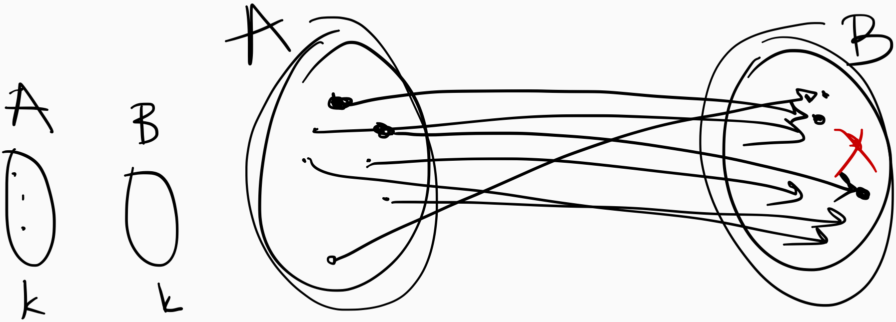
Bijections and cardinality

Recall that the cardinality of a finite set is the number of elements in the set.

Sets A and B have the same cardinality iff there is a bijection from A to B.



$A = \{1,2,3\}$ $Pow(A) = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \emptyset\}$

$(1,0,0)$ $(0,1,0)$

Definition The power set $Pow(A)$ of a set A is the set of all subsets of A. In other words,

$$Pow(A) = \{C \mid C \subseteq A\}.$$

For all $n \in \mathbb{Z}^+$ and all sets A: if $|A| = n$, then $|Pow(A)| = 2^n$.

There are 2^n bit vectors of length n.

Recall that if all elements of a set A are drawn from some ordered sequence $S = s_1, \ldots, s_n$: the characteristic vector of A is the sequence $(b_1, \ldots, b_n)$ where

$$b_i = \begin{cases} 1 & \text{if } s_i \in A \\ 0 & \text{if } s_i \notin A \end{cases}$$

We use the correspondence between bit vectors and subsets: $|Pow(A)|$ is the number of bit vectors of length n.

(0), (1)

$\exists$ 2^n bit vectors of length n

Proof

I will prove the statement by induction on n.

Base case | $n = 1$

There are $2 = 2^1 = 2^n$ bit vectors of length $n=1$, namely (0) and (1).

Inductive Step | Assume that there are 2^n bit vectors of length n, $\forall n \leq k$. I will show that there are 2^{k+1} bit vectors of length $k+1$.

The number of n-bit vectors is 2^n

We prove the statement by induction.

Base Case: Take $n = 1$. There are two bit vectors of length 1: (0) and (1).

Inductive Step: Assume that the property holds for $n = m$, so the number of m-bit vectors is 2^m. Now consider the set B of all $(m+1)$-bit vectors. We must show that $|B| = 2^{m+1}$.

Every $(b_1, b_2, \ldots, b_{m+1}) \in B$ starts with an m-bit vector $(b_1, b_2, \ldots, b_m)$ followed by b_{m+1}, which can be either 0 or 1.

Thus

$$= 2 \cdot 2^m = 2^{m+1}$$

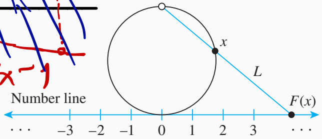
$$|B| = 2^m + 2^m = 2^{m+1}.$$

$(b_1, b_2, \ldots, b_m, b_{m+1})$

Sets A and B have the same cardinality iff there is a bijection from A to B.

Examples:

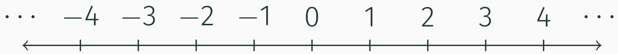
- $\mathbb{Z}$ and even integers
  - consider $f(n) = 2n$
- $\{x \in \mathbb{R} \mid 0 < x < 1\}$ and $\mathbb{R}^+$
  - consider $g(x) = \frac{1}{x} - 1$
- $\{x \in \mathbb{R} \mid 0 < x < 1\}$ and $\mathbb{R}$



Countable sets

A set that is either finite or has the same cardinality as $\mathbb{N}$ is called **countable**.

- $\mathbb{Z}$



$\mathbb{Q} = \{ x \in \mathbb{R} \mid x = \frac{a}{b}, \text{ for some } a \in \mathbb{Z}, b \in \mathbb{Z}^* \}$.

$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\cdots$
$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	$\frac{2}{6}$	$\cdots$
$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	$\cdots$
$\frac{4}{1}$	$\frac{4}{2}$	$\frac{4}{3}$	$\frac{4}{4}$	$\frac{4}{5}$	$\frac{4}{6}$	$\cdots$
$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{5}{5}$	$\frac{5}{6}$	$\cdots$
$\frac{6}{1}$	$\frac{6}{2}$	$\frac{6}{3}$	$\frac{6}{4}$	$\frac{6}{5}$	$\frac{6}{6}$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\ddots$

Uncountable sets

- A set that is not countable is called **uncountable**.
  - $S = \{x \in \mathbb{R} \mid 0 < x < 1\}$ is uncountable

Cantor's diagonal argument

Suppose S is countable. Then the decimal representations of these numbers can be written as a list

$$
\begin{aligned}
a_1 &= 0.a_{11}\ a_{12}\ a_{13} \dots\ a_{1n} \dots \\
a_2 &= 0.a_{21}\ a_{22}\ a_{23} \dots\ a_{2n} \dots \\
a_3 &= 0.a_{31}\ a_{32}\ a_{33} \dots\ a_{3n} \dots \\
&\vdots \\
a_n &= 0.a_{n1}\ a_{n2}\ a_{n3} \dots\ a_{nn} \dots \\
&\vdots
\end{aligned}
$$

0.12345...
0.351321....
0.011151...

Let $d = 0.d_1\ d_2\ d_3 \dots d_n \dots$ where

d = 0.212...

$$
d_i = \begin{cases} 1, & \text{if } a_{ii} \neq 1 \\ 2, & \text{if } a_{ii} = 1 \end{cases}
$$

Then d is not in the sequence $a_1, a_2, a_3 \dots$

Prove that if A then B,

Cantor's diagonal argument

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$$\begin{aligned}
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a_2 &= 0.a_{21}\ a_{22}\ a_{23} \ldots\ a_{2n} \ldots \\
a_3 &= 0.a_{31}\ a_{32}\ a_{33} \ldots\ a_{3n} \ldots \\
&\vdots \\
a_n &= 0.a_{n1}\ a_{n2}\ a_{n3} \ldots\ a_{nn} \ldots \\
&\vdots
\end{aligned}$$

Let $d = 0.d_1\ d_2\ d_3 \ldots d_n \ldots$ where

$$d_i = \begin{cases} 1, & \text{if } a_{ii} \neq 1 \\ 2, & \text{if } a_{ii} = 1 \end{cases}$$

Then d is not in the sequence a_1, a_2, a_3...

Cantor's diagonal argument

Suppose S is countable. Then the decimal representations of these numbers can be written as a list

$$\begin{aligned} a_1 &= 0.a_{11}\ a_{12}\ a_{13} \ldots\ a_{1n} \ldots \\ a_2 &= 0.a_{21}\ a_{22}\ a_{23} \ldots\ a_{2n} \ldots \\ a_3 &= 0.a_{31}\ a_{32}\ a_{33} \ldots\ a_{3n} \ldots \\ &\vdots \\ a_n &= 0.a_{n1}\ a_{n2}\ a_{n3} \ldots\ a_{nn} \ldots \\ &\vdots \end{aligned}$$

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