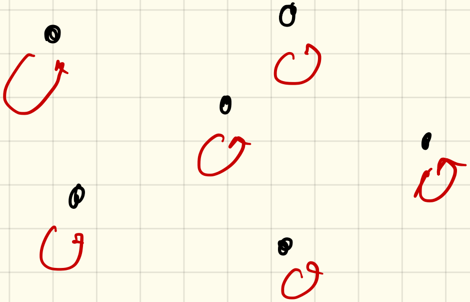
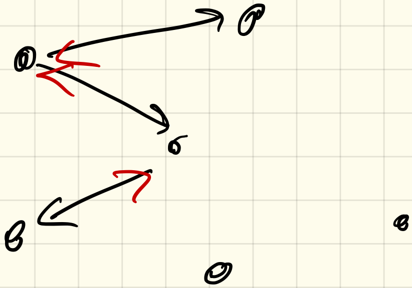
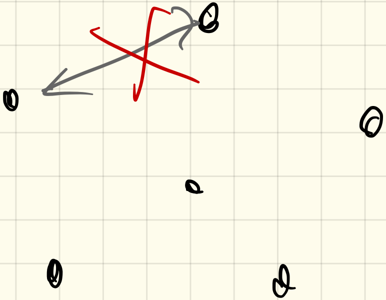
Reflexivity



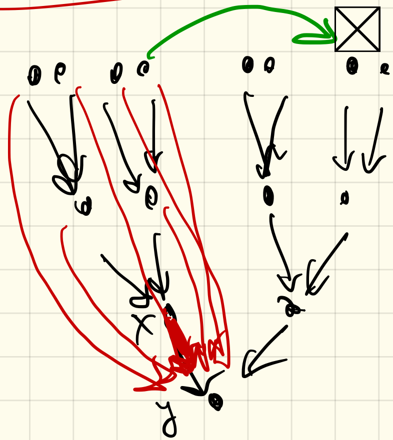
Symmetry



Anti-symmetry



Transitivity



Transitive closure

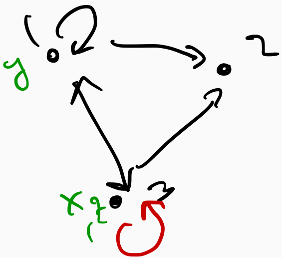
R^+

Given a binary relation R on a set A, the *transitive closure* R^* of R is the (uniquely determined) relation on A with the following properties:

- R^* is transitive;
- $R \subseteq R^*$;
- If S is a transitive relation on A and $R \subseteq S$, then $R^* \subseteq S$.

