For any numbers m and n if $m \neq 0$ and $n \neq 0$ then $m \cdot n \neq 0$

Proof Assume $m \neq 0$ and $n \neq 0$. Suppose for a proof by contradiction that $m \cdot n = 0$

As $m \neq 0$ $\frac{1}{m}$ is defined.

$$m \cdot n = 0$$

$$\frac{1}{m} \cdot m \cdot n = \frac{1}{m} \cdot 0$$

$$n = 0$$

Thus, we have derived a contradiction.

So for any $m \neq 0$ $n \neq 0$ we have $m \cdot n \neq 0$,

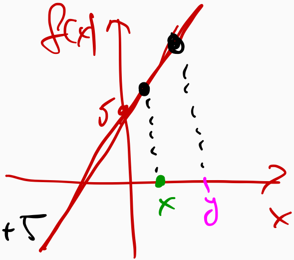
Direct proof

Assume $x \neq y$.

Therefore $2x \neq 2y$

Therefor $2x + 5 \neq 2y + 5$

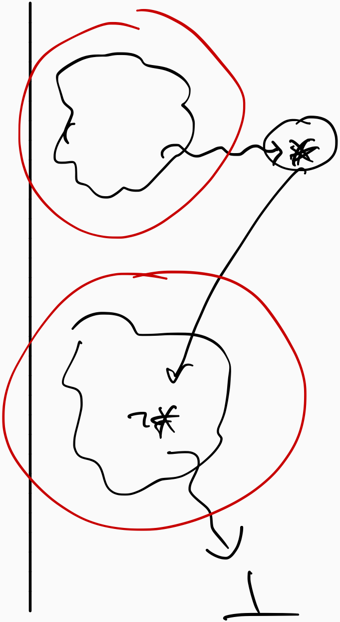
Thus $f(x) \neq f(y)$



Proof by contradiction

Assume $X \neq Y$. Assume for a proof by contradiction that $f(x) = f(y)$.

$2x + 5 = 2y + 5$. Then $2x = 2y$. Then $x = y$. A contradiction.



When to use indirect proof

- Many theorems can be proved either way. Usually, however, when both types of proof are possible, indirect proof is clumsier than direct proof.
- In the absence of obvious clues suggesting indirect argument, try first to prove a statement directly. Then, if that does not succeed, look for a counterexample.
- If the search for a counterexample is unsuccessful, look for a proof by contradiction

All (decimal) numbers — distances to points on a number line.

Examples.

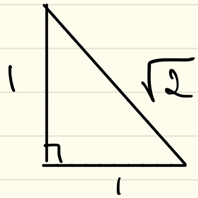
- -3.0 $\quad -\frac{3}{1}$
- 0 $\quad \frac{0}{1}$
- 1.6 $\quad \frac{16}{10} = \frac{8}{5}$
- $\pi = 3.14159\ldots$

A real number that is not rational is called irrational.

But are there any irrational numbers?

There exists irrational numbers

Proof We prove that $\sqrt{2}$ is irrational



Assume for a proof by contradiction that $\sqrt{2}$ is rational.

Then there exist integers m, n s.t.

$$\sqrt{2} = \frac{m}{n} \qquad n \neq 0$$

Assume that m and n do not have common factors.

$n\sqrt{2} = m$ Then $(n\sqrt{2})^2 = m^2$

$2n^2 = m^2$. Thus m^2 is even

So m is even. So there exists an integer k s.t. $m = 2k$.

Exercise:

If m^2 is even then m is even.

$2n^2 = m^2$ and $m = 2k$

$2n^2 = (2k)^2$

$2n^2 = 4k^2$

So $n^2 = 2k^2$

Thus n^2 is even. So n is even.

Thus there exists an integer l s.t $n = 2l$

So, 2 is a common factor of m and n and we have derived a contradiction.

Thus $\sqrt{2}$ is irrational.

Proving that $\sqrt{2}$ is not a rational number

Proof by contradiction.

- If $\sqrt{2}$ were rational then we could write it as $\sqrt{2} = x/y$ where x and y are integers and y is not 0.
- By repeatedly cancelling common factors, we can make sure that x and y have no common factors so they are not both even.
- Then $2 = x^2/y^2$ so $x^2 = 2y^2$ so x^2 is even. This means x is even, because the square of any odd number is odd.

the proof continued

- Let $x = 2w$ for some integer w.
- Then $x^2 = 4w^2$ so $4w^2 = 2y^2$ so $y^2 = 2w^2$ so y^2 is even so y is even.
- This contradicts the fact that x and y are not both even, so our original assumption, that $\sqrt{2}$ is rational, must have been wrong.

Prove that $1 + 3\sqrt{2}$ is irrational

If r and s are rational numbers then $r + s$ and $r \cdot s$ are both rational numbers.

Proof Assume that $1 + 3\sqrt{2}$ is rational.

let $r = 1 + 3\sqrt{2}$. Then $r - 1$ is rational

let $s = r - 1$, Then $\frac{1}{3} \cdot s$ is rational.

$$\frac{1}{3} \cdot s = \frac{1}{3}(r-1) = \frac{1}{3}(1 + 3\sqrt{2} - 1) = \sqrt{2}$$

A contradiction