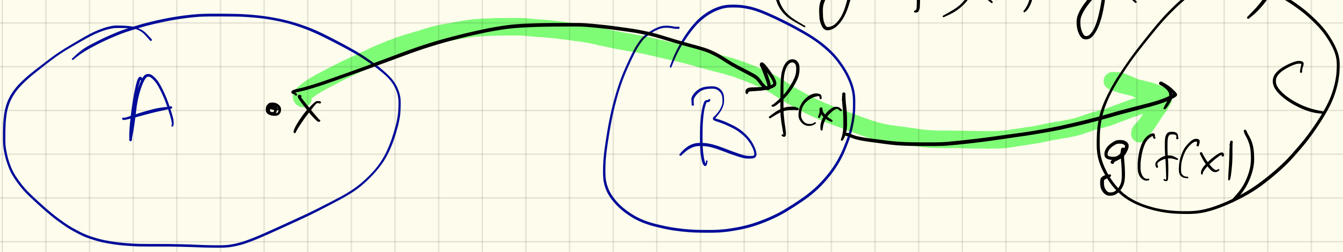
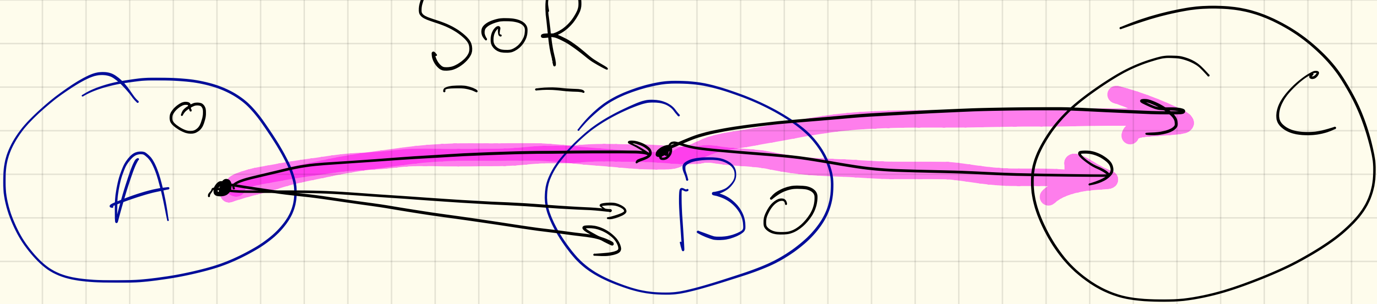
$f: A \to B \qquad g: B \to C$

$(g \circ f)(x) = g(f(x))$



$R \subseteq A \times B \qquad S \subseteq B \times C$

$\underline{S} \circ \underline{R}$



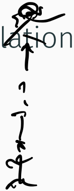
Composition of relations

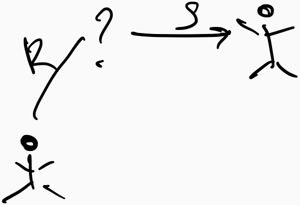
Definition Let $R \subseteq A \times B$ and $S \subseteq B \times C$. The (functional) composition of R and S, denoted by $S \circ R$, is the binary relation between A and C given by

$$S \circ R = \{(a, c) \mid \text{ exists } b \in B \text{ such that } aRb \text{ and } bSc\}.$$

Example: If R is the relation *is a sister of* and S is the relation *is a parent of*, then

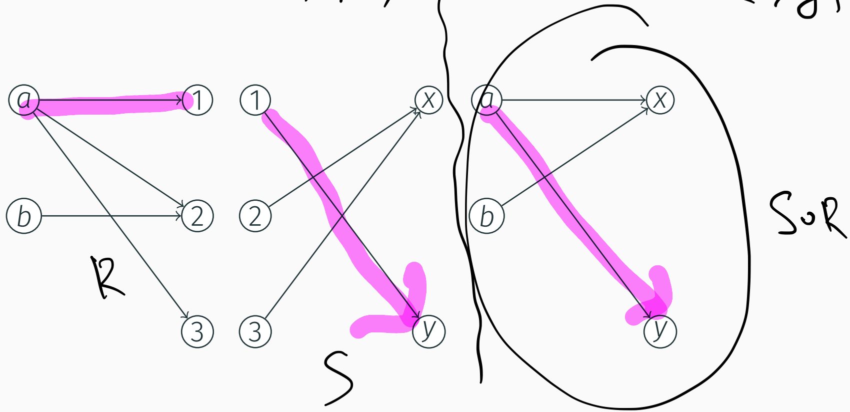
- $S \circ R$ is the relation *is an aunt of*;
- $S \circ S$ is the relation *is a grandparent of*.





