Foundations of Computer Science Comp109

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Part 3. Relations

Comp109 Foundations of Computer Science

Reading

Discrete Mathematics and Its Applications K. Rosen, Chapter 9.

Contents

Motivation

- Intuitively, there is a "relation" between two things if there is some connection between them.
  E.g.
  - 'friend of'
  - $a < b$
  - m divides n
- Relations are used in crucial ways in many branches of mathematics
  - Equivalence
  - Ordering
- Computer Science

Databases and relations

A database table ≈ relation

TABLE 1 Students.

Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

Definition The Cartesian product $A \times B$ of sets A and B is the *set* consisting of all pairs (a, b) with $a \in A$ and $b \in B$, i.e.,

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Note that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

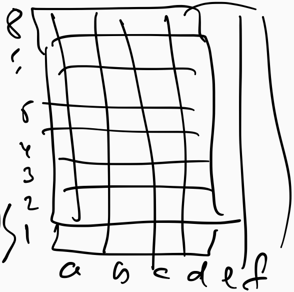
Note

- $\{1, 2\} = \{2, 1\}$ but $(1, 2) \neq (2, 1)$.

$A = \{a, b, c, d, e, f, g, h\}$

$(a, 1)$

$B = \{1, 2, 3, 4, 5, 6, 7, 8\}$



Example

- Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then

$$A \times B = \{(1, a), (2, a), (1, b), (2, b), (1, c), (2, c)\}.$$

- $B \times A =$

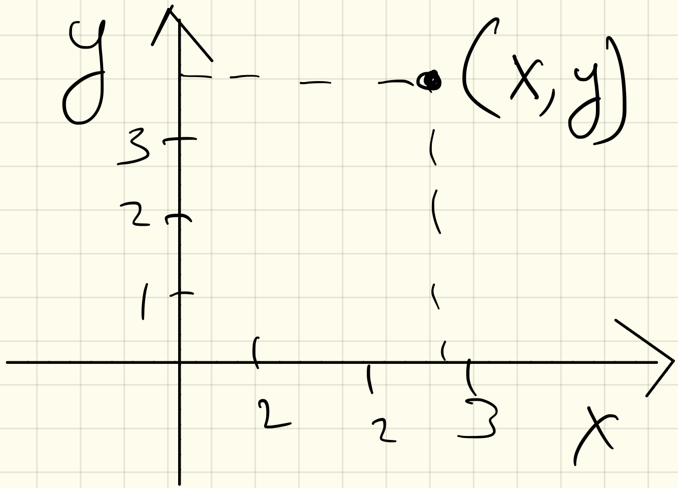
$$= \{(a,1), (b,1), (c,1), (a,2), (b,2), (c,2)\}$$

$A = \{1, 2, 3\}$

$B = \{3, 4\}$

$A \times B = \{(1,3), (1,4), (2,3), (2,4), (3,3), (3,4)\}$

$\mathbb{R} \times \mathbb{R} =$



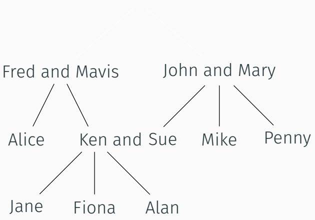
$\mathbb{R}^2$

Relations

Definition A binary relation between two sets A and B is a subset R of the Cartesian product $A \times B$.

If $A = B$, then R is called a binary relation on A.

Example: Family tree



Write down

- $R = \{(x, y) \mid x \text{ is a grandfather of } y\}$;

  $R = \{ (\text{Fred}, \text{Jane}), (\text{Fred}, \text{Fiona}), (\text{Fred}, \text{Alan}), (\text{John}, \text{Jane}), (\text{John}, \text{Fiona}), (\text{John}, \text{Alan}) \}$

- $S = \{(x, y) \mid x \text{ is a sister of } y\}$.

  $S = \{ (\text{Alice}, \text{Ken}) \ldots \}$

Example 2

Write down the ordered pairs belonging to the following binary relations between $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6\}$:

- $U = \{(x, y) \in A \times B \mid x + y = 9\}$;

$$U = \{(3,6), (5,4), (7,2)\}$$

- $V = \{(x, y) \in A \times B \mid x < y\}$.

$$V = \{(1,2), (1,4), (1,6), (3,4), (3,6), (5,6)\}$$

Example 3

Let $A = \{1, 2, 3, 4, 5, 6\}$. Write down the ordered pairs belonging to

$$R = \{(x, y) \in A \times A \mid \ x \text{ is a divisor of } y\ \}.$$

Homework

Representation of binary relations: directed graphs

- Let A and B be two finite sets and R a binary relation between these two sets (i.e., $R \subseteq A \times B$).
- We represent the elements of these two sets as vertices of a graph.
- For each $(a, b) \in R$, we draw an arrow linking the related elements.
- This is called the directed graph (or digraph) of R.

Example

Consider the relation V between $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6\}$ such that $V = \{(x, y) \in A \times B \mid x < y\}$.