Let $A = \{1, 2, 3\}$. Find the transitive closure of

$$R = \{(1,1), (1,2), (1,3), (2,3), (3,1)\}.$$

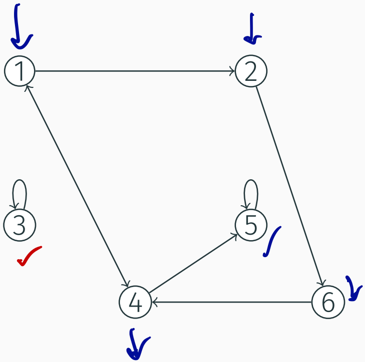


$(3,1) \in R$

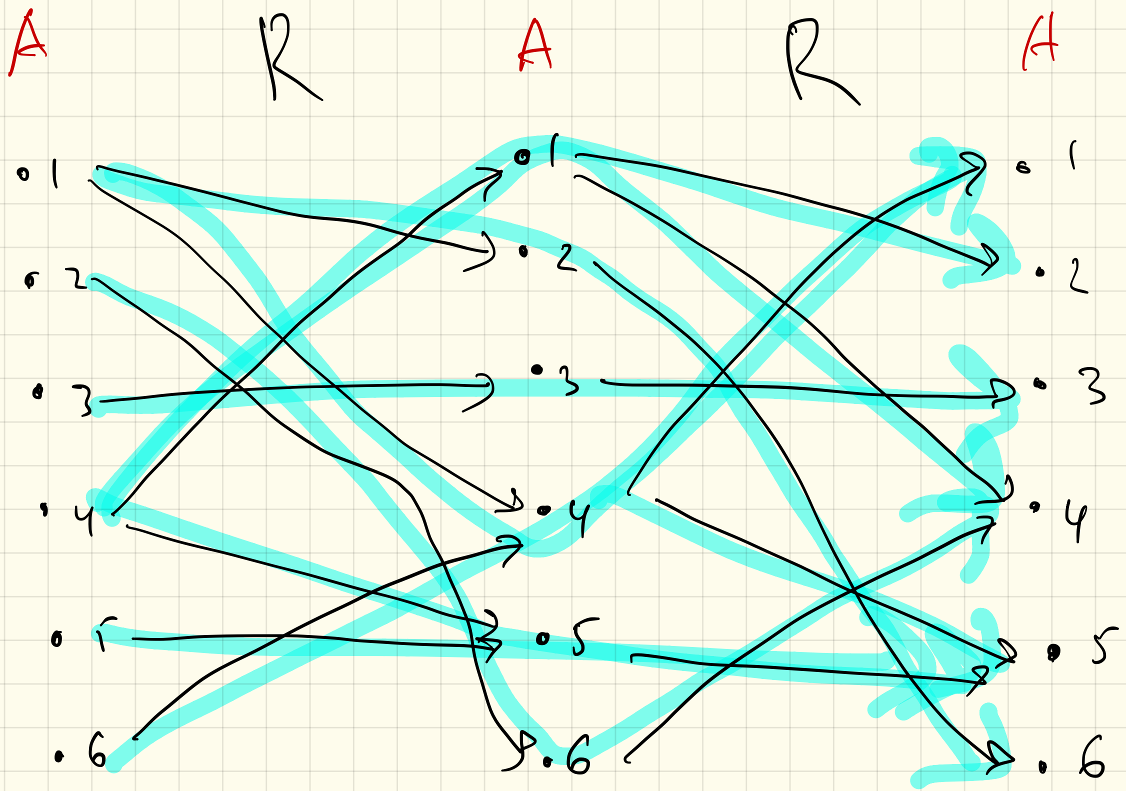
$(1,3) \in R$

$(3,3) \notin R$

Finding the transitive closure is easier with the digraph representation



Reachability relation


A
R
A
R
A
1
2
3
4
5
6
1
2
3
4
5
6
1
2
3
4
5
6

Transitivity and composition

A relation S is transitive if and only if $S \circ S \subseteq S$.

This is because

$$S \circ S = \{(a, c) \mid \text{ exists } b \text{ such that } aSb \text{ and } bSc\}.$$

Let S be a relation. Set $S^1 = S$, $S^2 = S \circ S$, $S^3 = S \circ S \circ S$, and so on.

Theorem Denote by S^* the transitive closure of S. Then xS^*y if and only if there exists $n > 0$ such that xS^ny.

Transitive closure in matrix form

The relation R on the set $A = \{1, 2, 3, 4, 5\}$ is represented by the matrix

$$
\begin{array}{c c}
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\begin{bmatrix}
T & F & F & T & F \\
F & T & F & F & T \\
F & F & T & F & F \\
T & F & T & F & F \\
F & T & F & T & F
\end{bmatrix}
\end{array}
\qquad (1,1),\ (1,4)
$$

Determine the matrix $R \circ R$ and hence explain why R is not transitive.

Computation

$$\begin{bmatrix} T & F & F & T & F \\ F & T & F & F & T \\ F & F & T & F & F \\ T & F & T & F & F \\ F & T & F & T & F \end{bmatrix} \begin{bmatrix} T & F & F & T & F \\ F & T & F & F & T \\ F & F & T & F & F \\ T & F & T & F & F \\ F & T & F & T & F \end{bmatrix} = \begin{bmatrix} T & F & T & T & F \\ F & T & F & T & T \\ F & F & T & F & F \\ T & F & T & T & F \\ T & T & T & F & T \end{bmatrix}$$

$$R \circ R = \{(a, c) \mid \text{ exists } b \in A \text{ such that } aRb \text{ and } bRc\}.$$

Note (in red) that there are pairs (a, c) that are in $R \circ R$ but not in R. Hence, R is not transitive.

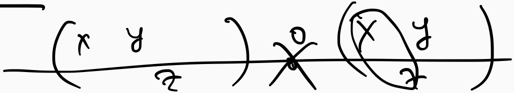
```
def warshall(a):
    assert (len(row) == len(a) for row in a)
    n = len(a)
    for k in range(n):
        for i in range(n):
            for j in range(n):
                a[i][j] = a[i][j] or
                    (a[i][k] and a[k][j])
    return a

print warshall([[1,0,0,1,0],
                [0,1,0,0,1],
                [0,0,1,0,0],
                [1,0,1,0,0],
                [0,1,0,1,0]])
```

Important relations: Equivalence relations

Definition A binary relation R on a set A is called an *equivalence relation* if it is reflexive, transitive, and symmetric.

Examples:



- the relation R on the non-zero integers given by xRy if $xy > 0$;
- the relation *has the same age* on the set of people.

Definition The *equivalence class* E_x of any $x \in A$ is defined by

$$E_x = \{y \mid yRx\}.$$