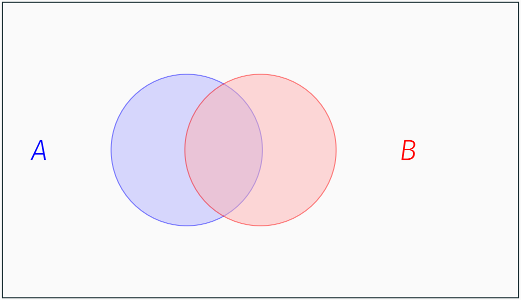
The algebra of sets

Suppose that A, B and U are sets with $A \subseteq U$ and $B \subseteq U$.

Commutative laws:

$$A \cup B = B \cup A, \quad A \cap B = B \cap A;$$



Proving the commutative law $A \cup B = B \cup A$

Definition: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ $B \cup A = \{x \mid x \in B \text{ or } x \in A\}$.

These are the same set. To see this, check all possible cases.

Case 1: Suppose $x \in A$ and $x \in B$. Since $x \in A$, the definitions above show that x is in both $A \cup B$ and $B \cup A$.

Case 2: Suppose $x \in A$ and $x \notin B$. Since $x \in A$, the definitions above show that x is in both $A \cup B$ and $B \cup A$.

Case 3: Suppose $x \notin A$ and $x \in B$. Since $x \in B$, the definitions above show that x is in both $A \cup B$ and $B \cup A$.

Case 4: Suppose $x \notin A$ and $x \notin B$. The definitions above show that x is not in $A \cup B$ and x is not in $B \cup A$.

So, for all possible x, either x is in both $A \cup B$ and $B \cup A$, or it is in neither. We conclude that the sets $A \cup B$ and $B \cup A$ are the same.

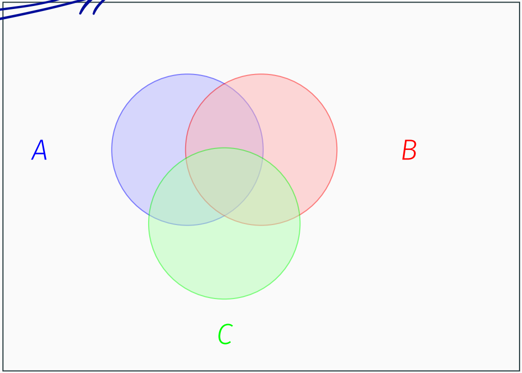
The algebra of sets

Suppose that A, B, C, U are sets with $A \subseteq U$, $B \subseteq U$, and $C \subseteq U$.

Associative laws:

$$A \cup (B \cup C) = (A \cup B) \cup C, \quad A \cap (B \cap C) = (A \cap B) \cap C;$$

x



Proving the associative law $A \cup (B \cup C) = (A \cup B) \cup C$

This is almost as easy as proving the commutative law, but now there are 8 cases to check, depending on whether $x \in A$, whether $x \in B$ and whether $x \in C$.

Definition: $X \cup Y = \{x \mid x \in X \text{ or } x \in Y\}$

Here is one case: Suppose $\underline{x \in A}$, $\underline{x \notin B}$ and $\underline{x \notin C}$. Since $x \in A$, we can use the definition with $X = A$ and $Y = B \cup C$ to show that $x \in A \cup (B \cup C)$.

Since $x \in A$, we can use the definition with $X = A$ and $Y = B$ to show that $x \in A \cup B$. Then we can use the definition with $X = A \cup B$ and $Y = C$ to show that $x \in (A \cup B) \cup C$.

Writing out all eight cases is tedious, but it is not difficult.

