Foundations of Computer Science Comp109

University of Liverpool
Boris Konev
konev@liverpool.ac.uk
http://www.csc.liv.ac.uk/~konev/COMP109

Part 3. Relations

Comp109 Foundations of Computer Science

Reading

Discrete Mathematics and Its Applications K. Rosen, Chapter 9.

Contents

Motivation

- Intuitively, there is a "relation" between two things if there is some connection between them.
  E.g.
  - 'friend of'
  - $a < b$
  - m divides n
- Relations are used in crucial ways in many branches of mathematics
  - Equivalence
  - Ordering
- Computer Science

Databases and relations

A database table ≈ relation

TABLE 1 Students.

Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

Ordered pairs

Definition The Cartesian product $A \times B$ of sets A and B is the *set* consisting of all pairs (a, b) with $a \in A$ and $b \in B$, i.e.,

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Note that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

Note

- $\{1, 2\} = \{2, 1\}$ but $(1, 2) \neq (2, 1)$.

Example

- Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then

$$A \times B = \{(1, a), (2, a), (1, b), (2, b), (1, c), (2, c)\}.$$

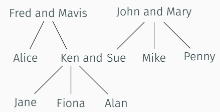
- $B \times A =$

Relations

Definition A binary relation between two sets A and B is a subset R of the Cartesian product $A \times B$.

If $A = B$, then R is called a binary relation on A.

Example: Family tree



Write down

- $R = \{(x, y) \mid x \text{ is a grandfather of } y\}$;
- $S = \{(x, y) \mid x \text{ is a sister of } y\}$.

Example 2

Write down the ordered pairs belonging to the following binary relations between $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6\}$:

- $U = \{(x, y) \in A \times B \mid x + y = 9\}$;
- $V = \{(x, y) \in A \times B \mid x < y\}$.

Example 3

Let $A = \{1, 2, 3, 4, 5, 6\}$. Write down the ordered pairs belonging to

$$R = \{(x, y) \in A \times A \mid x \text{ is a divisor of } y\}.$$

Representation of binary relations: directed graphs

- Let A and B be two finite sets and R a binary relation between these two sets (i.e., $R \subseteq A \times B$).
- We represent the elements of these two sets as vertices of a graph.
- For each $(a, b) \in R$, we draw an arrow linking the related elements.
- This is called the directed graph (or digraph) of R.

Example

Consider the relation V between $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6\}$ such that $V = \{(x, y) \in A \times B \mid x < y\}$.

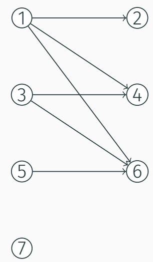


Figure 1: digraph of V

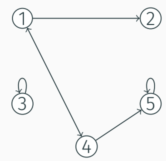
Digraphs of binary relations on a single set

A binary relation between a set A and itself is called "a binary relation on A".

To represent such a relation, we use a directed graph in which a single set of vertices represents the elements of A and arrows link the related elements.

Consider the relation $V \subseteq A \times A$ where $A = \{1, 2, 3, 4, 5\}$ and

$$V = \{(1, 2), (3, 3), (5, 5), (1, 4), (4, 1), (4, 5)\}.$$



Functions as relations

- Recall that a function f from a set A to a set B assigns exactly one element of B to each element of A.
  - Gives rise to the relation $R_f = \{(a, b) \in A \times B \mid b = f(a)\}$
- If a relation $S \subseteq A \times B$ is such that for every $a \in A$ there exists at most one $b \in B$ with $(a, b) \in S$, relation S is functional.
- (Sometimes in the literature, functions are introduced through functional relations.)

Inverse relation

Definition Given a relation $R \subseteq A \times B$, we define the *inverse relation* $R^{-1} \subseteq B \times A$ by

$$R^{-1} = \{(b, a) \mid (a, b) \in R\}.$$

Example: The inverse of the relation *is a parent of* on the set of people is the relation *is a child of*.

