Foundations of Computer Science Comp109

University of Liverpool
Boris Konev
konev@liverpool.ac.uk
http://www.csc.liv.ac.uk/~konev/COMP109

Part 4. Function

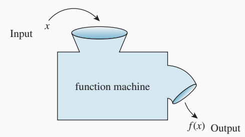
Comp109 Foundations of Computer Science

Reading

- Discrete Mathematics and Its Applications K. Rosen, Section 2.3.
- Discrete Mathematics with Applications S. Epp, Chapter 7.

Contents

Functions



Examples:

- $y = x^2$
- $y = \sin(x)$
- first letter of your name

Functions/methods on programming

Java
```
public int f(int x) {
    return x+5;
}
```

C/C++
```
int f(int x) {
    return x+5;
}
```

Python
```
def f(int x):
    return x+5
```


Definition

A **function** from a set A to a set B is an assignment of exactly one element of B to each element of A.

We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element of a.

If f is a function from A to B we write $f : A \to B$.

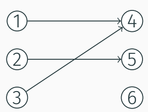


Figure 1: A function $f : \{1, 2, 3\} \to \{4, 5, 6\}$

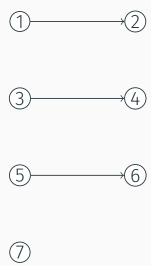


Figure 2: No function

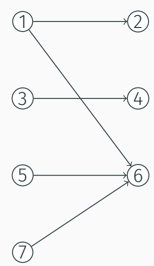


Figure 3: No function

Domain, codomain, and range

Suppose $f : A \to B$.

- A is called the *domain* of f. B is called the *codomain* of f.
- The *range* $f(A)$ of f is
$$f(A) = \{f(x) \mid x \in A\}.$$

Codomain vs range

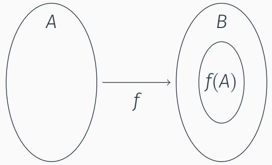
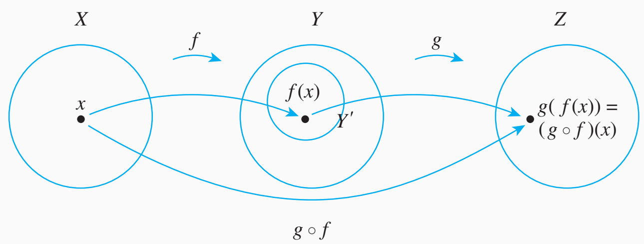


Figure 4: the range of f

Composition of functions

If $f : X \to Y$ and $g : Y \to Z$ are functions, then their **composition** $g \circ f$ is a function from X to Z given by
$$(g \circ f)(x) = g(f(x)).$$



Example

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ and the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = 4x + 3$. Calculate $g \circ f$, $f \circ g$, $f \circ f$ and $g \circ g$.

Injective (one-to-one) functions

Definition Let $f : A \to B$ be a function. We call f an *injective* (or *one-to-one*) function if
$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2 \text{ for all } a_1, a_2 \in A.$$

This is logically equivalent to $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$ and so injective functions never repeat values. In other words, different inputs give different outputs.

Examples

$f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$ is not injective.

$h : \mathbb{Z} \to \mathbb{Z}$ given by $h(x) = 2x$ is injective.

Surjective (or onto) functions

Definition $f : A \to B$ is *surjective* (or onto) if the range of f coincides with the codomain of f. This means that for every $b \in B$ there exists $a \in A$ with $b = f(a)$.

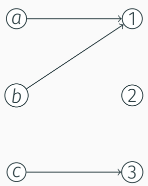
Examples

$f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$ is not surjective.

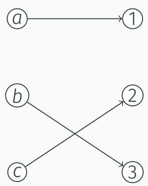
$h : \mathbb{Z} \to \mathbb{Z}$ given by $h(x) = 2x$ is not surjective.

$h' : \mathbb{Q} \to \mathbb{Q}$ given by $h'(x) = 2x$ is surjective.

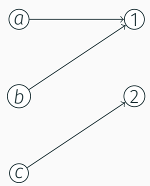
Classify $f : \{a, b, c\} \to \{1, 2, 3\}$ given by



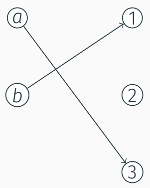
Classify $g : \{a, b, c\} \to \{1, 2, 3\}$ given by



Classify $h : \{a, b, c\} \to \{1, 2\}$ given by



Classify $h' : \{a, b, c\} \to \{1, 2, 3\}$ given by



Bijections

We call f *bijective* if f is both injective and surjective.

Examples

$f : \mathbb{Q} \to \mathbb{Q}$ given by $f(x) = 2x$ is bijective.

Inverse functions

If f is a bijection from a set X to a set Y, then there is a function f^{-1} from Y to X that “undoes” the action of f; that is, it sends each element of Y back to the element of X that it came from. This function is called the inverse function for f.

Then $f(a) = b$ if, and only if, $f^{-1}(b) = a$.

Example

$k : \mathbb{R} \to \mathbb{R}$ given by $k(x) = 4x + 3$ is invertible and

$$k^{-1}(y) = \frac{1}{4}(y - 3).$$

Example

Let $A = \{x \mid x \in \mathbb{R}, x \neq 1\}$ and $f : A \to A$ be given by

$$f(x) = \frac{x}{x - 1}.$$

Show that f is bijective and determine the inverse function.

Bijections and representations

Let $S = \{1, 2, \ldots, n\}$ and let B^n be the set of bit strings of length n. The function

$$f : Pow(S) \to B^n$$

which assigns each subset A of S to its characteristic vector is a bijection.

