Motivating example: “Mathematical trick”

Pick any number, add 5, multiply by 4, subtract 6, divide by 2, and subtract twice the original number. The answer is 7.

Step	Visual Result	Algebraic Result
Pick a number.	□ 5	x
Add 5.	□ | | | | |	$x+5$
Multiply by 4.	□ | | | | | □ | | | | | □ | | | | | □ | | | | |	$(x+5)\cdot 4 = 4x+20$
Subtract 6.	□ | | □ | | □ | | | | | □ | | | | |	$(4x+20)-6 = 4x+14$
Divide by 2.	□ | | □ | | | | |	$\frac{4x+14}{2} = 2x+7$
Subtract twice the original number.	| | | | | | |	$(2x+7)-2x = 7$

Generalising from the Generic Particular

The most powerful technique for proving a universal statement is one that works regardless of the choice of values for x.

To show that every x satisfies a certain property, suppose x is a particular but arbitrarily chosen and show that x satisfies the property.

Method of direct proof

- Express the statement to be proved in the form
$$\text{“}\forall x,\ \text{if } P(x) \text{ then } Q(x).\text{”}$$
(This step is often done mentally.)
- Start the proof by supposing x is a particular but arbitrarily chosen element for which the hypothesis $P(x)$ is true.
(This step is often abbreviated “Suppose P(x).”)
- Show that the conclusion $Q(x)$ is true by using definitions, previously established results, and the rules for logical inference.

If x in an odd integer then $x+1$ is even

Proof

$\forall x$ if x $\underbrace{\text{is an odd integer}}_{P(x)}$ then $\underbrace{x+1 \text{ is even}}_{Q(x)}$

Suppose that x is an odd integer

By definition of odd, there exists some k s.t.

$$x = 2k+1$$

$$x+1 = (2k+1)+1 = 2k+1+1 = 2k+2 = 2(k+1)$$

Thus, by definition of even, $x+1$ is even.

Q.E.D. $\square$

Prove that the sum of any two even integers is even

$\forall x, y$ if $\underbrace{x \text{ and } y \text{ are even integers}}_{P}$ then $\underbrace{x+y \text{ is even}}_{Q}$

Proof

Assume that x is even. Then $x = 2k$ for some k
and y is even. Then $y = 2\ell$ for some integer ℓ.

Then $x+y = 2k + 2\ell = 2(\underline{k+\ell})$

By definition of even, $x+y$ is even.

Prove that every integer is rational

$\forall x$ if x is integer then x is rational.

$\underbrace{\text{integer}}_{P}$ $\underbrace{\text{rational}}_{Q}$

Proof

Assume that x is an integer

Note $x = \frac{x}{1}$ ← fraction

By definition of a rational number x is rational.

Prove that the sum of any two rational numbers is rational

$\forall x, y$ if x and y are rational numbers

Then $x+y$ is rational.

Proof

Assume that x is rational

y is rational.

Then $x = \frac{m}{n}$ where m is an integer, n is an integer
$n \neq 0$

$y = \frac{k}{l}$ where k is an integer, l is an integer
$l \neq 0$.

Proof continued

$$x+y = \frac{m}{n} + \frac{k}{\ell} = \frac{\overbrace{\underbrace{m\cdot\ell}^{\text{integer}} + \underbrace{k\cdot n}^{\text{integer}}}^{\text{integer}}}{n\ell}$$

Since $n \neq 0$ and $\ell \neq 0$,

$n \cdot \ell \neq 0$

By definition of a rational number, $x+y$ is rational.

Prove that the product of any two rational numbers is rational

Homework

Prove that the double of a rational number is rational

Assume that x is rational.

$$2 = \frac{2}{1} \text{ is rational}$$

Thus $2x$ is the product of two rational numbers and so $2x$ is rational.