Foundations of Computer Science Comp109

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Part 5. Propositional Logic, digital circuits & computer arithmetic

Comp109 Foundations of Computer Science

Reading

- Discrete Mathematics and Its Applications, K.H. Rosen, Sections 1.1–1.3.
- Discrete Mathematics with Applications, S. Epp, Chapter 2.

Contents

- The language of propositional logic
- Semantics: interpretations and truth tables
- Semantic consequence
- Logical equivalence
- Logic and digital circuits
- Computer representation of numbers & computer arithmetic

Logic

Logic is concerned with

- the truth and falsity of statements;
- the question: when does a statement *follow from* a set of statements?

Propositional logic

Propositions

A proposition is a statement that can be true or false.
(but not both in the same time!)

- Logic is easy;
- I eat toast;
- $2 + 3 = 5$;
- $2 \cdot 2 = 5$.
- $4 + 5$;
- What is the capital of UK?

- Logic is not easy;
- Logic is easy or I eat toast;

Compound propositions

- More complex propositions formed using logical connectives (also called Boolean connectives)
- Basic logical connectives:
  1. $\neg$: negation (read "not")
  2. $\wedge$: conjunction (read "and"),
  3. $\vee$: disjunction (read "or")
  4. $\Rightarrow$: implication (read "if...then")
  5. $\Leftrightarrow$: equivalence (read "if, and only if,")
- Propositions formed using these logical connectives called compound propositions; otherwise atomic propositions
- A propositional formula is either an atomic or compound proposition

Giving meaning to propositions: Truth values

An *interpretation* I is a function which assigns to any atomic proposition p_i a *truth value*

$$I(p_i) \in \{0, 1\}.$$

- If $I(p_i) = 1$, then p_i is called *true* under the interpretation I.
- If $I(p_i) = 0$, then p_i is called *false* under the interpretation I.

Given an assignment I we can compute the truth value of compound formulas step by step using so-called truth tables.

Negation

The negation $\neg P$ of a formula P

It is not the case that P

Truth table:

P	$\neg P$
1	
0	

Conjunction

The conjunction $(P \wedge Q)$ of P and Q.

both P and Q are true

Truth table:

P	Q	$(P \wedge Q)$
1	1	
1	0	
0	1	
0	0	

Disjunction

The disjunction $(P \vee Q)$ of P and Q

at least one of P and Q is true

Truth table:

P	Q	$(P \vee Q)$
1	1	
1	0	
0	1	
0	0	

Equivalence

The equivalence $(P \Leftrightarrow Q)$ of P and Q

P and Q take the same truth value

Truth table:

P	Q	$(P \Leftrightarrow Q)$
1	1	
1	0	
0	1	
0	0	

Implication

The implication $(P \Leftrightarrow Q)$ of P and Q

if P then Q

Truth table:

P	Q	$(P \Rightarrow Q)$
1	1	
1	0	
0	1	
0	0	

Truth under an interpretation

So, given an interpretation I, we can compute the truth value of any formula P under I.

- If $I(P) = 1$, then P is called true under the interpretation I.
- If $I(P) = 0$, then P is called false under the interpretation I.

Example

List the Interpretations I such that $P = ((p_1 \vee \neg p_2) \wedge p_3)$ is true under I.

p_1	p_2	p_3	$\neg p_2$	$(p_1 \vee \neg p_2)$	$P = ((p_1 \vee \neg p_2) \wedge p_3)$

For values see http://www.csc.liv.ac.uk/~konev/COMP109/lecturelog.html

Logical puzzles

- An island has two kinds of inhabitants, knights, who always tell the truth, and knaves, who always lie.
- You go to the island and meet A and B.
  - A says "B is a knight."
  - B says "The two of us are of opposite types."
- What are A and B?

p: "A is a knight"; and q: "B is a knight"

- Options for A.
  - p is true $\quad p \Rightarrow q$
  - p is false $\quad \neg p \Rightarrow \neg q$
- Options for B.
  - q is true $\quad q \Rightarrow \neg p$
  - q is false $\quad \neg q \Rightarrow \neg p$

Truth table

p	q	$\neg p$	$\neg q$	$p \Rightarrow q$	$\neg p \Rightarrow \neg q$	$q \Rightarrow \neg p$	$\neg q \Rightarrow \neg p$
0	0						
0	1						
1	0						
1	1						

