1, 3, 5\}$ and $B = \{3, 4\}$.

- Compute $A \cup B$.

- Compute the union of the set C, represented by $(1, 0, 0, 0, 1)$, and the set D, represented by $(1, 1, 0, 0, 1)$.

The intersection of two sets

Definition The intersection of two sets A and B is the set

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

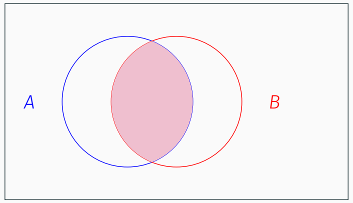


Figure 3: Venn diagram of $A \cap B$.

Example

Suppose

$$A = \{4, 7, 8\}$$

and

$$B = \{4, 9, 10\}.$$

Then

$$A \cap B = \{4\}$$

Detour: Set intersection in Python

```python
def intersection(A, B):
    result = set()
    for x in A:
        if x in B:
            result.add(x)
    return result
```

Testing the method:

```python
print intersection(m, n)
print intersection(n, {1})
```

But then there is a built-in operation:

```python
print n.intersection({1})
```


Intersection of sets represented by bit vectors

Let $S = \{1, 2, 3, 4, 5\}$, $A = \{1, 3, 5\}$ and $B = \{3, 4\}$.

- Compute $A \cap B$.

- Compute the intersection of the set C, represented by $(1, 0, 0, 0, 1)$, and the set D, represented by $(1, 1, 0, 0, 1)$.

The relative complement

Definition The relative complement of a set B relative to a set A is the set

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}.$$

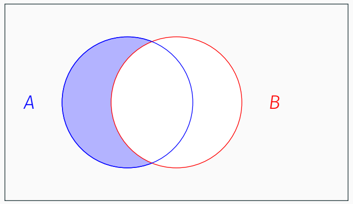


Figure 4: Venn diagram of $A - B$.

Example

Suppose

$$A = \{4, 7, 8\}$$

and

$$B = \{4, 9, 10\}.$$

Then

$$A - B = \{7, 8\}$$

Detour: Set complement in Python

```python
def complement(A, B):
    result = set()
    for x in A:
        if x not in B:
            result.add(x)
    return result
```

Testing the method:

```python
print complement(m, {'a'})
```

But then there is a built-in operation:

```python
print m-{'a'}
```

Relative complement and bit vectors

Let $S = \{1, 2, 3, 4, 5\}$, $A = \{1, 3, 5\}$ and $B = \{3, 4\}$.

- Compute $A - B$.
- Compute the relative complement of the set C, represented by $(1, 0, 0, 0, 1)$, related to the set D, represented by $(1, 1, 0, 0, 1)$.

The complement

When we are dealing with subsets of some large set U, then we call U the *universal set* for the problem in question.

Definition The complement of a set A is the set

$$\sim A = \{x \mid x \notin A\} = U - A.$$

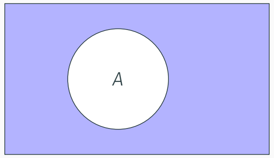


Figure 5: Venn diagram of $\sim A$. (The rectangle is U)

Complement and bit vectors

Let $S = \{1, 2, 3, 4, 5\}$, $A = \{1, 3, 5\}$ and $B = \{3, 4\}$.

- Compute $\sim A$.
- Compute $\sim B$.
- Compute the complement of the set C, represented by $(1, 0, 0, 0, 1)$.

The symmetric difference

Definition The symmetric difference of two sets A and B is the set

$$A \Delta B = \{x \mid (x \in A \text{ and } x \notin B) \text{ or } (x \notin A \text{ and } x \in B)\}.$$

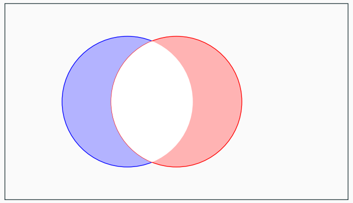


Figure 6: Venn diagram of $A \Delta B$.

Example

Suppose

$$A = \{4, 7, 8\}$$

and

$$B = \{4, 9, 10\}.$$

Then

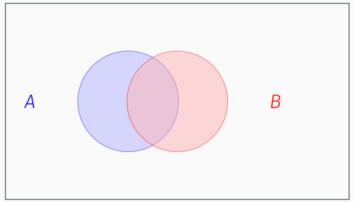
$$A \Delta B = \{7, 8, 9, 10\}$$

The algebra of sets

Suppose that A, B and U are sets with $A \subseteq U$ and $B \subseteq U$.

Commutative laws:

$$A \cup B = B \cup A, \quad A \cap B = B \cap A;$$



Proving the commutative law $A \cup B = B \cup A$

Definition: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ $B \cup A = \{x \mid x \in B \text{ or } x \in A\}$.

These are the same set. To see this, check all possible cases.

Case 1: Suppose $x \in A$ and $x \in B$. Since $x \in A$, the definitions above show that x is in both $A \cup B$ and $B \cup A$.

Case 2: Suppose $x \in A$ and $x \notin B$. Since $x \in A$, the definitions above show that x is in both $A \cup B$ and $B \cup A$.