$U = \{x \in \mathbb{N} \mid x \leq 15\} = \{0, 1, 2, \ldots, 15\}$

$A = \{1, 2, 5, 8, 11\}$

$B = \{1, 3, 4, 11, 12\}$

$A \cup B = \{1, 2, 3, 4, 5, 8, 11, 12\}$

$A \cap B = \{1, 11\}$

* $A - B = \{2, 5, 8\}$

$B - A = \{3, 4, 12\}$

$A \triangle B =$

$(A - B) \cup (B - A)$

$= \{2, 5, 8, 3, 4, 12\}$

$\sim A =$

$\{0, 3, 4, 6, 7, 9, 10, 12, 13, 14, 15\}$

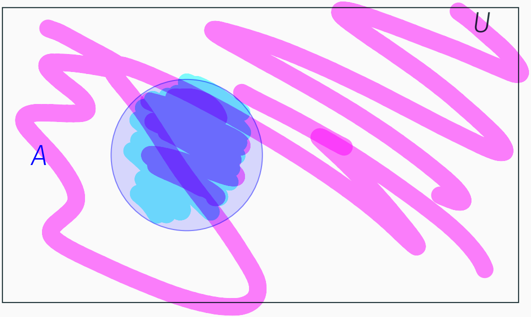
Suppose that A and U are sets with $A \subseteq U$.

Identity laws:

(similar : $a+0=a$)

$$A \cup \emptyset = A, \quad A \cup U = U, \quad A \cap U = A, \quad A \cap \emptyset = \emptyset;$$

(sim.: $a+(+\infty) = +\infty$)

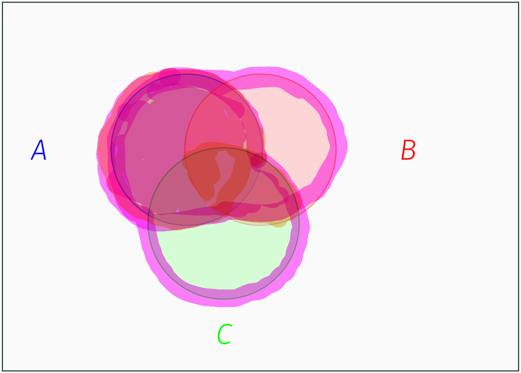


Suppose that A, B, C, U are sets with $A \subseteq U$, $B \subseteq U$, and $C \subseteq U$.

Distributive laws:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C);$$

$a \cdot (b + c) =$
$a \cdot b + a \cdot c$



The algebra of sets

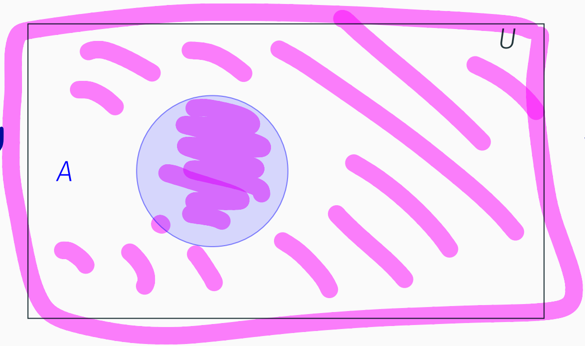
Suppose that A and U are sets with $A \subseteq U$. Let $\sim A = U - A$. Then

Complement laws:

$$A \cup \sim A = U, \; \sim U = \emptyset, \; \sim(\sim A) = A, \; A \cap \sim A = \emptyset, \; \sim \emptyset = U;$$