Digraph representation of compositions

$A = \{a, b\}$ $B = \{1, 2, 3\}$ $C = \{x, y\}$



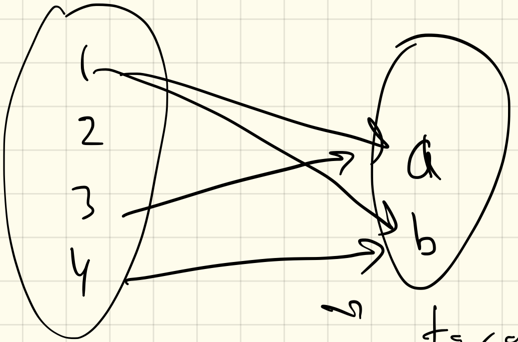
R S $S \circ R$

$R = \{(a,1), (a,2), (a,3), (b,2)\}$

Computer friendly representation of binary relations: matrices

- Another way of representing a binary relation between finite sets uses an array.
- Let $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_m\}$ and $R \subseteq A \times B$.
- We represent R by an array M of n rows and m columns. Such an array is called a *n by m matrix*.
- The entry in row i and column j of this matrix is given by $M(i, j)$ where

$$M(i,j) = \begin{cases} T & \text{if} \quad (a_i, b_j) \in R \\ F & \text{if} \quad (a_i, b_j) \notin R \end{cases}$$



target

Source

	a	b
1	T	T
2	F	F
3	T	F
4	F	T

Example 1

Let $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6\}$, and

$$U = \{(x, y) \in A \times B \mid x + y = 9\}$$

Assume an enumeration $a_1 = 1$, $a_2 = 3$, $a_3 = 5$, $a_4 = 7$ and $b_1 = 2$, $b_2 = 4$, $b_3 = 6$. Then M represents U, where

$$M = \begin{array}{c c} & \begin{array}{ccc} 2 & 4 & 6 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \\ 7 \end{array} & \begin{bmatrix} F & F & F \\ F & F & T \\ F & T & F \\ T & F & F \end{bmatrix} \end{array}$$

Example 2

Let $A = \{a, b, c, d\}$ and suppose that $R \subseteq A \times A$ has the following matrix representation:

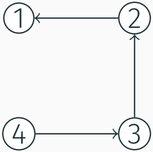
$$M = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} F & T & T & F \\ F & F & T & T \\ F & T & F & F \\ T & T & F & T \end{bmatrix} \end{array}$$

List the ordered pairs belonging to R.

$R = \{(a,b), (a,c), (b,c) \ldots\}$

Example

The binary relation R on $A = \{1, 2, 3, 4\}$ has the following digraph representation.



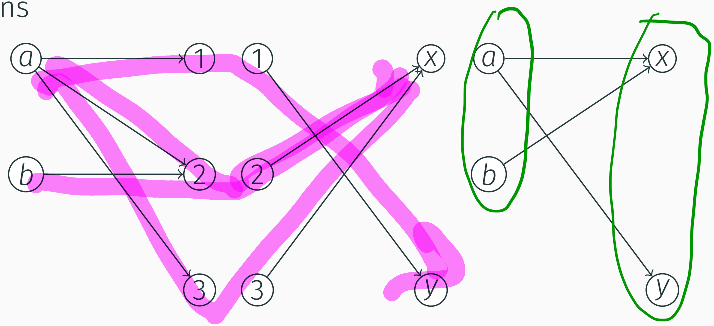
- The ordered pairs $R =$ $\{(4,3), (3,2), (2,1)\}$
- The matrix

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & & & & \end{array}$$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} 1\;\;2\;\;3\;\;4 \\ \begin{bmatrix} F & F & F & F \\ T & F & F & F \\ F & T & F & F \\ F & F & T & F \end{bmatrix} \end{array}$$

- In words: $\{(x,y) \mid y = x-1,\ x \in A, y \in A\}$

Now let’s go back and see how this works for matrices representing relations



	x	y
a	T	T
b		

$$R : \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} T & T & T \\ F & T & F \end{bmatrix} \end{array} \qquad S : \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} F & T \\ T & F \\ T & F \end{bmatrix} \qquad S \circ R : \begin{bmatrix} T & T \\ T & F \end{bmatrix}$$

The formal description

Given two matrices with entries "T" and "F" representing the relations we can form the matrix representing the composition. This is called the *logical (Boolean) matrix product*.

Let $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_m\}$ and $C = \{c_1, \ldots, c_p\}$.

The logical matrix M representing R is given by:

$$M(i,j) = \begin{cases} T & \text{if} \quad (a_i, b_j) \in R \\ F & \text{if} \quad (a_i, b_j) \notin R \end{cases}$$

The logical matrix N representing S is given by

$$N(i,j) = \begin{cases} T & \text{if} \quad (b_i, c_j) \in S \\ F & \text{if} \quad (b_i, c_j) \notin S \end{cases}$$

Matrix representation of compositions

Then the entries $P(i,j)$ of the logical matrix P representing $S \circ R$ are given by

- $P(i,j) = T$ if there exists l with $1 \leq l \leq m$ such that $M(i,l) = T$ and $N(l,j) = T$.
- $P(i,j) = F$, otherwise.

We write $P = MN$.

The example from before

Let R be the relation between $A = \{a, b\}$ and $B = \{1, 2, 3\}$ represented by the matrix

$$M = \begin{bmatrix} T & T & T \\ F & T & F \end{bmatrix}$$

Similarly, let S be the relation between B and $C = \{x, y\}$ represented by the matrix

$$N = \begin{bmatrix} F & T \\ T & F \\ T & F \end{bmatrix}$$

Example

Then the matrix $P = MN$ representing $S \circ R$ is

$$P = \begin{bmatrix} T & T \\ T & F \end{bmatrix}$$

M · N

	G	F	I	S
b	? T	? F		
j				
a			? T	

SoR

	b	j	a
b	F	T	T
j	T	F	F
a	T	F	F

M

	G	F	I	S
b	T	T	T	T
j	F	F	F	F
a	T	F	F	T

N