Semantic consequence

Definition Suppose Γ is a finite set of formulas and P is a formula. Then P follows from Γ ("is a semantic consequence of Γ") if the following implication holds for every interpretation I:

$$\text{If } I(Q) = 1 \text{ for all } Q \in \Gamma, \text{ then } I(P) = 1.$$

This is denoted by

$$\Gamma \models P.$$

Example

Show $\{(p_1 \wedge p_2)\} \models (p_1 \vee p_2)$.

p_1	p_2	$(p_0 \wedge p_1)$	$(p_0 \vee p_1)$
1	1		
1	0		
0	1		
0	0		

Example

Show $\{p_1\} \not\models p_2$.

p_1	p_2
1	1
1	0
0	1
0	0

Example

Show $\{p_1\} \models (p_1 \vee p_2)$.

p_1	p_2	$(p_1 \vee p_2)$
1	1	
1	0	
0	1	
0	0	

Logic and proof principles I

- *Modus Ponens*
  Direct proof corresponds to the following semantic consequence
  $$\{P, (P \Rightarrow Q)\} \models Q;$$
- *Reductio ad absurdum*
  Proof by contradiction corresponds to
  $$\{(\neg P \Rightarrow \perp)\} \models P,$$
  where $\perp$ is a special proposition, which is false under every interpretation.

Logic and proof principles II

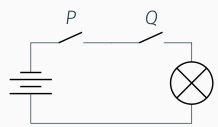
- *Modus Tollens*
  Another look at proof by contradiction
  $$\{(P \Rightarrow Q), \neg Q\} \models \neg P$$
- Case analysis
  $$\{(P \Rightarrow Q), (R \Rightarrow Q), (P \vee R)\} \models Q$$

Proof theory

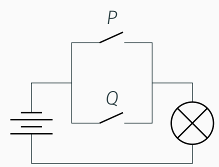
- We have studied proofs as carefully reasoned arguments to convince a sceptical listener that a given statement is true.
  - "Social" proofs
- Proof theory is a branch of mathematical logic dealing with proofs as mathematical objects
  - Strings of symbols
  - Rules for manipulation
  - Mathematics becomes a 'game' played with strings of symbols
  - Can be read and interpreted by computer

Application: Digital logic circuits

Logic and electric circuits



"in series"



"in parallel"

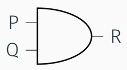
P	Q	light
closed	*closed*	on
closed	*open*	off
open	*closed*	off
open	*open*	off

Q	Q	light
closed	*closed*	on
closed	*open*	on
open	*closed*	on
open	*open*	off

Modern computers use logic gates

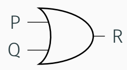
Basic logic gates

AND gate



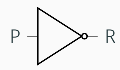
P	Q	R
1	1	1
1	0	0
0	1	0
0	0	0

OR gate



P	Q	R
1	1	1
1	0	1
0	1	1
0	0	0

NOT gate



P	R
1	0
0	1

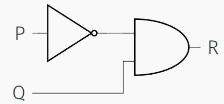
Rules for a combinatorial circuit

- Never combine two input wires.
- A single input wire can be split partway and used as input for two separate gates.
- An output wire can be used as input.
- *No output of a gate can eventually feed back into that gate.*

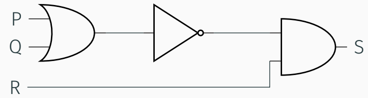
Determining output for a given circuit

Input signals: $P = 0$ and $Q = 1$

Boolean expression



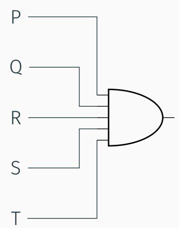
Input signals: $P = 1$, $Q = 1$ and $R = 1$

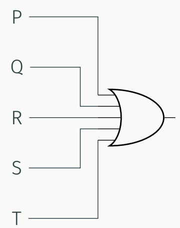


Constructing circuits for Boolean expressions

- $(\neg P \wedge Q) \vee \neg Q$
- $((P \wedge Q) \wedge (R \wedge S)) \wedge T$

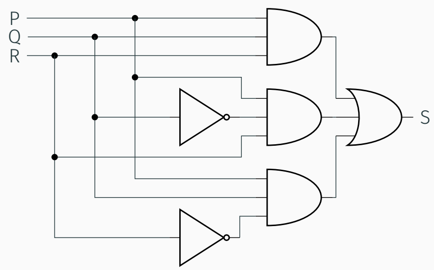
Multi-input AND and OR gates





Designing a circuit for a given input/output table

Input			Output
P	Q	R	S
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0



$(P \wedge Q \wedge R) \vee (P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R)$

