Mathematical induction

- Mathematical induction is one of the more *recently* developed techniques of proof in the history of mathematics.
- It is used to check conjectures about the outcomes of processes that occur repeatedly and according to definite patterns.
- In general, mathematical induction is a method for proving that a property defined for integers n is true for all values of n that are greater than or equal to some initial integer

Example: Domino effect

One domino for each natural number, arranged in order.

- I will push domino 0 (the one at the front of the picture) towards the others.
- For every natural number m, if the m'th domino falls, then the $(m+1)$st domino will fall.

Conclude: All of the Dominoes will fall.

Proving by induction that a property holds for every natural number n

- Prove that the property holds for the natural number $n = 0$.
- Prove that if the property holds for $n = m$ (for any natural number m) then it holds for $n = m + 1$.

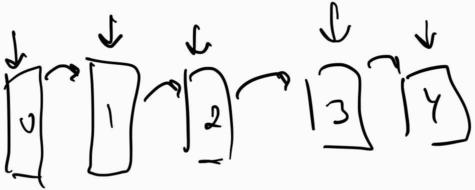
The validity of proof by mathematical induction is generally taken as an axiom. That is why it is referred to as the principle of mathematical induction rather than as a theorem.

A proof of a property by induction looks like this

Base Case: Show that the property holds for $n = 0$.

Inductive Step: Assume that the property holds for $n = m$. Show that it holds for $n = m + 1$.

Conclusion: You can now conclude that the property holds for every natural number n.



$$1 + 2 + 3 + 4 \dots + 49 + 50 +$$

$$100 + 99 + 98 + 97 + \dots 52 + 51$$

$$101 + \dots + 101 + 101 =$$

$$50 \times 101$$

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

For all natural numbers n, $0+1+2+\ldots+n=\frac{n(n+1)}{2}$

Proof

We prove this statement by mathematical induction.

Base case $n=0$.

$LHS=0$

$RHS=\frac{0\cdot(0+1)}{2}=0$

$LHS=RHS$ and we are done

Inductive step Assume that the statement holds true for $n=m$. We need to show that it holds for $n=m+1$.

So $0+1+2+\ldots+m = \frac{m\cdot(m+1)}{2}$

Goal: the prop. holds for $n=m+1$

That is $0+1+2+\ldots+(m+1) = \frac{(m+1)((m+1)+1)}{2}$

Consider the LHS

$$0+1+2+\ldots+m+m+1 = \frac{m(m+1)}{2}+m+1 =$$

$$\frac{m(m+1)+2m+2}{2} = \frac{m(m+1)+2(m+1)}{2} = \frac{(m+1)(m+2)}{2}$$

By the principle of mathematical induction,

for every natural n we have

$$0+1+2+\ldots+n = \frac{n(n+1)}{2}$$

Example: Proof by induction

For every natural number n,

$$0 + 1 + \cdots + n = \frac{n(n+1)}{2}.$$

Base Case: Take $n = 0$. The left-hand-side and the right-hand-side are both 0 so they are equal.

Inductive Step: Assume that the property holds for $n = m$, so

$$0 + 1 + \cdots + m = \frac{m(m+1)}{2}.$$

Now consider $n = m + 1$. We must show that

$$0 + 1 + \cdots + m + (m+1) = \frac{(m+1)(m+2)}{2}.$$

Proof continued

Since

$$0 + 1 + \cdots + m = \frac{m(m+1)}{2}.$$

$$\begin{aligned} 0 + 1 + \cdots + m + (m+1) &= \frac{m(m+1)}{2} + m + 1 \\ &= \frac{m(m+1) + 2(m+1)}{2} \\ &= \frac{(m+1)(m+2)}{2} \end{aligned}$$

$n = 2k+1$ for some k

k			
0	1	1	$= 1^2$
1	$1+3$	4	$= 2^2$
2	$1+3+5$	9	$= 3^2$
3	$1+3+5+7$	16	$= 4^2$
4	$1+3+5+7+9$	25	$= 5^2$

Hypothesis The sum of odd numbers

$$1+3+5+\ldots+(2k+1) = (k+1)^2$$

$\uparrow$

Proof We prove this statement by mathematical induction.

Base case $k=0$ $1 = (k+1)^2 = 1^2$

Inductive step: Suppose that the property holds for $k = m$.

That is $1+3+\dots+(2m+1) = (m+1)^2$.

Goal prove that the statement holds for $k=m+1$

$$1+3+\dots+(2(m+1)+1) = ((m+1)+1)^2$$

Consider the LHS of my goal

$$1+3+\dots+(2m+1)+(2(m+1)+1) =$$

$$(m+1)^2 + (2(m+1)+1) =$$

$$m^2+2m+1+2m+2+1$$

$$= m^2+4m+4 = (m+2)^2$$

Other starting values

Suppose you want to prove a statement not for all natural numbers, but for all integers greater than or equal to some particular natural number b

Base Case: Show that the property holds for $n = b$.

