Foundations of Computer Science Comp109

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Part 1. Number Systems and Proof Techniques

Comp109 Foundations of Computer Science

Reading

- S. Epp. *Discrete Mathematics with Applications*
  Chapter 4, Sections 5.2 and 5.3.
- E. Bloch. *Proofs and Fundamentals*
  Chapter 2, Section 6.3.
- K. Rosen. *Discrete Mathematics and Its Applications*
  Section 5.1.

Contents

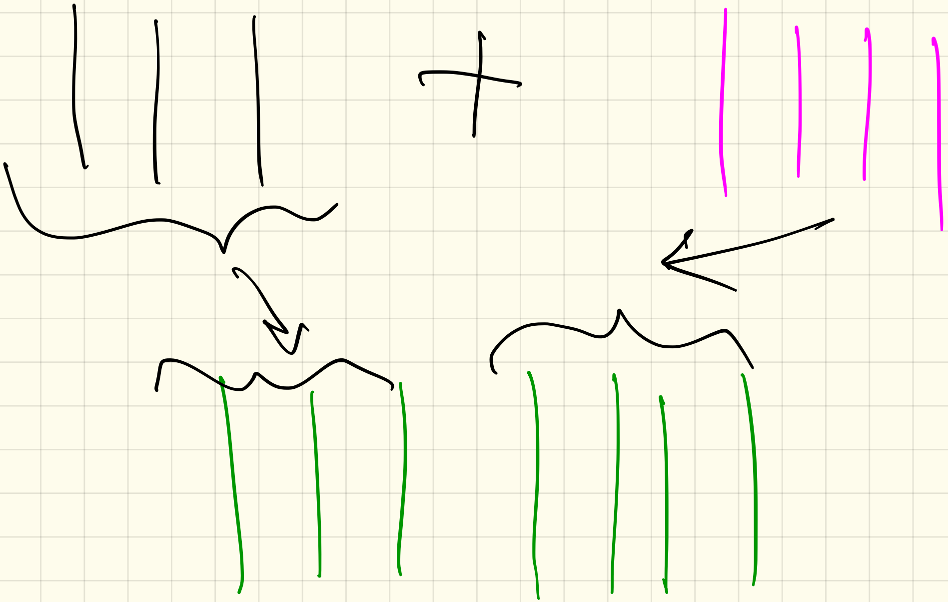
- The most basic datatypes
  - Natural Numbers
  - Integers
  - Rationals
  - Real Numbers
  - Prime Numbers
- Proof Techniques
  - Direct proof and disproof
    - Disproof by counterexample
    - Existence proof
    - Generalising from the generic particular
    - ...
  - Indirect Proof
    - Proof by contradiction
    - ...
  - Proof by mathematical induction