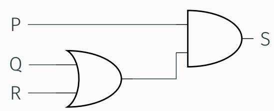
disjunctive normal form (DNF)

Another example

Input			Output
P	Q	R	S
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	1
0	0	0	0

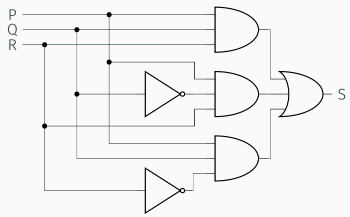
Reopen the first case

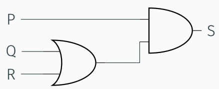
Input			Output
P	Q	R	S
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0



$P \wedge (Q \vee R)$

Circuit equivalence





- Two digital circuits are equivalent if they produce the same output given the same inputs.

Logical equivalence

Definition Two formulas P and Q are called equivalent if they have the same truth value under every possible interpretation. In other words, P and Q are equivalent if $I(P) = I(Q)$ for every interpretation I. This is denoted by

$$P \equiv Q.$$

$$(P \land Q \land R) \lor (R \land \neg Q \land R) \lor (P \land Q \land \neg R) \equiv P \land (Q \lor R)$$

On logical equivalence

Theorem The relation $\equiv$ is an equivalence relation on $\mathcal{P}$.

Proof

- $\equiv$ is reflexive, since, trivially, $I(P) = I(P)$ for every interpretation I.
- $\equiv$ is transitive, since $P \equiv Q$ and $Q \equiv R$ implies $P \equiv R$.
- $\equiv$ is symmetric, since $P \equiv Q$ implies $Q \equiv P$.

Simplifying propositional formulae

Exercises:

- $(P \Rightarrow Q) \equiv (\neg P \lor Q)$
- $\neg(P \Rightarrow Q) \equiv (P \land \neg Q)$
- $(P \Leftrightarrow Q) \equiv ((P \Rightarrow Q) \land (Q \Rightarrow P))$
- $(P \Leftrightarrow Q) \equiv (\neg P \Leftrightarrow \neg Q)$
- $(P \land (P \lor Q)) \equiv P$
- $\neg(P \lor (\neg P \land Q)) \equiv (\neg P \land \neg Q)$

Useful equivalences

The following equivalences can be checked by truth tables:

- Associative laws:

$$(P \lor (Q \lor R)) \equiv ((P \lor Q) \lor R),$$

$$(P \land (Q \land R)) \equiv ((P \land Q) \land R);$$

- Commutative laws:

$$(P \lor Q) \equiv (Q \lor P), \quad (P \land Q) \equiv (Q \land P);$$

- Identity laws:

$$(P \lor \bot) \equiv P, \ (P \lor \top) \equiv \top, \ (P \land \top) \equiv P, \ (P \land \bot) \equiv \bot;$$

- Distributive laws:

$$(P \land (Q \lor R)) \equiv ((P \land Q) \lor (P \land R))$$

$$(P \lor (Q \land R)) \equiv ((P \lor Q) \land (P \lor R));$$

- Complement laws:

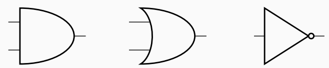
$$P \lor \neg P \equiv \top, \ \neg\top \equiv \bot, \ \neg\neg P \equiv P, P \land \neg P \equiv \bot, \ \neg\bot \equiv \top;$$

- De Morgan's laws:

$$\neg(P \lor Q) \equiv (\neg P \land \neg Q), \ \neg(P \land Q) \equiv (\neg P \lor \neg Q).$$

Boolean functions

- A function $F : \{0,1\}^k \to \{0,1\}$, where $k \in \mathbb{Z}^+$ is arity of F, is called a Boolean function
- Any Boolean function can be expressed as a combination of $\land$, $\lor$, $\neg$



- Do we need all three types of gates?

