Cardinality of sets

Definition The cardinality of a *finite* set S is the number of elements in S, and is denoted by $|S|$.

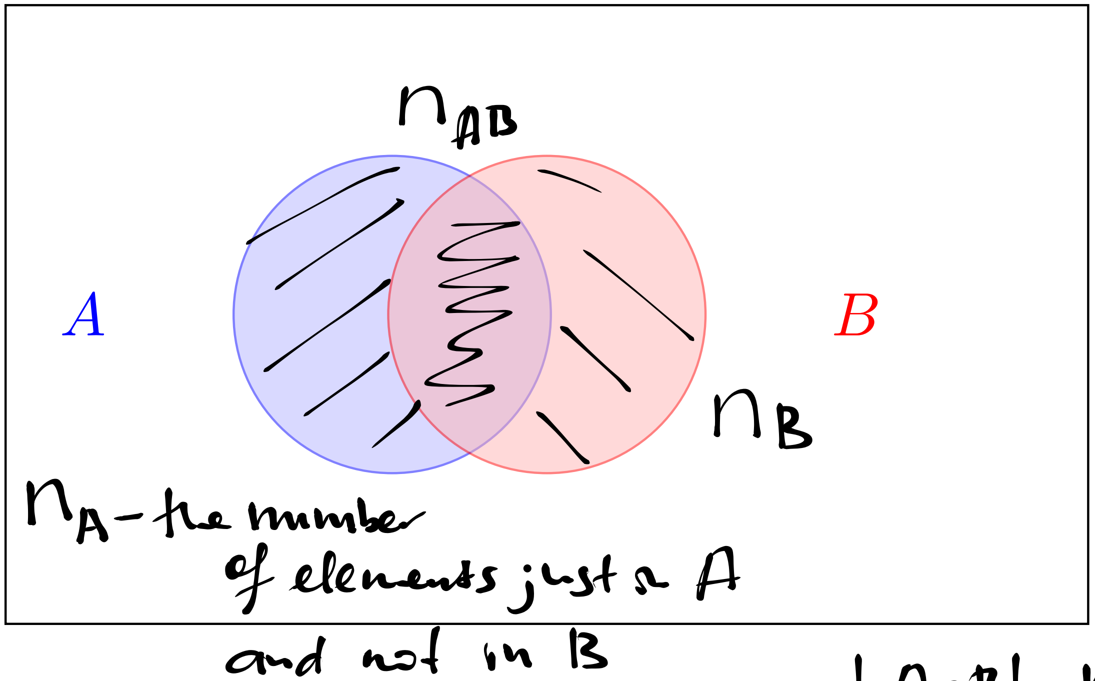
$$|\{1,1,1,1\}| = 1$$

$$|\{1,2,3\}| = 3 \qquad |\emptyset| = 0$$

Computing the cardinality of a union of two sets

If A and B are sets then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$



$$|A \cup B| = n_A + n_B + n_{AB}$$

$$|A| = n_A + n_{AB}$$

$$|B| = n_B + n_{AB}$$

$$|A| + |B| = n_A + n_B + 2n_{AB}$$

$$|A \cap B| = n_{AB}$$

Example

Suppose there are 100 third-year students. 40 of them take the module "Sequential Algorithms" and 80 of them take the module "Multi-Agent Systems". 25 of them took both modules. How many students took neither modules?

S – set of 3rd y. students $|S| = 100$

A – set of 3rd y. st. on the module.. $|A| = 80$

M – " – " – " $|M| = 40$

$|A \cap M| = 25$

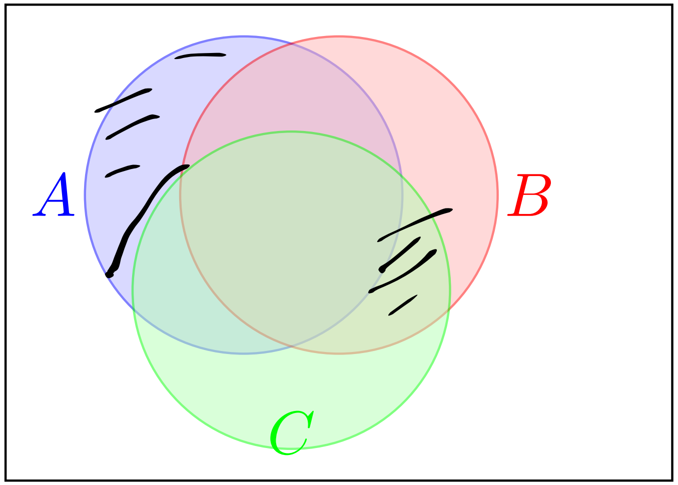
$|S - (A \cup M)|$

$|(A \cup M)| = |A| + |M| - |A \cap M| =$

$80 + 40 - 25 = 95$

Computing the cardinality of a union of three sets

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



These are special cases of the principle of inclusion and exclusion which we will study later.

Proof: We need lots of notation.