Cardinality of finite sets and functions

Recall: *The cardinality of a finite set S is the number of elements in S*

A bijection $f : S \to \{1, \ldots, n\}$.

For finite sets A and B

- $|A| \geq |B|$ iff there is a surjective function from A to B.
- $|A| \leq |B|$ iff there is a injective function from A to B.
- $|A| = |B|$ iff there is a bijection from A to B.

The pigeonhole principle

Let $f : A \to B$ be a function where A and B are finite sets.

The *pigeonhole principle* states that if $|A| > |B|$ then at least one value of f occurs more than once.

In other words, we have $f(a) = f(b)$ for some distinct elements a, b of A.

Pigeons and pigeonholes

If (N+1) pigeons occupy N holes, then some hole must have at least 2 pigeons.

Example

Problem. There are 15 people on a bus. Show that at least two of them have a birthday in the same month of the year.

Example

Problem. How many different surnames must appear in a telephone directory to guarantee that at least two of the surnames begin with the same letter of the alphabet and end with the same letter of the alphabet?

Example

Problem. Five numbers are selected from the numbers $1, 2, 3, 4, 5, 6, 7$ and 8. Show that there will always be two of the numbers that sum to 9.

Extended pigeonhole principle

Consider a function $f : A \to B$ where A and B are finite sets and $|A| > k|B|$ for some natural number k. Then, there is a value of f which occurs at least $k + 1$ times.

Example

Problem. How many different surnames must appear in a telephone directory to guarantee that at least five of the surnames begin with the same letter of the alphabet and end with the same letter of the alphabet?

Example

Problem. Show that in any group of six people there are either three who all know each other or three complete strangers.

Bijections and cardinality

Recall that the cardinality of a finite set is the number of elements in the set.

Sets A and B have the same cardinality iff there is a bijection from A to B.

Example: The cardinality of the power set.

Definition The power set $Pow(A)$ of a set A is the set of all subsets of A. In other words,

$$Pow(A) = \{C \mid C \subseteq A\}.$$

For all $n \in \mathbb{Z}^+$ and all sets A: if $|A| = n$, then $|Pow(A)| = 2^n$.

Power set and bit vectors

Recall that if all elements of a set A are drawn from some ordered sequence $S = s_1, \ldots, s_n$: the characteristic vector of A is the sequence $(b_1, \ldots, b_n)$ where

$$b_i = \begin{cases} 1 & \text{if} \quad s_i \in A \\ 0 & \text{if} \quad s_i \notin A \end{cases}$$

We use the correspondence between bit vectors and subsets: $|Pow(A)|$ is the number of bit vectors of length n.

The number of n-bit vectors is 2^n

We prove the statement by induction.

Base Case: Take $n = 1$. There are two bit vectors of length 1: (0) and (1).

The number of n-bit vectors is 2^n

Inductive Step: Assume that the property holds for $n = m$, so the number of m-bit vectors is 2^m. Now consider the set B of all $(m+1)$-bit vectors. We must show that $|B| = 2^{m+1}$.

Every $(b_1, b_2, \ldots, b_{m+1}) \in B$ starts with an m-bit vector $(b_1, b_2, \ldots, b_m)$ followed by b_{m+1}, which can be either 0 or 1.

Thus

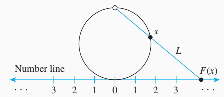
$$|B| = 2^m + 2^m = 2^{m+1}.$$

Infinite sets

Sets A and B have the same cardinality iff there is a bijection from A to B.

Examples:

- $\mathbb{Z}$ and even integers
  - consider $f(n) = 2n$
- $\{x \in \mathbb{R} \mid 0 < x < 1\}$ and $\mathbb{R}^+$
  - consider $g(x) = \frac{1}{x} - 1$
- $\{x \in \mathbb{R} \mid 0 < x < 1\}$ and $\mathbb{R}$



Countable sets

A set that is either finite or has the same cardinality as $\mathbb{N}$ is called **countable**.

- $\mathbb{Z}$

$$\cdots \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \cdots$$

Countable Sets: $\mathbb{Q}$

$$\begin{array}{ccccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \frac{2}{6} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \frac{3}{6} & \cdots \\
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \frac{4}{6} & \cdots \\
\frac{5}{1} & \frac{5}{2} & \frac{5}{3} & \frac{5}{4} & \frac{5}{5} & \frac{5}{6} & \cdots \\
\frac{6}{1} & \frac{6}{2} & \frac{6}{3} & \frac{6}{4} & \frac{6}{5} & \frac{6}{6} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots
\end{array}$$

Uncountable sets

- A set that is not countable is called **uncountable**.
  - $S = \{x \in \mathbb{R} \mid 0 < x < 1\}$ is uncountable

Cantor's diagonal argument

Suppose S is countable. Then the decimal representations of these numbers can be written as a list

$$\begin{aligned}
a_1 &= 0.a_{11}\, a_{12}\, a_{13} \ldots\, a_{1n} \ldots \\
a_2 &= 0.a_{21}\, a_{22}\, a_{23} \ldots\, a_{2n} \ldots \\
a_3 &= 0.a_{31}\, a_{32}\, a_{33} \ldots\, a_{3n} \ldots \\
&\vdots \\
a_n &= 0.a_{n1}\, a_{n2}\, a_{n3} \ldots\, a_{nn} \ldots \\
&\vdots
\end{aligned}$$

Let $d = 0.d_1\, d_2\, d_3 \ldots d_n \ldots$ where

$$d_i = \begin{cases} 1, & \text{if } a_{ii} \neq 1 \\ 2, & \text{if } a_{ii} = 1 \end{cases}$$

Then d is not in the sequence $a_1, a_2, a_3 \ldots$