$\sim U = U - U$

$= \emptyset$



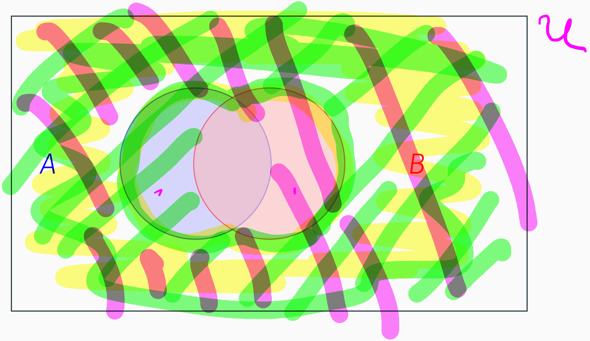
$\sim(\sim A) =$

A

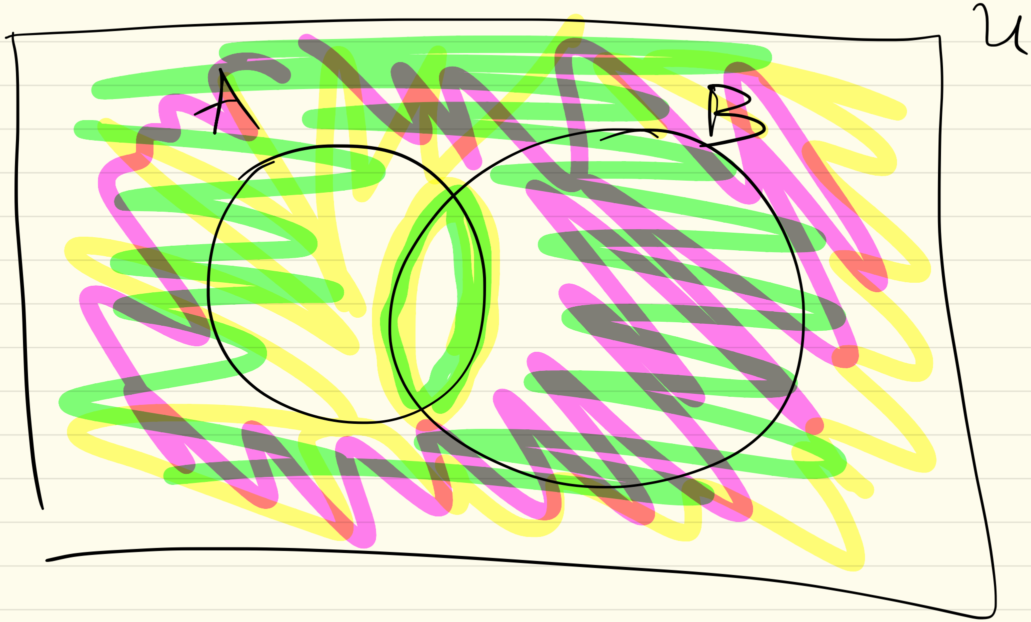
Suppose that A, B and U are sets with $A \subseteq U$, and $B \subseteq U$. Recall that $\sim X = U - X$ and $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ and $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$. Then

De Morgan's laws:

$$\sim (A \cup B) = \sim A \cap \sim B, \quad \sim (A \cap B) = \sim A \cup \sim B.$$



$$\sim(A \cap B) = \sim A \cup \sim B$$



$\sim(A \cap B) = \sim A \cup \sim B$ (for any sets A, B)

Proof

We will show that there can be no element x that belongs in $\sim(A \cap B)$ but not in $\sim A \cup \sim B$ (and vice versa).

Case 1) $x \in A$, $x \in B$: $x \in A \cap B$. Therefore, $x \notin \sim(A \cap B)$

$x \notin \sim A$ $\quad x \notin \sim B$. Therefore, $x \notin \sim A \cup \sim B$.

Case 2) $x \in A$, $x \notin B$.

Case 3) $x \notin A$, $x \in B$. Because $x \notin A$, it is $x \notin A \cap B$.
So, it is $x \in \sim(A \cap B)$.
Also, $x \notin A \implies x \in \sim A \implies x \in \sim A \cup \sim B$

Case 4) $x \notin A$, $x \notin B$.

A proof of De Morgan's law $\sim(A \cap B) = \sim A \cup \sim B$

Case 1: Suppose $x \in A$ and $x \in B$. From the definition of $\cap$, $x \in A \cap B$. So from the definition of $\sim$, $x \notin \sim(A \cap B)$. From the definition of $\sim$, $x \notin \sim A$ and also $x \notin \sim B$. So from the definition of $\cup$, $x \notin \sim A \cup \sim B$.