Composition of relations

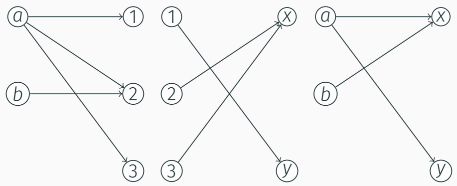
Definition Let $R \subseteq A \times B$ and $S \subseteq B \times C$. The (functional) composition of R and S, denoted by $S \circ R$, is the binary relation between A and C given by

$$S \circ R = \{(a, c) \mid \text{ exists } b \in B \text{ such that } aRb \text{ and } bSc\}.$$

Example: If R is the relation *is a sister of* and S is the relation *is a parent of*, then

- $S \circ R$ is the relation *is an aunt of*;
- $S \circ S$ is the relation *is a grandparent of*.

Digraph representation of compositions



R $\qquad\qquad$ S $\qquad\qquad$ $S \circ R$

Computer friendly representation of binary relations: matrices

- Another way of representing a binary relation between finite sets uses an array.
- Let $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_m\}$ and $R \subseteq A \times B$.
- We represent R by an array M of n rows and m columns. Such an array is called a *n by m matrix.*
- The entry in row i and column j of this matrix is given by $M(i, j)$ where

$$M(i,j) = \begin{cases} T & \text{if} \quad (a_i, b_j) \in R \\ F & \text{if} \quad (a_i, b_j) \notin R \end{cases}$$

Example 1

Let $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6\}$, and

$$U = \{(x, y) \in A \times B \mid x + y = 9\}$$

Assume an enumeration $a_1 = 1$, $a_2 = 3$, $a_3 = 5$, $a_4 = 7$ and $b_1 = 2$, $b_2 = 4$, $b_3 = 6$. Then M represents U, where

$$M = \begin{bmatrix} F & F & F \\ F & F & T \\ F & T & F \\ T & F & F \end{bmatrix}$$

Example 2

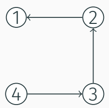
Let $A = \{a, b, c, d\}$ and suppose that $R \subseteq A \times A$ has the following matrix representation:

$$M = \begin{bmatrix} F & T & T & F \\ F & F & T & T \\ F & T & F & F \\ T & T & F & T \end{bmatrix}$$

List the ordered pairs belonging to R.

Example

The binary relation R on $A = \{1, 2, 3, 4\}$ has the following digraph representation.



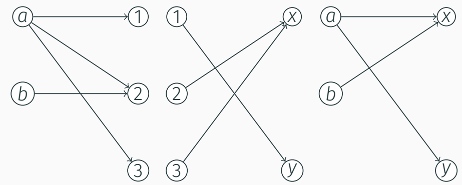
- The ordered pairs $R =$
- The matrix

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

- In words:

Matrices and composition

Now let's go back and see how this works for matrices representing relations



$$R : \begin{bmatrix} T & T & T \\ F & T & F \end{bmatrix} \qquad S : \begin{bmatrix} F & T \\ T & F \\ T & F \end{bmatrix} \qquad S \circ R : \begin{bmatrix} T & T \\ T & F \end{bmatrix}$$

The formal description

Given two matrices with entries "T" and "F" representing the relations we can form the matrix representing the composition. This is called the *logical (Boolean) matrix product*.

Let $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_m\}$ and $C = \{c_1, \ldots, c_p\}$.

The logical matrix M representing R is given by:

$$M(i,j) = \begin{cases} T & \text{if} \quad (a_i, b_j) \in R \\ F & \text{if} \quad (a_i, b_j) \notin R \end{cases}$$

The logical matrix N representing S is given by

$$N(i,j) = \begin{cases} T & \text{if} \quad (b_i, c_j) \in S \\ F & \text{if} \quad (b_i, c_j) \notin S \end{cases}$$

Matrix representation of compositions

Then the entries $P(i, j)$ of the logical matrix P representing $S \circ R$ are given by

- $P(i, j) = T$ if there exists l with $1 \leq l \leq m$ such that $M(i, l) = T$ and $N(l, j) = T$.
- $P(i, j) = F$, otherwise.

We write $P = MN$.