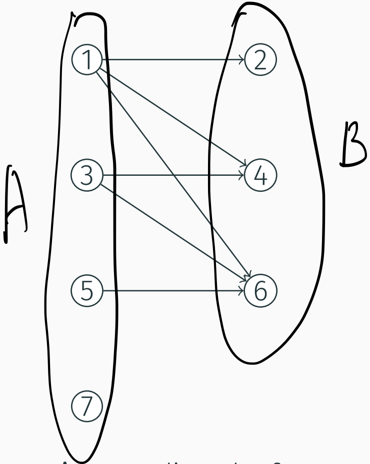
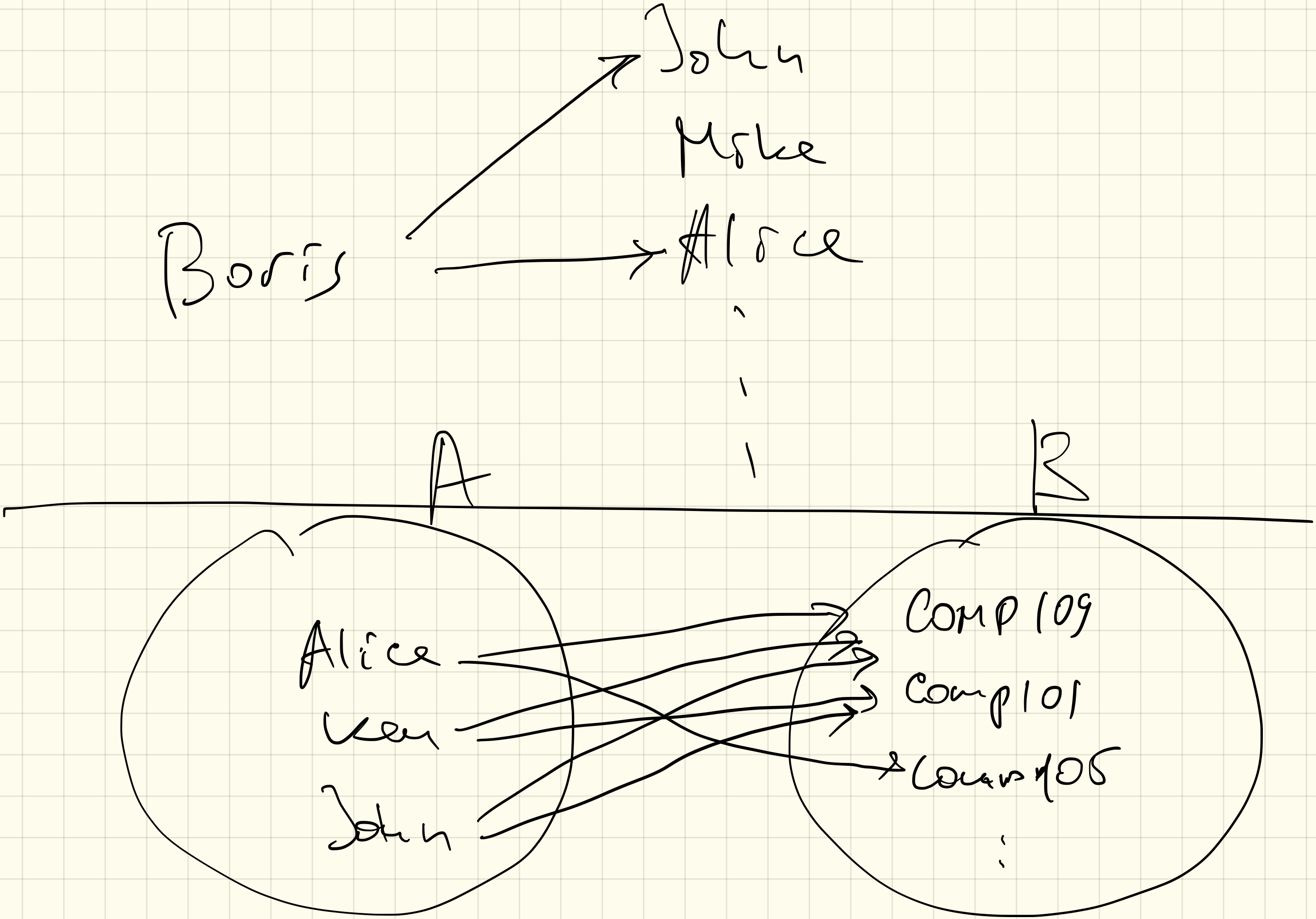


Figure 1: digraph of V


John
Mike
Alice
Boris
A
B
Alice
Ken
John
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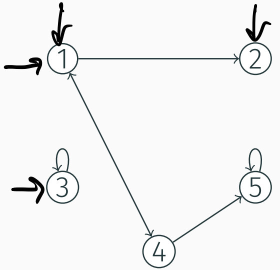
Comp105

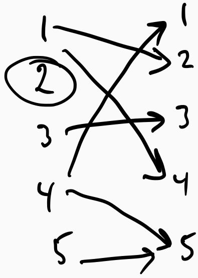
A binary relation between a set A and itself is called "a binary relation on A".

