Part 5. Propositional Logic, digital circuits & computer arithmetic

Comp109 Foundations of Computer Science

Reading

- Discrete Mathematics and Its Applications, K.H. Rosen, Sections 1.1–1.3.
- Discrete Mathematics with Applications, S. Epp, Chapter 2.

Contents

- The language of propositional logic
- Semantics: interpretations and truth tables
- Semantic consequence
- Logical equivalence
- Logic and digital circuits
- Computer representation of numbers & computer arithmetic

Logic

Logic is concerned with

- the truth and falsity of statements;
- the question: when does a statement follow from a set of statements?

Propositional logic

Propositions

A proposition is a statement that can be true or false. (but not both in the same time!)

- Logic is easy; ✓
- I eat toast; ✓
- \blacksquare 2 + 3 = 5; \checkmark

- What is the capital of UK? ➤

■ Logic is not easy;

Logic is easy or I eat toast;

logic o easy

(easy

(easy

Compound propositions

- More complex propositions formed using logical connectives (also called Boolean connectives)
- Basic logical connectives:
 - 1. ¬: negation (read "not")
 - 2. ∧: conjunction (read "and"),
 - 3. ∨: disjunction (read "or")
 - 4. ⇒: implication (read "if...then")
 - 5. ⇔: equivalence (read "if, and only if,")
- Propositions formed using these logical connectives called compound propositions; otherwise atomic propositions
- A propositional formula is either an atomic or compound proposition

Giving meaning to propositions: Truth values

An *interpretation* I is a function which assigns to any atomic proposition p_i a truth value

$$I(p_i) \in \{0,1\}.$$

- If $I(p_i) = 1$, then p_i is called true under the interpretation I.
- If $I(p_i) = 0$, then p_i is called *false* under the interpretation I.

Given an assignment *I* we can compute the truth value of compound formulas step by step using so-called truth tables.

Negation

The negation $\neg P$ of a formula P

It is not the case that P

Р	$\neg P$
1	O
0	1

Conjunction

The conjunction $(P \land Q)$ of P and Q.

both P and Q are true

Р	Q	$(P \wedge Q)$
1	1	1
1	0	0
0	1	0
0	0	Ö

Disjunction

The disjunction $(P \lor Q)$ of P and Q at least one of P and Q is true

Р	Q	$(P \lor Q)$
1	1	(
1	0	(
0	1	(
0	0	0

Equivalence

The equivalence $(P \Leftrightarrow Q)$ of P and Q

P and Q take the same truth value

Р	Q	$(P \Leftrightarrow Q)$
1	1	1
1	0	0
0	1	0
0	0	١

Implication

The implication $(P \Leftrightarrow Q)$ of P and Q

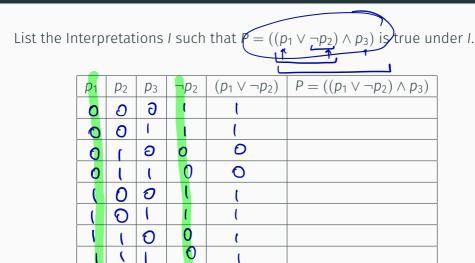
if P then Q

Р	Q	$(P \Rightarrow Q)$
1	1	l
1	0	Ð
0	1	
0	0	(

Truth under an interpretation

So, given an interpretation *I*, we can compute the truth value of any formula *P* under *I*.

- If I(P) = 1, then P is called true under the interpretation I.
- If I(P) = 0, then P is called false under the interpretation I.



For values see http://www.csc.liv.ac.uk/~konev/COMP109/lecturelog.html

Logical puzzles

- An island has two kinds of inhabitants, knights, who always tell the truth, and knaves, who always lie.
- You go to the island and meet A and B.
 - A says "B is a knight."
 - B says "The two of us are of opposite types."
- What are A and B?

p: "A is a knight"; and q: "B is a knight"

- Options for A.
 - *p* is true

$$p \Rightarrow q$$

 \blacksquare *p* is false

$$\neg p \Rightarrow \neg q$$

- \blacksquare Options for B.
 - \blacksquare q is true

 $q \Rightarrow \neg p$

 \blacksquare q is false

 $\neg q \Rightarrow \neg p$

	р	q	$\neg p$	$\neg q$	$p \Rightarrow q$	$\neg p \Rightarrow \neg q$	$q \Rightarrow \neg p$	$\neg q \Rightarrow \neg p$
	0	0	1	((1	
	0	1						·
Ì	1	0						
Ì	1	1						

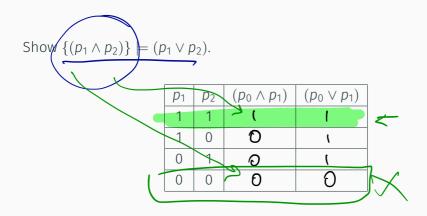
Semantic consequence

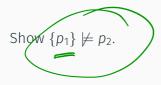
Definition Suppose Γ is a finite set of formulas and P is a formula. Then P follows from Γ ("is a semantic consequence of Γ ") if the following implication holds for every interpretation I:

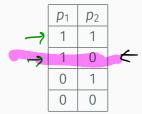
If
$$I(Q) = 1$$
 for all $Q \in \Gamma$, then $I(P) = 1$.

This is denoted by









Show $\{p_1\} \models (p_1 \lor p_2)$.

<i>p</i> ₁	p ₂	$(p_1 \vee p_2)$
1	1	
1	0	
0	1	
0	0	

Logic and proof principles I

Modus Ponens
 Direct proof corresponds to the following semantic consequence

$$\{P, (P \Rightarrow Q)\} \models Q;$$

Reductio ad absurdumProof by contradiction corresponds to

where \perp is a special proposition, which is false under every interpretation.

Logic and proof principles II

Modus TollensAnother look at proof by contradiction

$$\{(P \Rightarrow Q), \neg Q\} \models \neg P$$

Case analysis

$$\{(P \Rightarrow Q), (R \Rightarrow Q), (P \lor R)\} \models Q$$

Proof theory

- We have studied proofs as carefully reasoned arguments to convince a sceptical listener that a given statement is true.
 - "Social" proofs
- Proof theory is a branch of mathematical logic dealing with proofs as mathematical objects
 - Strings of symbols
 - Rules for manipulation
 - Mathematics becomes a 'game' played with strings of symbols
 - Can be read and interpreted by computer