The example from before

Let R be the relation between $A = \{a, b\}$ and $B = \{1, 2, 3\}$ represented by the matrix

$$M = \begin{bmatrix} T & T & T \\ F & T & F \end{bmatrix}$$

Similarly, let S be the relation between B and $C = \{x, y\}$ represented by the matrix

$$N = \begin{bmatrix} F & T \\ T & F \\ T & F \end{bmatrix}$$

Example

Then the matrix $P = MN$ representing $S \circ R$ is

$$P = \begin{bmatrix} T & T \\ T & F \end{bmatrix}$$

Infix notation for binary relations

If R is a binary relation then we write xRy whenever $(x, y) \in R$. The predicate xRy is read as x is R-related to y.

Properties of binary relations (1)

A binary relation R on a set A is

- *reflexive* when xRx for all $x \in A$.

$$\forall x\, A(x) \implies xRx$$

- *symmetric* when xRy implies yRx for all $x, y \in A$;

$$\forall x, y\; xRy \implies yRx$$

Properties of binary relations (2)

A binary relation R on a set A is

- *antisymmetric* when xRy and yRx imply $x = y$ for all $x, y \in A$;

$$\forall x, y\; xRy \text{ and } yRx \implies y = x$$

- *transitive* when xRy and yRz imply xRz for all $x, y, z \in A$.

$$\forall x, y, z\; xRy \text{ and } yRz \implies xRz$$

Example

- *reflexive* xRx
- *symmetric* $xRy \implies yRx$
- *antisymmetric* $xRy, yRx \implies x = y$
- *transitive* $xRy, yRz \implies xRz$

Let $A = \{1, 2, 3\}$.

$R_1 = \{(1,1), (2,2), (3,3), (2,3), (3,2)\}$

$R_2 = \{(2,2), (2,3), (3,2), (3,3)\}$

$R_3 = \{(1,1), (2,2), (3,3), (1,3)\}$

$R_4 = \{(1,3), (3,2), (2,3)\}$

Digraf representation

In the directed graph representation, R is

- *reflexive* if there is always an arrow from every vertex to itself;
- *symmetric* if whenever there is an arrow from x to y there is also an arrow from y to x;
- *antisymmetric* if whenever there is an arrow from x to y and $x \neq y$, then there is no arrow from y to x;
- *transitive* if whenever there is an arrow from x to y and from y to z there is also an arrow from x to z.

Example

Which of the following define a relation that is reflexive, symmetric, antisymmetric or transitive?

- x divides y on the set $\mathbb{Z}^+$ of positive integers;
- $x \neq y$ on the set $\mathbb{Z}$ of integers;
- x *has the same age as* y on the set of people.

Transitive closure

Given a binary relation R on a set A, the *transitive closure* R^* of R is the (uniquely determined) relation on A with the following properties:

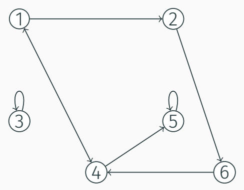
- R^* is transitive;
- $R \subseteq R^*$;
- If S is a transitive relation on A and $R \subseteq S$, then $R^* \subseteq S$.

Example

Let $A = \{1, 2, 3\}$. Find the transitive closure of

$$R = \{(1,1), (1,2), (1,3), (2,3), (3,1)\}.$$

Finding the transitive closure is easier with the digraph representation



Reachability relation

Transitivity and composition

A relation S is transitive if and only if $S \circ S \subseteq S$.

This is because

$$S \circ S = \{(a,c) \mid \text{ exists } b \text{ such that } aSb \text{ and } bSc\}.$$

Let S be a relation. Set $S^1 = S$, $S^2 = S \circ S$, $S^3 = S \circ S \circ S$, and so on.

Theorem Denote by S^* the transitive closure of S. Then xS^*y if and only if there exists $n > 0$ such that xS^ny.

Transitive closure in matrix form

The relation R on the set $A = \{1,2,3,4,5\}$ is represented by the matrix

$$\begin{bmatrix} T & F & F & T & F \\ F & T & F & F & T \\ F & F & T & F & F \\ T & F & T & F & F \\ F & T & F & T & F \end{bmatrix}$$

Determine the matrix $R \circ R$ and hence explain why R is not transitive.