To represent such a relation, we use a directed graph in which a single set of vertices represents the elements of A and arrows link the related elements.

Consider the relation $V \subseteq A \times A$ where $A = \{1, 2, 3, 4, 5\}$ and

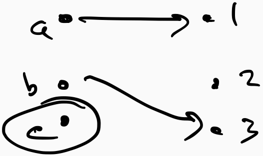
$$V = \{(1, 2), (3, 3), (5, 5), (1, 4), (4, 1), (4, 5)\}.$$





Functions as relations

- Recall that a function f from a set A to a set B assigns exactly one element of B to each element of A.
  - Gives rise to the relation $R_f = \{(a, b) \in A \times B \mid b = f(a)\}$
- If a relation $S \subseteq A \times B$ is such that for every $a \in A$ there exists at most one $b \in B$ with $(a, b) \in S$, relation S is functional.
- (Sometimes in the literature, functions are introduced through functional relations.)



Inverse relation

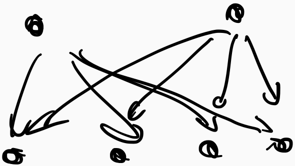
Definition Given a relation $R \subseteq A \times B$, we define the *inverse relation* $R^{-1} \subseteq B \times A$ by

$$R^{-1} = \{(b, a) \mid (a, b) \in R\}.$$

Example: The inverse of the relation *is a parent of* on the set of people is the relation *is a child of*.

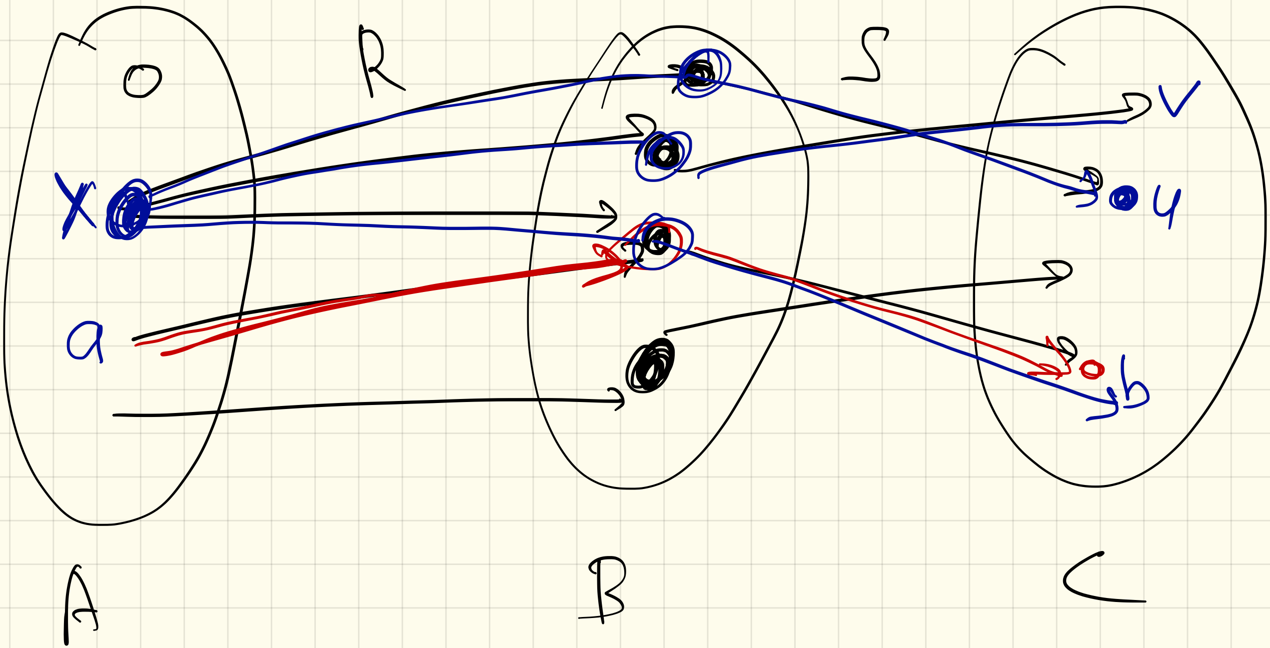
Real Donald Trump

Boris Konev



(a,b) $\subseteq A \times C$

$(x,u), (x,v), (x,b$



$S \circ R$

Composition of relations

Definition Let $R \subseteq A \times B$ and $S \subseteq B \times C$. The (functional) composition of R and S, denoted by $S \circ R$, is the binary relation between A and C given by

$$S \circ R = \{(a, c) \mid \text{ exists } b \in B \text{ such that } aRb \text{ and } bSc\}.$$

Example: If R is the relation *is a sister of* and S is the relation *is a parent of*, then

- $S \circ R$ is the relation *is an aunt of*;
- $S \circ S$ is the relation *is a grandparent of*.