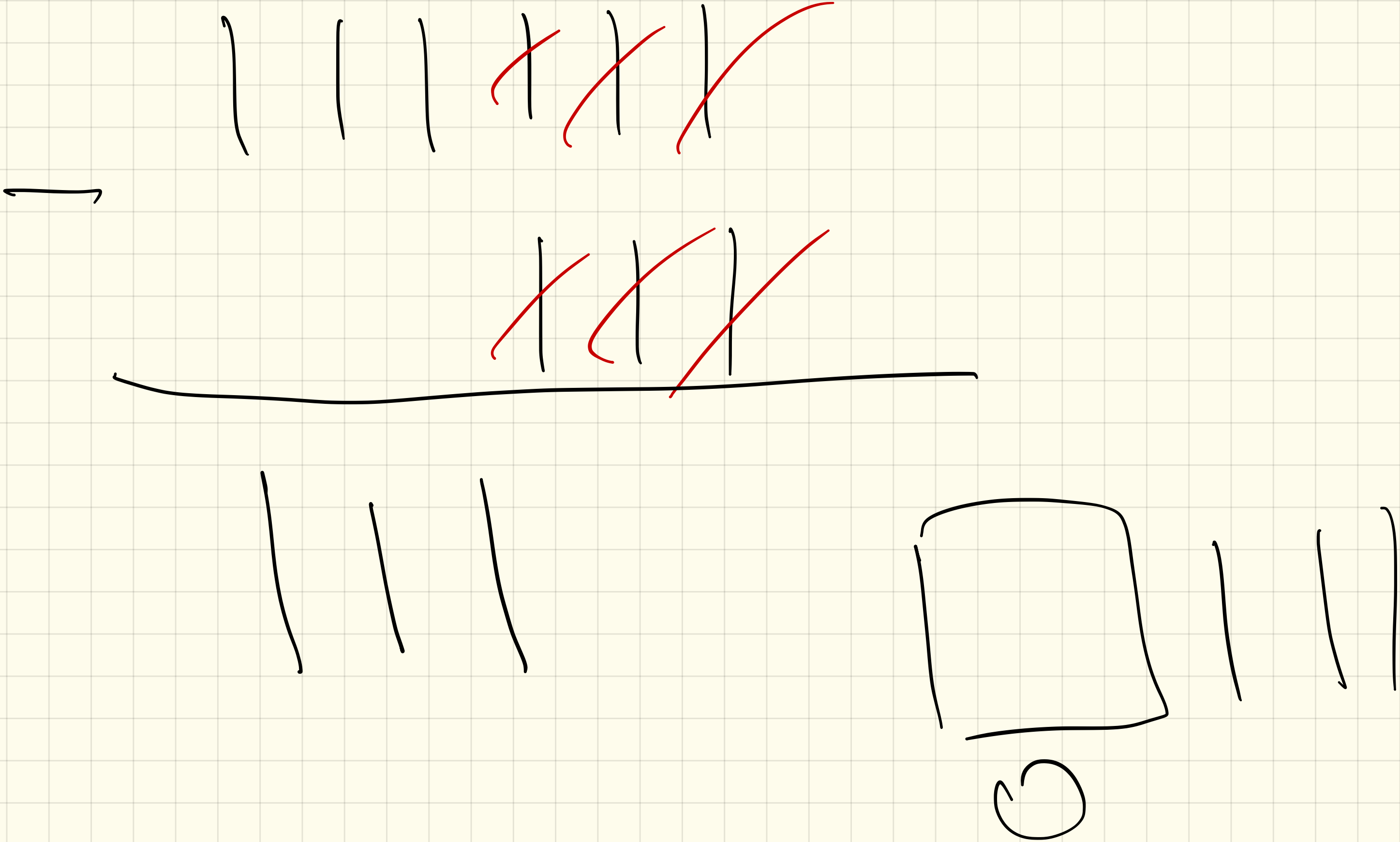
What is a number?

$\underbrace{||||}_{4}$

$\overline{\underline{\mathrm{IV}}}$

$\overline{\underline{\mathrm{VII}}}$





The natural numbers

$$0, 1, 2, 3, \ldots$$

Key property: Any natural number can be obtained from 0 by applying the operation $S(n) = n + 1$ some number times.

Examples: $S(0) = 1$.

$S(S(0)) = 2$.

$S(S(S(0))) = 3$.

Prime numbers

A prime number is a integer greater than 1 which has exactly two divisors that are positive integers: 1 and itself.

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, \ldots$$

Every natural number greater than 1 can be written as a unique product of prime numbers.

Examples: $6 = 2 \times 3$. $15 = 3 \times 5$. $1400 = 2^3 \times 5^2 \times 7$.

Example: prime and composite numbers

1. Is 1 prime?
2. Is every integer greater than 1 either prime or composite?
3. Write the first six prime numbers.
4. Write the first six composite numbers.

The Integers $\ldots, -2, -1, 0, 1, 2, \ldots$

The Rational Numbers all numbers that can be written as $\dfrac{m}{n}$ where m and n are integers and n is not 0.

$$\frac{1}{2} = \frac{2}{4} = \frac{-3}{-6}$$

Reminder: Algebraic manipulation

$$\frac{1}{2}+\frac{3}{4}=\frac{2}{4}+\frac{3}{4}=\frac{2+3}{4}=\frac{5}{4}=$$

$$=1\frac{1}{4}$$

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+cb}{b\cdot d}$$

$$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$$

$$\frac{a}{b}:\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{a\cdot d}{b\cdot c}$$

Solving and computing

Mathematics underpins STEM subjects. In many cases, we are concerned with solving and computing

Complete the table of values for $y = 3 - x^2$

x	-3	-2	-1	0	1	2	3
y		-1	2		2		-6

The quadratic equation $2x^2 + 6x + 7 = 0$ has roots α and β.

Write down the value of $\alpha + \beta$ and the value of $\alpha\beta$.