Computation

$$\begin{bmatrix} T & F & F & T & F \\ F & T & F & F & T \\ F & F & T & F & F \\ T & F & T & F & F \\ F & T & F & T & F \end{bmatrix} \begin{bmatrix} T & F & F & T & F \\ F & T & F & F & T \\ F & F & T & F & F \\ T & F & T & F & F \\ F & T & F & T & F \end{bmatrix} = \begin{bmatrix} T & F & T & T & F \\ F & T & F & T & T \\ F & F & T & F & F \\ T & F & T & T & F \\ T & T & T & F & T \end{bmatrix}$$

$$R \circ R = \{(a,c) \mid \text{ exists } b \in A \text{ such that } aRb \text{ and } bRc\}.$$

Note (in red) that there are pairs (a,c) that are in $R \circ R$ but not in R. Hence, R is not transitive.

Detour: Warshall's algorithm

```
def warshall(a):
    assert (len(row) == len(a) for row in a)
    n = len(a)
    for k in range(n):
        for i in range(n):
            for j in range(n):
                a[i][j] = a[i][j] or
                    (a[i][k] and a[k][j])
    return a

print warshall([[1,0,0,1,0],
                [0,1,0,0,1],
                [0,0,1,0,0],
                [1,0,1,0,0],
                [0,1,0,1,0]])
```


Important relations: Equivalence relations

Definition A binary relation R on a set A is called an *equivalence relation* if it is reflexive, transitive, and symmetric.

Examples:

- the relation R on the non-zero integers given by xRy if $xy > 0$;
- the relation *has the same age* on the set of people.

Definition The *equivalence class* E_x of any $x \in A$ is defined by

$$E_x = \{y \mid yRx\}.$$

Example

Define a relation R on the set $\mathbb{R}$ of real numbers by setting xRy if and only if $x - y$ is an integer. Prove that R is an equivalence relation. Moreover,

- $E_0 = \mathbb{Z}$ is the equivalence class of 0;
- $E_{\frac{1}{2}} = \{\ldots, -2\frac{1}{2} - 1\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, \ldots\}$ is the equivalence class of $\frac{1}{2}$.

Functions and equivalence relations

Let $f : A \to B$ be a function. Define a relation R on A by

$$a_1Ra_2 \Leftrightarrow f(a_1) = f(a_2).$$

Then R is an equivalence relation on A. The equivalence class E_a of $a \in A$ is given by

$$E_a = \{a' \in A \mid f(a') = f(a)\}.$$

Example: A is a set of cars, B is the set of real numbers, and f assigns to any car in A its length. Then a_1Ra_2 if and only if a_1 and a_2 are of the same length.

Partition of a set

A *partition* of a set A is a collection of non-empty subsets $A_1, \ldots, A_n$ of A satisfying:

- $A = A_1 \cup A_2 \cup \cdots \cup A_n$;
- $A_i \cap A_j = \emptyset$ for $i \neq j$.

The A_i are called the blocks of the partition.

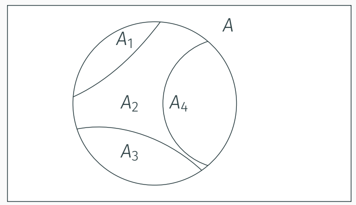


Figure 3: Partition of A

Connecting partitions and equivalence relations

Theorem Let R be an equivalence relation on a non-empty set A. Then the equivalence classes $\{E_x \mid x \in A\}$ form a partition of A.

Proof (Optional)

The proof is in four parts:

(1) We show that the equivalence classes $E_x = \{y \mid yRx\}$, $x \in A$, are non-empty subsets of A: by definition, each E_x is a subset of A. Since R is reflexive, xRx. Therefore $x \in E_x$ and so E_x is non-empty.

(2) We show that A is the union of the equivalence classes $E_x, x \in A$: We know that $E_x \subseteq A$, for all E_x, $x \in A$. Therefore the union of the equivalence classes is a subset of A. Conversely, suppose $x \in A$. Then $x \in E_x$. So, A is a subset of the union of the equivalence classes.