Case 3: Suppose $x \notin A$ and $x \in B$. Since $x \in B$, the definitions above show that x is in both $A \cup B$ and $B \cup A$.

Case 4: Suppose $x \notin A$ and $x \notin B$. The definitions above show that x is not in $A \cup B$ and x is not in $B \cup A$.

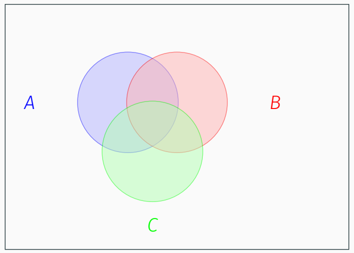
So, for all possible x, either x is in both $A \cup B$ and $B \cup A$, or it is in neither. We conclude that the sets $A \cup B$ and $B \cup A$ are the same.

The algebra of sets

Suppose that A, B, C, U are sets with $A \subseteq U$, $B \subseteq U$, and $C \subseteq U$.

Associative laws:

$$A \cup (B \cup C) = (A \cup B) \cup C, \quad A \cap (B \cap C) = (A \cap B) \cap C;$$



Proving the associative law $A \cup (B \cup C) = (A \cup B) \cup C$

This is almost as easy as proving the commutative law, but now there are 8 cases to check, depending on whether $x \in A$, whether $x \in B$ and whether $x \in C$.

Definition: $X \cup Y = \{x \mid x \in X \text{ or } x \in Y\}$

Here is one case: Suppose $x \in A$, $x \notin B$ and $x \notin C$. Since $x \in A$, we can use the definition with $X = A$ and $Y = B \cup C$ to show that $x \in A \cup (B \cup C)$.

Since $x \in A$, we can use the definition with $X = A$ and $Y = B$ to show that $x \in A \cup B$. Then we can use the definition with $X = A \cup B$ and $Y = C$ to show that $x \in (A \cup B) \cup C$.

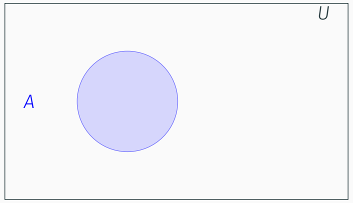
Writing out all eight cases is tedious, but it is not difficult.

The algebra of sets

Suppose that A and U are sets with $A \subseteq U$.

Identity laws:

$$A \cup \emptyset = A, \ A \cup U = U, \ A \cap U = A, \ A \cap \emptyset = \emptyset;$$

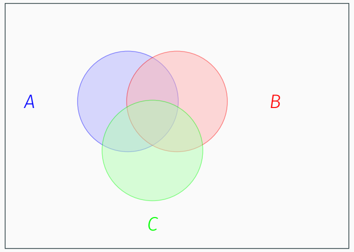


The algebra of sets

Suppose that A, B, C, U are sets with $A \subseteq U$, $B \subseteq U$, and $C \subseteq U$.

Distributive laws:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \ A \cup (B \cap C) = (A \cup B) \cap (A \cup C);$$

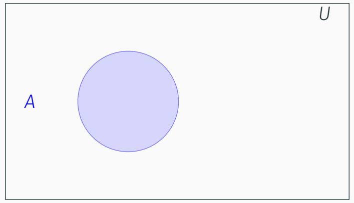


The algebra of sets

Suppose that A and U are sets with $A \subseteq U$. Let $\sim A = U - A$. Then

Complement laws:

$$A \cup \sim A = U, \ \sim U = \emptyset, \ \sim(\sim A) = A, A \cap \sim A = \emptyset, \ \sim \emptyset = U;$$

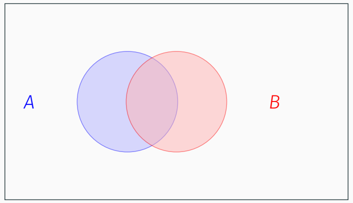


The algebra of sets

Suppose that A, B and U are sets with $A \subseteq U$, and $B \subseteq U$. Recall that $\sim X = U - X$ and $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ and $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$. Then

De Morgan's laws:

$$\sim(A \cup B) = \sim A \cap \sim B, \ \sim(A \cap B) = \sim A \cup \sim B.$$



A proof of De Morgan's law $\sim(A \cap B) = \sim A \cup \sim B$

Case 1: Suppose $x \in A$ and $x \in B$. From the definition of $\cap$, $x \in A \cap B$. So from the definition of $\sim$, $x \notin \sim(A \cap B)$. From the definition of $\sim$, $x \notin \sim A$ and also $x \notin \sim B$. So from the definition of $\cup$, $x \notin \sim A \cup \sim B$.

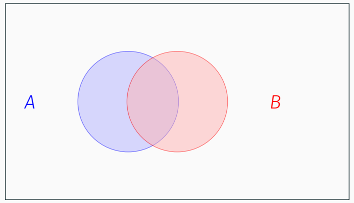
Case 2: Suppose $x \in A$ and $x \notin B$. From the definition of $\cap$, $x \notin A \cap B$. So from the definition of $\sim$, $x \in \sim(A \cap B)$. From the definition of $\sim$, $x \notin \sim A$ but $x \in \sim B$. So from the definition of $\cup$, $x \in \sim A \cup \sim B$.

