Cardinality of sets

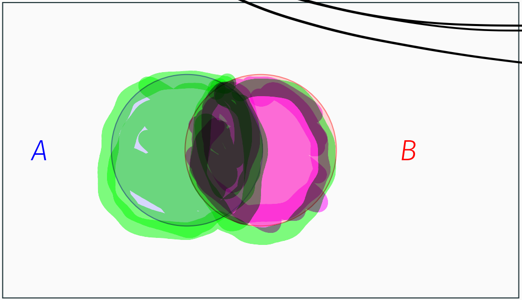
Definition The cardinality of a *finite* set S is the number of elements in S, and is denoted by $|S|$.

$|S|$

Computing the cardinality of a union of two sets

If A and B are sets then

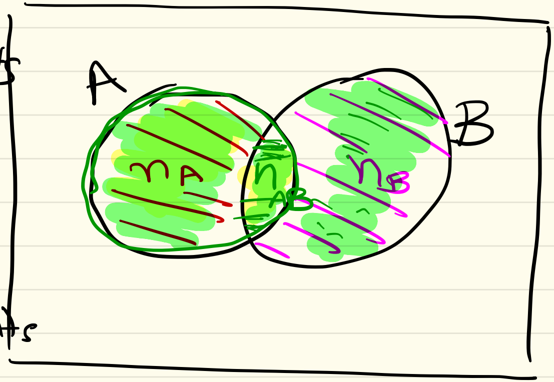
$$|A \cup B| = |A| + |B| - |A \cap B|.$$



For all sets A, B, it holds that $|A \cup B| = |A| + |B| - |A \cap B|$.

Proof.

Let n_A be the number of elements in A but not in B (in $A - B$).



Let n_B be the # of elements in B but not in A (in $B - A$)

Let also n_{AB} be the # of elements in $A \cap B$.

$$|A| + |B| - |A \cap B| = (n_A + n_{AB}) + (n_B + n_{AB}) - (n_{AB})$$

$$= n_A + n_B + n_{AB}$$

$$= |A \cup B|$$

$\square$.

Example

S : set of students

SA : set of students that took S.A.

M.A : -"- -"- -"- -"- M.A.S.

SA ∩ MA : -"- -"- -"- -"- both modules

X : set of -"- -"- -"- neither

Suppose there are 100 third-year students. 40 of them take the module “Sequential Algorithms” and 80 of them take the module “Multi-Agent Systems”. 25 of them took both modules. How many students took neither modules?

$|S| = 100$

$|SA| = 40$ (→ A)

$|MA| = 80$ (→ B)

$|SA \cap MA| = 25$

$|SA| + |MA| - |SA \cap MA|$

$= 40 + 80 - 25$

$= 95 = |SA \cup MA| = |\sim X|$

$SA \cup MA \to \sim X$

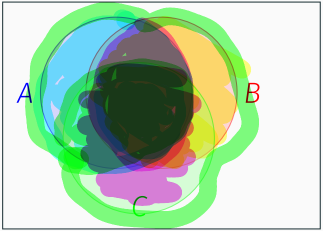
$|A \cup B| = |A| + |B| - |A \cap B|$

$|U| = |S| = 100$

$|X| = 100 - 95 = 5$

Computing the cardinality of a union of three sets

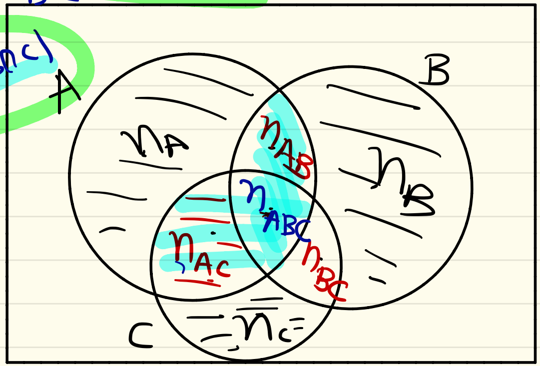
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



These are special cases of the principle of inclusion and exclusion which we will study later.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$|A \cup B \cup C| = n_A + n_B + n_C + n_{AB} + n_{AC} + n_{BC} + n_{ABC}$$



$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| =$$

$$(n_A + n_{AB} + n_{ABC} + n_{AC}) +$$

$$(n_B + n_{AB} + n_{ABC} + n_{BC}) +$$

$$(n_C + n_{AC} + n_{ABC} + n_{BC}) -$$

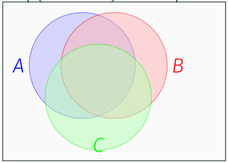
$$(n_{AB} + n_{ABC}) - (n_{AC} + n_{ABC}) - (n_{BC} + n_{ABC}) +$$

$$(n_{ABC}) = n_A + n_B + n_C + n_{AB} + n_{AC} + n_{BC} + n_{ABC}$$

Proof (optional)

We need lots of notation.

- $|A - (B \cup C)| = n_a$, $|B - (A \cup C)| = n_b$, $|C - (A \cup B)| = n_c$,
- $|(A \cap B) - C| = n_{ab}$, $|(A \cap C) - B| = n_{ac}$, $|(B \cap C) - A| = n_{bc}$,
- $|A \cap B \cap C| = n_{abc}$.



Then

$$\begin{aligned}|A \cup B \cup C| &= n_a + n_b + n_c + n_{ab} + n_{ac} + n_{bc} + n_{abc} \\ &= (n_a + n_{ab} + n_{ac} + n_{abc}) + (n_b + n_{ab} + n_{bc} + n_{abc}) \\ &+ (n_c + n_{ac} + n_{bc} + n_{abc}) - (n_{ab} + n_{abc}) \\ &- (n_{ac} + n_{abc}) - (n_{bc} + n_{abc}) + n_{abc}\end{aligned}$$

Reflection

The following statements hold:

- $\emptyset \in \{\emptyset\}$ but $\emptyset \notin \emptyset$; $\emptyset \subseteq \emptyset$
- $\emptyset \subseteq \{5\}$;
- $\{2\} \not\subseteq \{\{2\}\}$ but $\{2\} \in \{\{2\}\}$;
- $\{3, \{3\}\} \neq \{3\}$.

FACT: $\emptyset \subseteq A$, for every set A.

Why is this set theory "naive"

It suffers from paradoxes.

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X = {people in town except barber}

It suffers from paradoxes.

A leading example:

A barber is the man who shaves all those, and only those, men who do not shave themselves.

- Who shaves the barber?

If b shaves himself, then b ∈ {those shaved by b} ⟹ b ∈ {those that don't shave themselves} ⟹ b does not shave himself

If b doesn't shave himself, then b ∈ {those that don't shave them} ⟹ b ∈ {those shaved by b} ⟹ b shaves himself

Russell's Paradox

Russell's paradox shows that the 'object' $\{x \mid P(x)\}$ is not always meaningful.

Set $A = \{B \mid B \notin B\}$

Problem: do we have $A \in A$?

Abbreviate, for any set C, by $P(C)$ the statement $C \notin C$. Then $A = \{B \mid P(B)\}$.

- If $A \in A$, then (from the definition of P), not $P(A)$. Therefore $A \notin A$.
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