- $|A-(B\cup C)| = n_a$,$n_b = |B-(A\cup C)| = n_b$,$|C-(A\cup B)| = n_c$,
- $|(A\cap B)-C| = n_{ab}$,$|(A\cap C)-B| = n_{ac}$,$|(B\cap C)-A| = n_{bc}$,
- $|A\cap B\cap C| = n_{abc}$.

Then

$$\begin{aligned}|A\cup B\cup C| &= n_a + n_b + n_c + n_{ab} + n_{ac} + n_{bc} + n_{abc}\\ &= (n_a + n_{ab} + n_{ac} + n_{abc}) + (n_b + n_{ab} + n_{bc} + n_{abc})\\ &+ (n_c + n_{ac} + n_{bc} + n_{abc}) - (n_{ab} + n_{abc})\\ &- (n_{ac} + n_{abc}) - (n_{bc} + n_{abc}) + n_{abc}\end{aligned}$$

The cardinality of the power set.

For all $n \in \mathbb{N}$ and all sets A: if $|A| = n$, then $|Pow(A)| = 2^n$.

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Base Case: Take $n = 0$. A must be the empty set.

$Pow(\emptyset) = \{\emptyset\}$, which contains $2^0 = 1$ element.

$|A| = 1$

$A = \{1\}$

$\emptyset, \{1\}$

$2 = 2^1$

For all $n \in \mathbb{N}$ and all sets A: if $|A| = n$, then $|Pow(A)| = 2^n$.

Inductive Step: Assume that the property holds for $n = m$, so any set A with $|A| = m$ has $|Pow(A)| = 2^m$. Now consider a set B with $|B| = m + 1$. We must show that $|Pow(B)| = 2^{m+1}$.

Here is how we do it. Give names to the elements of B like this: $B = \{b_1, \ldots, b_{m+1}\}$. Let $B' = \{b_1, \ldots, b_m\}$. Then $|B'| = m$ and, by the induction hypothesis, $|Pow(B')| = 2^m$.

$$Pow(B) = Pow(B') \cup \{C \cup \{b_{m+1}\} \mid C \in Pow(B')\}.$$

$$|Pow(B)| = 2^m + 2^m = 2^{m+1}.$$

Ordered pairs

In discussing sets, the order in which elements are listed is unimportant. In order to handle ordered lists of objects we first introduce:

Definition The cartesian product $A \times B$ of sets A and B is the set consisting of all pairs (a, b) with $a \in A$ and $b \in B$, i.e.,

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Note that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

Example

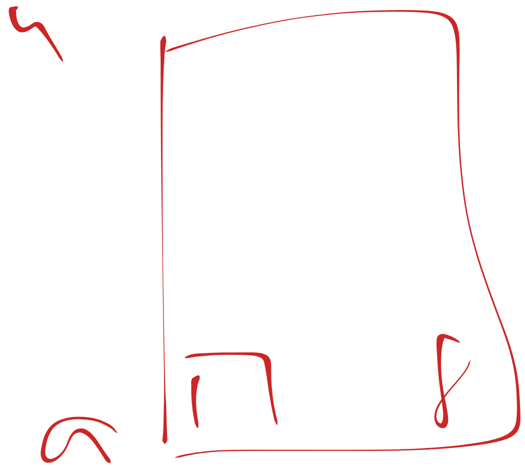
- $\{1, 2\} = \{2, 1\}$ but $(1, 2) \neq (2, 1)$.

Example

Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then

$$A \times B = \{(1, a), (2, a), (1, b), (2, b), (1, c), (2, c)\}.$$

$\{1, \dots 8\} \times \{a, \dots h\}$



Cartesian plane

The set $\mathbb{R} \times \mathbb{R}$, or $\mathbb{R}^2$ as it is often written, consists of all pairs of real numbers (x, y). $\mathbb{R}^2$ is called the *Cartesian plane*.