Case 3: Suppose $x \notin A$ and $x \in B$. From the definition of $\cap$, $x \notin A \cap B$. So from the definition of $\sim$, $x \in \sim(A \cap B)$. From the definition of $\sim$, $x \in \sim A$ but $x \notin \sim B$. So from the definition of $\cup$, $x \in \sim A \cup \sim B$.

Case 4: Suppose $x \notin A$ and $x \notin B$. From the definition of $\cap$, $x \notin A \cap B$. So from the definition of $\sim$, $x \in \sim(A \cap B)$. From the definition of $\sim$, $x \in \sim A$ and $x \in \sim B$. So from the definition of $\cup$, $x \in \sim A \cup \sim B$.

Using the algebra of sets

Prove that $A \Delta B = (A \cup B) \cap \sim(A \cap B)$. (See the next slide.)



$$\begin{aligned}
(A \cup B) \cap \sim(A \cap B) &= (A \cup B) \cap (\sim A \cup \sim B) && \text{De Morgan} \\
&= ((A \cup B) \cap \sim A) \cup ((A \cup B) \cap \sim B) && \text{distributive} \\
&= (\sim A \cap (A \cup B)) \cup (\sim B \cap (A \cup B)) && \text{commutative} \\
&= ((\sim A \cap A) \cup (\sim A \cap B)) \cup ((\sim B \cap A) \cup (\sim B \cap B)) && \text{distributive} \\
&= ((A \cap \sim A) \cup (B \cap \sim A)) \cup ((A \cap \sim B) \cup (B \cap \sim B)) && \text{commutative} \\
&= (\emptyset \cup (B \cap \sim A)) \cup ((A \cap \sim B) \cup \emptyset) && \text{complement} \\
&= (A \cap \sim B) \cup (B \cap \sim A) && \text{commutative and identity} \\
&= A \Delta B && \text{definition}
\end{aligned}$$

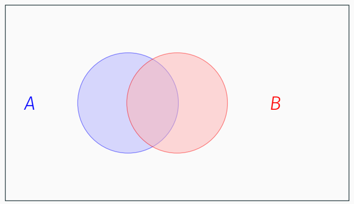
Cardinality of sets

Definition The cardinality of a *finite* set S is the number of elements in S, and is denoted by $|S|$.

Computing the cardinality of a union of two sets

If A and B are sets then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

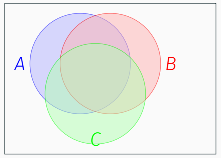


Example

Suppose there are 100 third-year students. 40 of them take the module “Sequential Algorithms” and 80 of them take the module “Multi-Agent Systems”. 25 of them took both modules. How many students took neither modules?

Computing the cardinality of a union of three sets

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

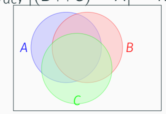


These are special cases of the principle of inclusion and exclusion which we will study later.

Proof (optional)

We need lots of notation.

- $|A - (B \cup C)| = n_a$, $|B - (A \cup C)| = n_b$, $|C - (A \cup B)| = n_c$,
- $|(A \cap B) - C| = n_{ab}$, $|(A \cap C) - B| = n_{ac}$, $|(B \cap C) - A| = n_{bc}$,
- $|A \cap B \cap C| = n_{abc}$.



Then

$$\begin{aligned} |A \cup B \cup C| &= n_a + n_b + n_c + n_{ab} + n_{ac} + n_{bc} + n_{abc} \\ &= (n_a + n_{ab} + n_{ac} + n_{abc}) + (n_b + n_{ab} + n_{bc} + n_{abc}) \\ &+ (n_c + n_{ac} + n_{bc} + n_{abc}) - (n_{ab} + n_{abc}) \\ &- (n_{ac} + n_{abc}) - (n_{bc} + n_{abc}) + n_{abc} \end{aligned}$$

Reflection

The following statements hold:

- $\emptyset \in \{\emptyset\}$ but $\emptyset \notin \emptyset$;
- $\emptyset \subseteq \{5\}$;
- $\{2\} \not\subseteq \{\{2\}\}$ but $\{2\} \in \{\{2\}\}$;
- $\{3, \{3\}\} \neq \{3\}$.

Why is this set theory “naive”

It suffers from paradoxes.

Why is this set theory “naive”

It suffers from paradoxes.

A leading example:

> *A barber is the man who shaves all those, and only those, men who do not shave themselves.*

- Who shaves the barber?

Russell’s Paradox

Russell’s paradox shows that the ‘object’ $\{x \mid P(x)\}$ is not always meaningful.

Set $A = \{B \mid B \notin B\}$

Problem: do we have $A \in A$?

Abbreviate, for any set C, by $P(C)$ the statement $C \notin C$. Then $A = \{B \mid P(B)\}$.

- If $A \in A$, then (from the definition of P), not $P(A)$. Therefore $A \notin A$.
- If $A \notin A$, then (from the definition of P), $P(A)$. Therefore $A \in A$.