(Optional) Proof (continued)

The purpose of the last two parts is to show that distinct equivalence classes are disjoint, satisfying (ii) in the definition of partition.

(3) We show that if xRy then $E_x = E_y$: Suppose that xRy and let $z \in E_x$. Then, zRx and xRy. Since R is a transitive relation, zRy. Therefore, $z \in E_y$. We have shown that $E_x \subseteq E_y$. An analogous argument shows that $E_y \subseteq E_x$. So, $E_x = E_y$.

(4) We show that any two distinct equivalence classes are disjoint: To this end we show that if two equivalence classes are not disjoint then they are identical. Suppose $E_x \cap E_y \neq \emptyset$. Take a $z \in E_x \cap E_y$. Then, zRx and zRy. Since R is symmetric, xRz and zRy. But then, by transitivity of R, xRy. Therefore, by (3), $E_x = E_y$.

Connecting partitions and equivalence relations

Theorem Suppose that $A_1, \ldots, A_n$ is a partition of A. Define a relation R on A by setting: xRy if and only if there exists i such that $1 \leq i \leq n$ and $x, y \in A_i$. Then R is an equivalence relation.

Proof (Optional)

- Reflexivity: if $x \in A$, then $x \in A_i$ for some i. Therefore xRx.
- Transitivity: if xRy and yRz, then there exists A_i and A_j such that $x, y \in A_i$ and $y, z \in A_j$. $y \in A_i \cap A_j$ implies $i = j$. Therefore $x, z \in A_i$ which implies xRz.
- Symmetry: if xRy, then there exists A_i such that $x, y \in A_i$. Therefore yRx.

Important relations: Partial orders

Definition A binary relation R on a set A which is reflexive, transitive and antisymmetric is called a partial order.

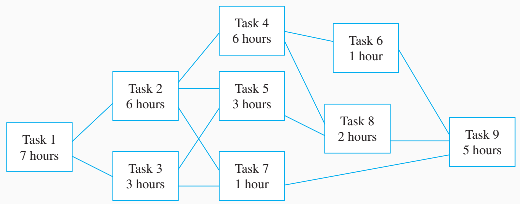
Partial orders are important in situations where we wish to characterise precedence.

Examples:

- the relation $\leq$ on the the set $\mathbb{R}$ of real numbers;
- the relation $\subseteq$ on $Pow(A)$;
- "*is a divisor of*" on the set $\mathbb{Z}^+$ of positive integers.

Example: Job scheduling

Task	Immediately Preceding Tasks
1	
2	1
3	1
4	2
5	2, 3
6	4
7	2, 3
8	4, 5
9	6, 7, 8



Predecessors in partial orders

If R is a partial order on a set A and xRy, $x \neq y$ we call x a predecessor of y.

If x is a predecessor of y and there is no $z \notin \{x, y\}$ for which xRz and zRy, we call x an immediate predecessor of y.

Important relations: Total orders

Definition A binary relation R on a set A is a total order if it is a partial order such that for any $x, y \in A$, xRy or yRx.

The Hasse diagram of a total order is a chain.

Examples

- the relation $\leq$ on the set $\mathbb{R}$ of real numbers;
- the usual lexicographical ordering on the words in a dictionary;
- the relation "is a divisor of" is *not* a total order.

n-ary relations

The Cartesian product $A_1 \times A_2 \times \cdots \times A_n$ of sets $A_1, A_2, \ldots, A_n$ is defined by

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, \ldots, a_n) \mid a_1 \in A_1, \ldots, a_n \in A_n\}.$$

Here $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$ if and only if $a_i = b_i$ for all $1 \leq i \leq n$.

An n-ary relation is a subset of $A_1 \times \ldots\ldots A_n$

Databases and relations

A database table $\approx$ relation

TABLE 1 Students.

Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

Students $= \{$

Unary relations

Unary relations are just subsets of a set.

Example: The unary relation `EvenPositiveIntegers` on the set $\mathbb{Z}^+$ of positive integers is

$$\{x \in \mathbb{Z}^+ \mid x \text{ is even}\}.$$