Inductive Step: Assume that the property holds for $n = m$ for any $m \geq b$. Show that it holds for $n = m + 1$.

Conclusion: You can now conclude that the property holds for every integer $n \geq b$.

For all integers $n \geq 8$, n¢ can be obtained using 3¢ and 5¢ coins.

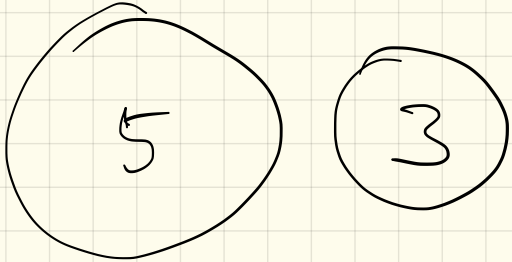
Base Case: For $n = 8$, 8¢ = 3¢ + 5¢.

Inductive Step: Suppose that m¢ can be obtained using 3¢ and 5¢ coins for any $m \geq 8$. We must show that $(m + 1)$¢ can be obtained using 3¢ and 5¢ coins.

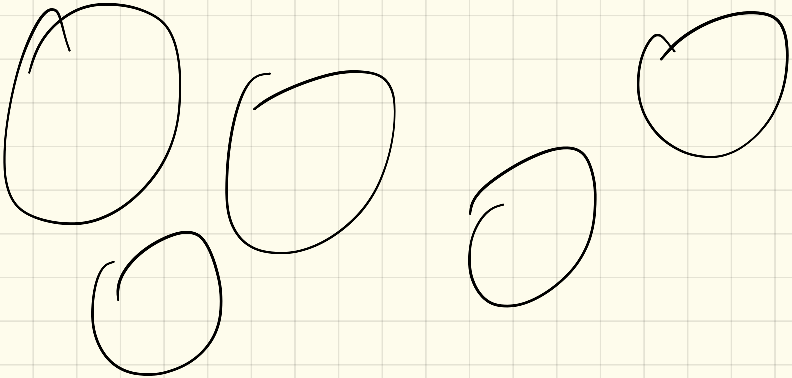
Consider cases

- There is a 5¢ coin among those used to make up the m¢.
  - Replace the 5¢ coin with two 3¢ coins. We obtain $(m + 1)$¢.
- There is no 5¢ coin among those used to make up the m¢.
  - There are three 3¢ coins ($m \geq 8$).
    - Replace the three 3¢ coins with two 5¢ coins

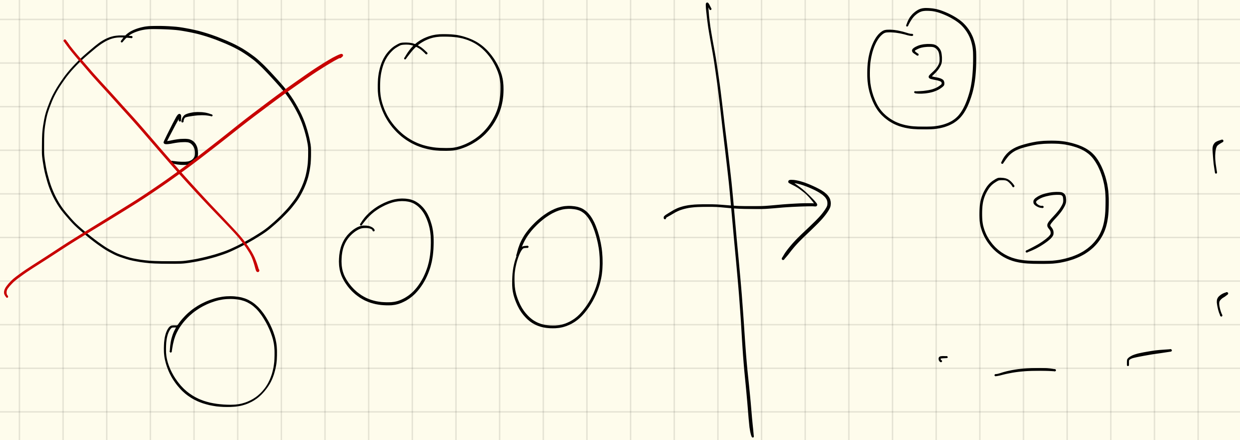
Base case for $n=8$



Inductive step Suppose you can give $n=m$ cents in 3 and 5 cent coins. Then you can give $n=m+1$ cents.



Case 1 There's one 5¢ coin



Case 2: No 5¢ coin on the table