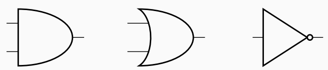
Boolean functions of arity 2

P	Q								
1	1	0	1	0	1	0	1	0	1
1	0	0	0	1	1	0	0	1	1
0	1	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0

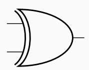
P	Q								
1	1	0	1	0	1	0	1	0	1
1	0	0	0	1	1	0	0	1	1
0	1	0	0	0	0	1	1	1	1
0	0	1	1	1	1	1	1	1	1

Logic gates

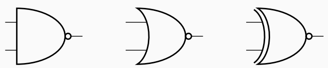
- AND, OR, NOT



- XOR



- NAND, NOR, XNOR



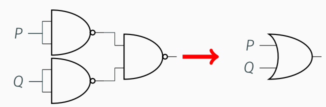
Universality of NAND and NOR

- NAND (AKA Sheffer stroke) $P \mid Q = \neg(P \land Q)$
- and NOR (AKA Pierce arrow) $P \downarrow Q = \neg(P \lor Q)$

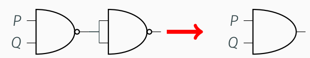
are universal:

$$\neg P \equiv P \mid P$$

$$P \lor Q \equiv (P \mid P) \mid (Q \mid Q)$$



$$P \land Q \equiv (P \mid Q) \mid (P \mid Q)$$



Application: Number systems and circuits for addition

Binary number system

Positional system: multiply each digit by its place value

- Decimal notation:

$$4268_{10} = 4 \cdot 10^3 + 2 \cdot 10^2 + 6 \cdot 10^1 + 8 \cdot 10^0$$

- Binary notation

$$\begin{aligned} 1100\,0111_2 &= 1 \cdot 2^7 + 1 \cdot 2^6 + 0 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 \\ &= 128 + 64 + 0 + 0 + 0 + 4 + 2 + 1 = 199_{10} \end{aligned}$$

Here indices 10 and 2 are used to highlight the base of the number system

Convert decimal numbers to binaries: divide by 2

Rule: divide repeatedly by 2, writing down the reminder from each stage from right to left.

Example:

	533/2 = 266	remainder = 1
	266/2 = 133	remainder = 0
	133/2 = 66	remainder = 1
	66/2 = 33	remainder = 0
	33/2 = 16	remainder = 1
	16/2 = 8	remainder = 0
	8/2 = 4	remainder = 0
	4/2 = 2	remainder = 0
	2/2 = 1	remainder = 0
	1/2 = 0	remainder = 1

$$533_{10} = 1000010101_2$$

Alternative method

- If you know powers of 2, continually subtract largest power value from the number

$$\begin{aligned} 123_{10} &= 64 + (123 - 64) = 64 + 59 \\ &= 64 + 32 + (59 - 32) = 64 + 32 + 27 \\ &= 64 + 32 + 16 + (27 - 16) = 64 + 32 + 16 + 11 \\ &= 64 + 32 + 16 + 8 + (11 - 8) = 64 + 32 + 16 + 8 + 3 = \\ &= 64 + 32 + 16 + 8 + 2 + (3 - 2) \\ &= 64 + 32 + 16 + 8 + 2 + 1 = \\ &= 1 \cdot 2^6 + 1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = \\ &= 1111011_2 \end{aligned}$$

Binary addition

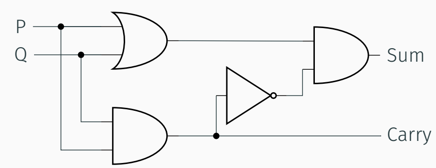
$$0_2 + 0_2 = 0_2 \qquad 0_2 + 1_2 = 1_2$$
$$1_2 + 0_2 = 1_2 \qquad 1_2 + 1_2 = 10_2$$

$$\begin{array}{rccccc} & & {}^{1}1 & {}^{1}1 & {}^{1}1 & 1 \\ + & & 1 & 0 & 1 & 1 \\ \hline & 1 & 1 & 0 & 1 & 0 \end{array}$$

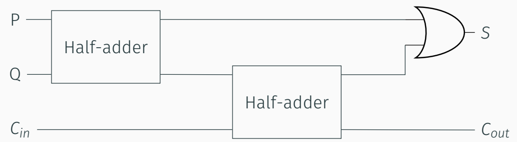
Half-adder

$$\begin{array}{rccccc} & & {}^{1}1 & {}^{1}1 & {}^{1}1 & 1 \\ + & & 1 & 0 & 1 & 1 \\ \hline & 1 & 1 & 0 & 1 & 0 \end{array}$$

