Part 5. Propositional Logic, digital circuits & computer arithmetic

Comp109 Foundations of Computer Science

Reading

- *Discrete Mathematics and Its Applications*, K.H. Rosen, Sections 1.1–1.3.
- *Discrete Mathematics with Applications*, S. Epp, Chapter 2.

Contents

Logic

Logic is concerned with

- the truth and falsity of statements;
- the question: when does a statement *follow from* a set of statements?

Propositional logic

Propositions

A proposition is a statement that can be true or false.
(but not both in the same time!)

- Logic is easy;
- I eat toast;
- $2 + 3 = 5$;
- $2 \cdot 2 = 5$.
- $4 + 5$;
- What is the capital of UK?

- Logic is not easy;
- Logic is easy or I eat toast;

$\neg$ logic is easy

logic is easy $\vee$ I eat toast

Compound propositions

- More complex propositions formed using logical connectives (also called Boolean connectives)
- Basic logical connectives:
  1. $\neg$: negation (read "not")
  2. $\wedge$: conjunction (read "and"),
  3. $\vee$: disjunction (read "or")
  4. $\Rightarrow$: implication (read "if...then")
  5. $\Leftrightarrow$: equivalence (read "if, and only if,")
- Propositions formed using these logical connectives called compound propositions; otherwise atomic propositions
- A propositional formula is either an atomic or compound proposition

Giving meaning to propositions: Truth values

An *interpretation* I is a function which assigns to any atomic proposition p_i a *truth value*

$$I(p_i) \in \{0, 1\}.$$

- If $I(p_i) = 1$, then p_i is called *true* under the interpretation I.
- If $I(p_i) = 0$, then p_i is called *false* under the interpretation I.

Given an assignment I we can compute the truth value of compound formulas step by step using so-called truth tables.

Negation

The negation $\neg P$ of a formula P

It is not the case that P

Truth table:

P	$\neg P$
1	0
0	1

Conjunction

The conjunction $(P \wedge Q)$ of P and Q.

both P and Q are true

Truth table:

P	Q	$(P \wedge Q)$
1	1	1
1	0	0
0	1	0
0	0	0

Disjunction

The disjunction $(P \vee Q)$ of P and Q

at least one of P and Q is true

Truth table:

P	Q	$(P \vee Q)$
1	1	1
1	0	1
0	1	1
0	0	0

Equivalence

The equivalence $(P \Leftrightarrow Q)$ of P and Q

P and Q take the same truth value

Truth table:

P	Q	$(P \Leftrightarrow Q)$
1	1	1
1	0	0
0	1	0
0	0	1

Implication

The implication $(P \Leftrightarrow Q)$ of P and Q

if P then Q

Truth table:

P	Q	$(P \Rightarrow Q)$
1	1	1
1	0	0
0	1	1
0	0	1

Truth under an interpretation

So, given an interpretation I, we can compute the truth value of any formula P under I.

- If $I(P) = 1$, then P is called **true** under the interpretation I.
- If $I(P) = 0$, then P is called **false** under the interpretation I.

List the Interpretations I such that $P = ((p_1 \lor \neg p_2) \land p_3)$ is true under I.

p_1	p_2	p_3	$\neg p_2$	$(p_1 \lor \neg p_2)$	$P = ((p_1 \lor \neg p_2) \land p_3)$
0	0	0	1	1	
0	0	1	1	1	
0	1	0	0	0	
0	1	1	0	0	
1	0	0	1	1	
1	0	1	1	1	
1	1	0	0	1	
1	1	1	0	1	

For values see `http://www.csc.liv.ac.uk/~konev/COMP109/lecturelog.html`

Logical puzzles

- An island has two kinds of inhabitants, knights, who always tell the truth, and knaves, who always lie.
- You go to the island and meet A and B.
  - A says "B is a knight."
  - B says "The two of us are of opposite types."
- What are A and B?

p: "A is a knight"; and q: "B is a knight"

- Options for A.
  - p is true $\quad p \Rightarrow q$
  - p is false $\quad \neg p \Rightarrow \neg q$
- Options for B.
  - q is true $\quad q \Rightarrow \neg p$
  - q is false $\quad \neg q \Rightarrow \neg p$

Truth table

p	q	$\neg p$	$\neg q$	$p \Rightarrow q$	$\neg p \Rightarrow \neg q$	$q \Rightarrow \neg p$	$\neg q \Rightarrow \neg p$
0	0	1	1	1	1	1	1
0	1						
1	0						
1	1						

Definition Suppose Γ is a finite set of formulas and P is a formula. Then P follows from Γ ("is a semantic consequence of Γ") if the following implication holds for every interpretation I:

$$\text{If } I(Q) = 1 \text{ for all } Q \in \Gamma, \text{ then } I(P) = 1.$$

This is denoted by

$$\Gamma \models P.$$

Show $\{(p_1 \land p_2)\} \models (p_1 \lor p_2)$.

p_1	p_2	$(p_0 \land p_1)$	$(p_0 \lor p_1)$
1	1	1	1
1	0	0	1
0	1	0	1
0	0	0	0

Example

Show $\{p_1\} \not\models p_2$.

p_1	p_2
1	1
1	0
0	1
0	0

Example

Show $\{p_1\} \models (p_1 \vee p_2)$.

p_1	p_2	$(p_1 \vee p_2)$
1	1	
1	0	
0	1	
0	0	

Logic and proof principles I

- *Modus Ponens*

  Direct proof corresponds to the following semantic consequence

$$\{P, (P \Rightarrow Q)\} \models Q;$$

- *Reductio ad absurdum*

  Proof by contradiction corresponds to

$$\{(\neg P \Rightarrow \bot)\} \models P,$$

  where $\bot$ is a special proposition, which is false under every interpretation.

Logic and proof principles II

- *Modus Tollens*

  Another look at proof by contradiction

$$\{(P \Rightarrow Q), \neg Q\} \models \neg P$$

- Case analysis

$$\{(P \Rightarrow Q), (R \Rightarrow Q), (P \vee R)\} \models Q$$

Proof theory

- We have studied proofs as carefully reasoned arguments to convince a sceptical listener that a given statement is true.
  - “Social” proofs
- Proof theory is a branch of mathematical logic dealing with proofs as mathematical objects
  - Strings of symbols
  - Rules for manipulation
  - Mathematics becomes a ‘game’ played with strings of symbols
  - Can be read and interpreted by computer