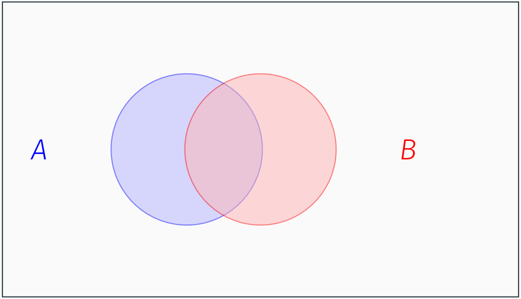
Case 2: Suppose $x \in A$ and $x \notin B$. From the definition of $\cap$, $x \notin A \cap B$. So from the definition of $\sim$, $x \in \sim(A \cap B)$. From the definition of $\sim$, $x \notin \sim A$ but $x \in \sim B$. So from the definition of $\cup$, $x \in \sim A \cup \sim B$.

Case 3: Suppose $x \notin A$ and $x \in B$. From the definition of $\cap$, $x \notin A \cap B$. So from the definition of $\sim$, $x \in \sim(A \cap B)$. From the definition of $\sim$, $x \in \sim A$ but $x \notin \sim B$. So from the definition of $\cup$, $x \in \sim A \cup \sim B$.

Case 4: Suppose $x \notin A$ and $x \notin B$. From the definition of $\cap$, $x \notin A \cap B$. So from the definition of $\sim$, $x \in \sim(A \cap B)$. From the definition of $\sim$, $x \in \sim A$ and $x \in \sim B$. So from the definition of $\cup$, $x \in \sim A \cup \sim B$.

Using the algebra of sets

Prove that $A \Delta B = (A \cup B) \cap \sim (A \cap B)$. (See the next slide.)



$$A \triangle B = (A \cup B) \cap \sim(A \cap B)$$

Proof.

$$(A \cup B) \cap \sim(A \cap B) =$$

$$(A \cup B) \cap (\sim A \cup \sim B) =$$

$$((A \cup B) \cap \sim A) \cup ((A \cup B) \cap \sim B) =$$

$$(\sim A \cap (A \cup B)) \cup (\sim B \cap (A \cup B)) =$$

$$((\underbrace{\cancel{\sim A \cap A}}_{=\emptyset}) \cup (\sim A \cap B)) \cup ((\sim B \cap A) \cup (\underbrace{\cancel{\sim B \cap B}}_{\emptyset})) =$$