Work out $\frac{1}{3} \times \frac{1}{5}$

Find the general solution, in degrees, of the equation

$$2\sin(3x + 45°) = 1$$

5 miles = 8 kilometres

Which is longer, 26 miles or 45 km?

Statements

Which of the following are true?

- "26 miles is longer than 45 km."
- An integer doubled is larger than the integer.
- The sum of any two odd numbers is even.

The moral of the story

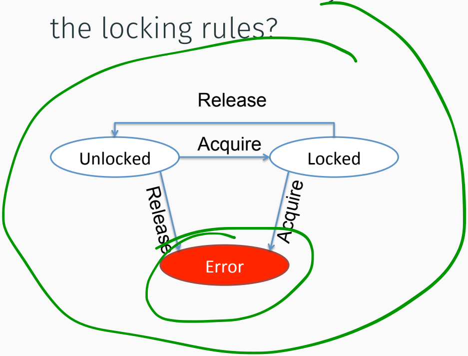
- We can't believe a statement just because it appears to be true.

We need a proof that the statement is true or a proof that it is false.

Do we care?

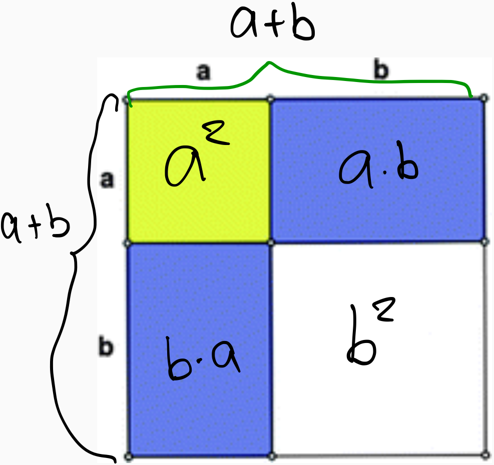
```
do {
    KeAcquireSpinLock();
    nPacketsOld = nPackets;
    if (request) {
        request = request->Next;
        KeReleaseSpinLock();
        nPackets++;
    }
} while (nPackets != nPacketsOld);
KeReleaseSpinLock();
```

Does this code obey the locking rules?

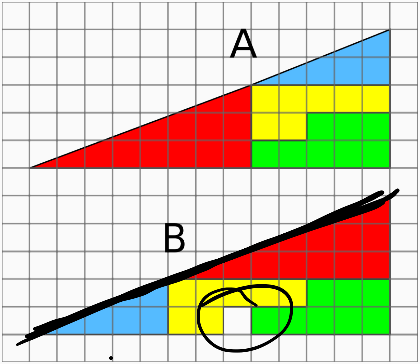


You don't need to understand the actual code!

[1] from Microsoft presentations on Static Driver Verifier (part of Visual Studio)



Visual proof of
$(a+b)^2 = a^2 + 2ab + b^2$



Visual "proof" of
32.5 = 31.5

$\frac{1}{2}\,13 \times 5$

Proofs

- A mathematical proof is as a **carefully reasoned argument** to convince a sceptical listener (often yourself) that a given statement is true.
- Both discovery and proof are integral parts of problem solving. When you think you have discovered that a certain statement is true, try to figure out why it is true.
- If you succeed, you will know that your discovery is genuine. Even if you fail, the process of trying will give you insight into the nature of the problem and may lead to the discovery that the statement is false.

Example: Odd and even numbers

Definition

An integer n is **even** if, and only if, n equals twice some integer.

An integer n is **odd** if, and only if, n equals twice some integer plus 1.

Symbolically, if n is an integer, then

n is even $\Leftrightarrow \exists$ an integer k such that $n = 2k$.

n is odd $\Leftrightarrow \exists$ an integer k such that $n = 2k + 1$.

Notice the use of $\quad \Leftrightarrow \quad \exists \quad \forall$.

Example: Properties of odd and even numbers

Use the definitions of even and odd to justify your answers to the following questions.

Definition

n is even $\Leftrightarrow \exists$ an integer k such that $n = 2k$.
n is odd $\Leftrightarrow \exists$ an integer k such that $n = 2k + 1$.

1. Is 0 even?

2. Is 301 odd?

Is it true that the the sum of two even numbers is even?

Example:

$4+6$:

$4 = 2 \cdot 2$

$6 = 2 \cdot 3$

$\underline{4+6} = 2 \cdot 2 + 2 \cdot 3 = 2(2+3) = \underline{\underline{2 \cdot 5}}$