P	Q	Carry	Sum
1	1	1	0
1	0	0	1
0	1	0	1
0	0	0	0

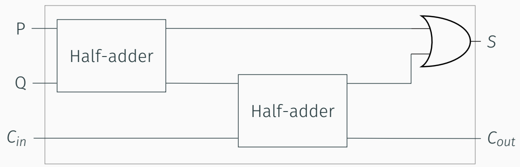


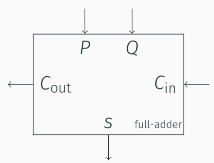
Full-adder



P	Q	C_{in}	C_{out}	S
1	1	1	1	1
1	1	0	1	0
1	0	1	1	0
1	0	0	0	1
0	1	1	1	0
0	1	0	0	1
0	0	1	0	1
0	0	0	0	0

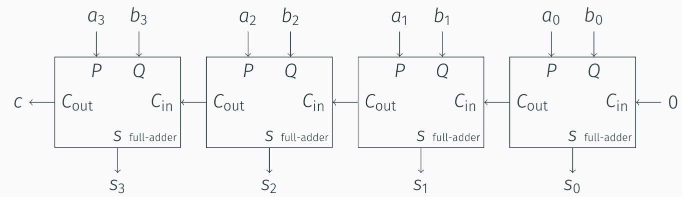
'Black box' notation





4-bit adder

$$\begin{array}{rccccc} & & a_3 & a_2 & a_1 & a_0 \\ + & & b_3 & b_2 & b_1 & b_0 \\ \hline & c & s_3 & s_2 & s_1 & s_0 \end{array}$$



Computer representation of negative integers

- Typically a fixed number of bits is used to represent integers: 8, 16, 32 or 64 bits
  - Unsigned integer can take all space available
- Signed integers
  - Leading sign

$$\begin{aligned} 0\,000\,0001_2 &= 1_{10} \\ 1\,000\,0001_2 &= -1_{10} \end{aligned}$$

    but then

$$1\,000\,0000_2 = -0_{10} \text{ (?!)}$$

  - Two's complement:
    given a positive integer a, the **two's complement of** a **relative to a fixed bit length** n is the binary representation of

$$2^n - a.$$

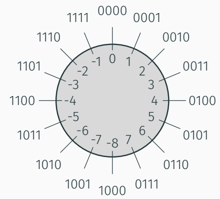
Example: 4-bit two's complement (n=4)

- $a = 1$, two's complement: $2^4 - 1 = 15 = 1111_2 = -1$
- $a = 2$, two's complement: $2^4 - 2 = 14 = 1110_2 = -2$
- $a = 3$, two's complement: $2^4 - 3 = 13 = 1101_2 = -3$
- ...
- $a = 8$, two's complement: $2^4 - 8 = 8 = 1000_2 = -8$

Properties

- Positive numbers start with 0, negative numbers start with 1
- 0 is always represented as a string of zeros
- -1 is always represented as a string of ones

Example: 4-bits



- The number range is split unevenly between +ve and -ve numbers
- The range of numbers we can represent in n bits is -2^{n-1} to $2^{n-1} - 1$

Addition

- Easy for computers
- Example: 2+3

$$\begin{array}{rcccc} & 0 & 0 & 1 & 0 \\ + & 0 & 0 & 1 & 1 \\ \hline & 0 & 1 & 0 & 1 \end{array}$$

- A carry that goes off the end can often be ignored
- Example: $-1 + -3$

$$\begin{array}{rcccc} & 1 & 1 & 1 & 1 \\ + & 1 & 1 & 0 & 1 \\ \hline {}_1 & 1 & 1 & 0 & 0 \end{array}$$

Subtraction

- Treat as an addition by negating second operand
- Example: $4 - 3 = 4 + (-3)$

$$\begin{array}{rcccc} & 0 & 1 & 0 & 0 \\ + & 1 & 1 & 0 & 1 \\ \hline {}_1 & 0 & 0 & 0 & 1 \end{array}$$

Overflow

- Example: $4 + 7$

$$\begin{array}{rcccc} & 0 & 1 & 0 & 0 \\ + & 0 & 1 & 0 & 1 \\ \hline & 1 & 0 & 0 & 1 \end{array}$$

- The correct result 9 is too big to fit into 4-bit representation
- Testing for overflow:
  If both inputs to an addition have the same sign, and the output sign is different, overflow has occurred
  - Overflow cannot occur if inputs have opposite sign.