$$(\emptyset \cup (\sim A \cap B)) \cup ((\sim B \cap A) \cup \emptyset) =$$

$$(\sim A \cap B) \cup (\sim B \cap A)$$

$$(B - A) \cup (A - B) = (A - B) \cup (B - A) = A \triangle B$$

$$A = B$$

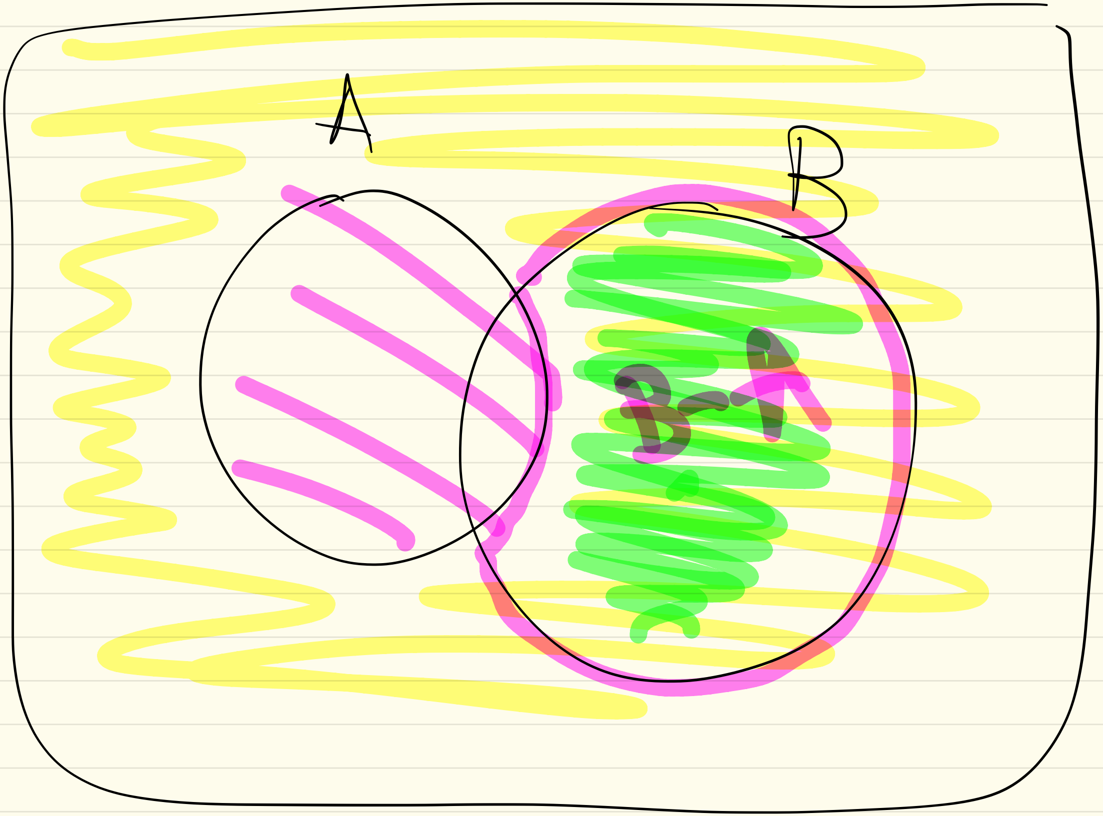
$$A = \ldots = \text{sth}$$

$$B = \ldots = \text{sth} \quad (1)$$

$$A = \ldots = B \quad (2)$$

$$B = \ldots = A \quad (3)$$

$$C \cap (D \cup E) = (C \cap D) \cup (C \cap E)$$


A
B
B-A

$$
\begin{aligned}
(A \cup B) \cap \sim (A \cap B) &= (A \cup B) \cap (\sim A \cup \sim B) \text{ De Morgan} \\
&= ((A \cup B) \cap \sim A) \cup ((A \cup B) \cap \sim B) \text{ distributive} \\
&= (\sim A \cap (A \cup B)) \cup (\sim B \cap (A \cup B)) \text{ commutative} \\
&= ((\sim A \cap A) \cup (\sim A \cap B)) \cup ((\sim B \cap A) \cup (\sim B \cap B)) \text{ distributive} \\
&= ((A \cap \sim A) \cup (B \cap \sim A)) \cup ((A \cap \sim B) \cup (B \cap \sim B)) \text{ commutative} \\
&= (\emptyset \cup (B \cap \sim A)) \cup ((A \cap \sim B) \cup \emptyset) \text{ complement} \\
&= (A \cap \sim B) \cup (B \cap \sim A) \text{ commutative and identity} \\
&= A \Delta B \text{ definition}
\end{aligned}
$$

$S = \{1, 2, 3\}$ $\quad |S| = |\{1,2,3\}| = 3$

$|\{1, 1, 1\}| = 1$ $\quad |\{A, \{A,B\}, \{D\}, E\}| = 4$

$\{1,1,1\} = \{1\}$

Definition The cardinality of a *finite* set S is the number of elements in S, and is denoted by $|S|$.

$|\mathbb{N}| = \aleph_0$ $\quad |\{A, \{A, A\}, \{A\}\}| =$

$= |\{A, \{A\}\}| = 2$

$|\mathbb{R}| = \aleph_1$ $\quad |\mathbb{Z}| = \aleph_0$ $\quad |\mathbb{Q}| = \aleph_0$