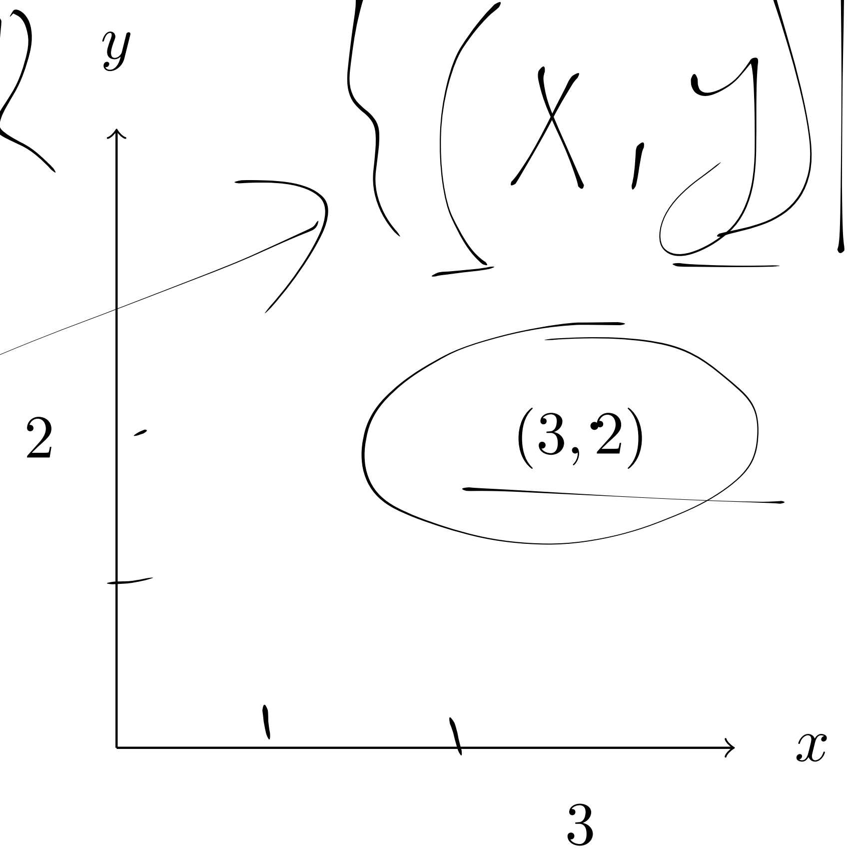


Figure: The Cartesian plane

(x, y, z)

$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$

Definition

The cartesian product $A_1 \times A_2 \times \cdots \times A_n$ of sets $A_1, A_2, \ldots, A_n$ is defined by

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, \ldots, a_n) \mid a_1 \in A_1, \ldots, a_n \in A_n\}.$$

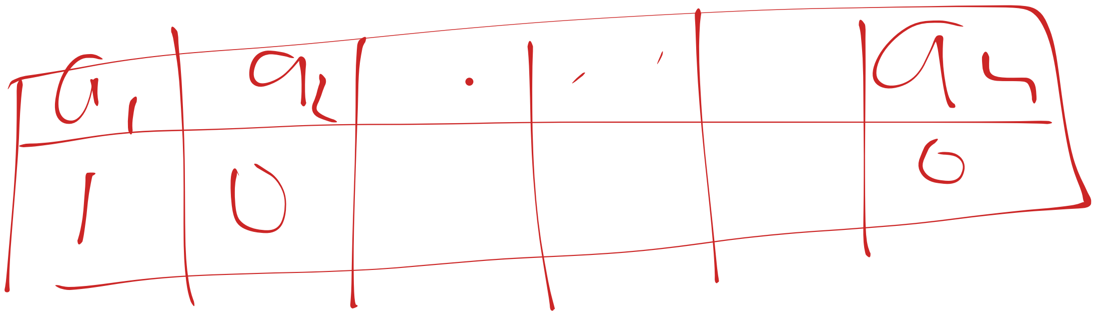
Here $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$ if and only if $a_i = b_i$ for all $1 \leq i \leq n$.

If $A_1, \ldots, A_n$ are all the same set A, then we write A^n to refer to the set

$$A_1 \times \cdots \times A_n.$$

Bit strings of length n

Let $B = \{0, 1\}$. B^n consists of all lists of zeros and ones of length n. These are called *bit strings* of length n.



Bit strings can be used to represent the subsets of a set.

Suppose we have a set $S = \{s_1, \ldots, s_n\}$.

(I have given the n elements of the set names. s_1 is the name of the first element. s_2 is the name of the second element (which is different from the first element) and so on.)

If we have a subset $A \subseteq S$, the characteristic vector of A is the n-bit string $(b_1, \ldots, b_n) \in B^n$ where

$$b_i = \begin{cases} 1 & \text{if} \quad s_i \in A \\ 0 & \text{if} \quad s_i \notin A \end{cases}$$

Example

Let $S = \{1, 2, 3, 4, 5\}$, $A = \{1, 3, 5\}$ and $B = \{3, 4\}$.

- The characteristic vector of A is $(1, 0, 1, 0, 1)$.
- The characteristic vector of B is $(0, 0, 1, 1, 0)$.
- The characteristic vector of $A \cap B$ is $(0, 0, 1, 0, 0)$.
- The characteristic vector of $A \cup B$ is $(1, 0, 1, 1, 1)$.

$\{1, 3, 5\} \cap \{3, 4\} = \{3\}$

Why is this set theory "naive"

It suffers from paradoxes.

A leading example:

A barber is the man who shaves all those, and only those, men who do not shave themselves.

- Who shaves the barber?

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Russell's Paradox

Russell's paradox shows that the 'object' $\{x \mid P(x)\}$ is not always meaningful.

Set $A = \{B \mid B \notin B\}$

Problem: do we have $A \in A$?

Abbreviate, for any set C, by $P(C)$ the statement $C \notin C$. Then $A = \{B \mid P(B)\}$.

- If $A \in A$, then (from the definition of P), not $P(A)$. Therefore $A \notin A$.
- If $A \notin A$, then (from the definition of P), $P(A)$. Therefore $A \in A$.