Two's complement and bit negation

Example $n = 4$

- $2^4 - a = ((2^4 - 1) - a) + 1.$
- The binary representation of $(2^4 - 1)$ is 1111_2
- Subtracting a 4-bit number a from 1111_2 just switches all the 0's in a to 1's and all the 1's to 0's.
  For example,

$$\begin{array}{rcccc} & 1 & 1 & 1 & 1 \\ - & 1 & 0 & 0 & 1 \\ \hline & 0 & 1 & 1 & 0 \end{array}$$

- So, to compute the two's complement of a, flip the bits and add 1.

Example

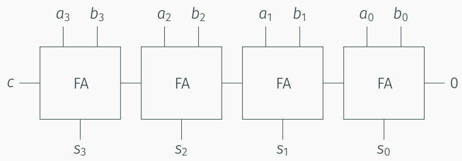
- Find the 8-bit two's complement of 19.
- Conversely, observe that

$$2^n - (2^n - a) = a$$

  so to find the decimal representation of the integer with a given two's complement
  - Find the two's complement of the given two's complement
  - Write the decimal equivalent of the result

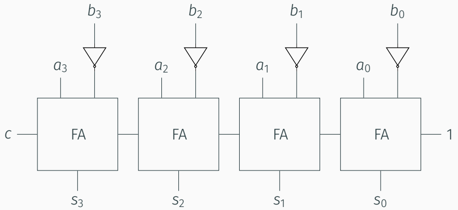
  Example: Which number is represented by 1010 1001?

Recall: 4-bit adder

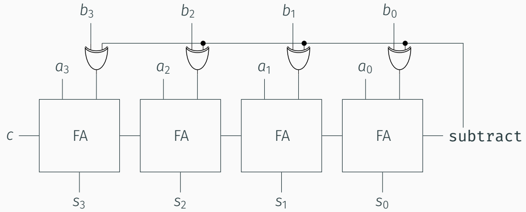


4-bit subtractor

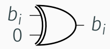
- Implementing $a + b$ as the sum of a and two's complement of b



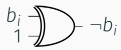
4-bit adder / subtractor



- When `subtract` is 0:



- When `subtract` is 1:



Integer types in high-level languages

E.g. Java has the following integer data types, using 2's complement:

byte	8-bit	-128 to $+127$
short	16-bit	$-32\,768$ to $+32\,767$
int	32-bit	$-2\,147\,483\,648$ to $+2\,147\,483\,647$
long	64-bit	-2^{63} to $+2^{63} - 1$

Floating point numbers

- It is not always possible to express numbers in integer form.
- Real, or floating point numbers are used in the computer when:
  - the number to be expressed is outside of the integer range of the computer, like
  $$3.6 \times 10^{40} \text{ or } 1.6 \times 10^{-19}$$
  - or, when the number contains a decimal fraction, like
  $$123.456$$

Scientific notation (AKA standard form)

The number is written in two parts:

- Just the digits (with the decimal point placed after the first digit), followed by
- $\times 10$ to a power that puts the decimal point where it should be (i.e. it shows how many places to move the decimal point).

$$123.456 = 1.23456 \times 10^2$$

In this example, 123.456 is written as 1.23456×10^2 because $123.456 = 1.23456 \times 100 = 1.23456 \times 10^2$

Binary fractions

Likewise, fractions can be represented base 2.

$$\begin{aligned} 10.01_2 &= 1 \times 2^1 + 0 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2} \\ &= 1 \times 2 + 0 + 0 + 1 \times 0.25 \\ &= 1.25_{10} \end{aligned}$$

Scientific representation: $10.01_2 = 1.001 \times 2^1$

Note: in binary, for any non-zero number the leading digit is always 1

Computer representation

To represent a number in scientific notation:

- The sign of the number.
- The magnitude of the number, known as the mantissa or significand
- The sign of the exponent
- The magnitude of the exponent

Example: eight characters

S EE MMMMM

- **S** is the sign of the number
- **EE** are two characters encoding the exponent
  - both sign and magnitude
- **MMMMM** are five characters for the mantissa

IEEE 754

- IEEE standard for floating-point arithmetic
- Implemented in many hardware units
- Stipulates computer representation of numbers
- For binary:
  - 16 bit half precision numbers: 5 for exponent, 11 for mantissa
  - 32 bit single precision numbers: 8 for exponent, 24 for mantissa
  - 64 bit double precision numbers: 11 for exponent 53 for mantissa
  - 128 bit quadruple precision numbers: 15 for exponent 113 for mantissa
  - 256 bit octuple precision numbers: 19 for exponent 237 for mantissa