Efficient (Parallel) Sorting

- One of the most frequent operations performed by computers is organising (sorting) data
- The access to sorted data is more convenient/faster
- There is a constant need for good sorting algorithms including sequential, parallel and distributed solutions
- There is a plethora of sorting algorithms. We already know that one can use *heaps* for sorting. Here we focus on two sorting procedures including *quick-sort* and *merge-sort*

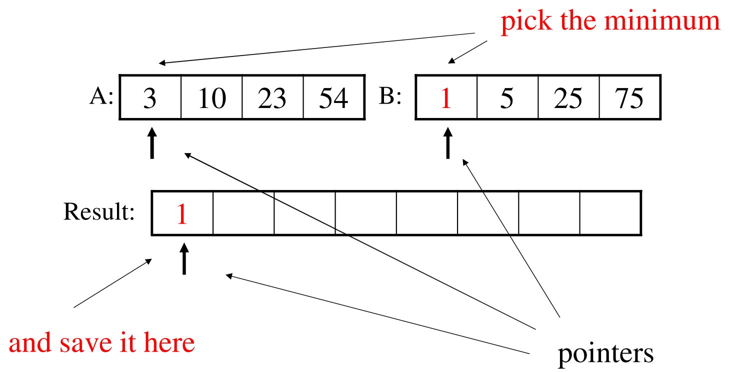
Merging to ordered sequences

- The key to *merge-sort* is merging procedure *merge*, s.t., having two input *sequences*
  - A= $<a_1 \le a_2 \le \cdots \le a_m>$ and B= $<b_1 \le b_2 \le \cdots \le b_n>$
  - it produces combined C= $<c_1 \le c_2 \le \cdots \le c_{m+n}>$
- Example:

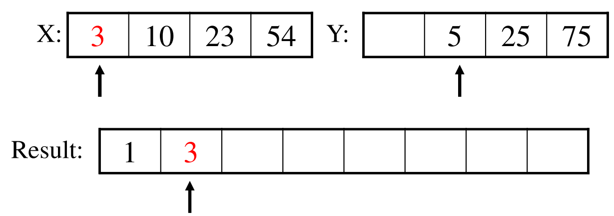
A = < 3, 8, 9 >   B = < 1, 5, 7 >

merge(A, B) = < 1, 3, 5, 7, 8, 9 >

Merging (cont.)



Merging (cont.)



Merging (cont.)

X:		10	23	54

Y:		5	25	75

Result:	1	3	5					

Merging (cont.)

X:		10	23	54

Y:			25	75

Result:	1	3	5	10				

Merging (cont.)

X:			23	54

Y:			25	75

Result:	1	3	5	10	23			

Merging (cont.)

X:				54

Y:			25	75

Result:	1	3	5	10	23	25		

Merging (cont.)

X:

			54

Y:

			75

Result:

1	3	5	10	23	25	54	

Merging (cont.)

X:

Y:

			75

Result:

1	3	5	10	23	25	54	75

- Single run of *merge* procedure produces combined sorted sequence. Thus the time complexity is linear $O(m+n)$.

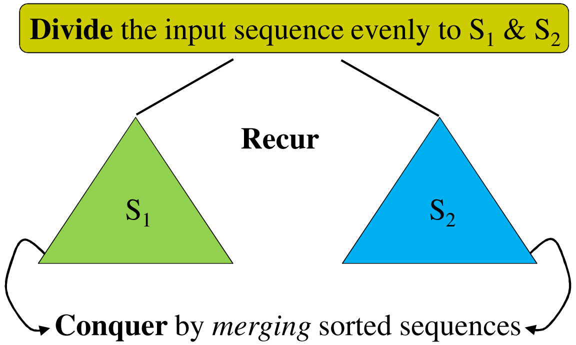
Divide-and-Conquer Method

- A very natural recursive approach
  - Divide
    - if the input size is *small* then solve the problem directly;
    - otherwise *divide* the input data into *two or more disjoint subsets*
  - Recur
    - *recursively* solve the sub-problems associated with the subsets
  - Conquer
    - take the *solutions to the sub-problems* and merge them into a *solution to the original problem*

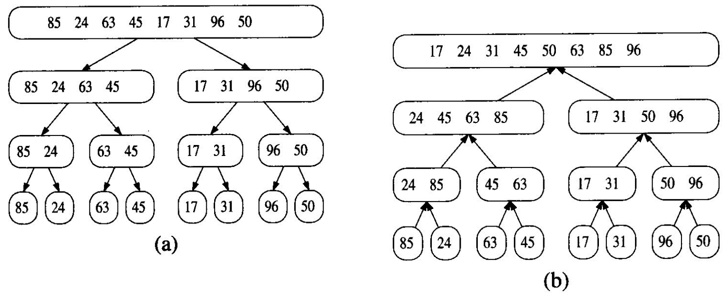
Merge-Sorting

- Divide: if input sequence S has 0 or 1 element then return S; otherwise *split* S into *two* sequences S_1 and S_2, each containing about ½ elements of S
- Recur: *recursively* sort sequences S_1 and S_2
- Conquer: Put the elements back into S by merging the *sorted* sequences S_1 and S_2 into a *single sorted sequence*

Merge-Sorting (top down approach)



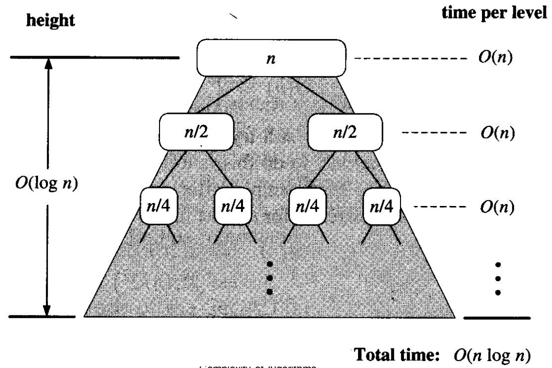
Merge-Sorting (example)



Merge-Sorting (analysis)

- Recall that merging two *sorted sequences* S_1 and S_2 takes $O(n_1+n_2)$ time, where n_1 is the size of S_1 and n_2 is the size of S_2
- The depth of the recursion is $O(\log n)$ due to the halving process
- Thus *merge-sort* runs in $O(n \log n)$ time in the *worst* (and *average*) case

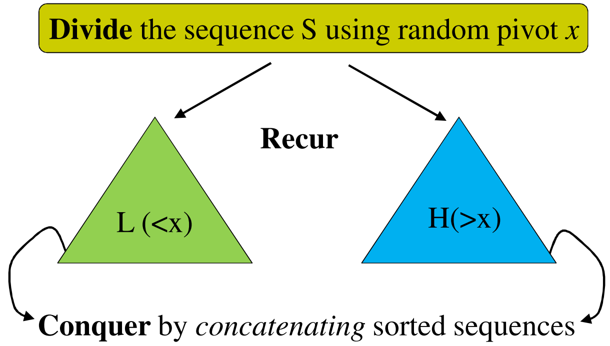
Merge-Sorting (analysis)



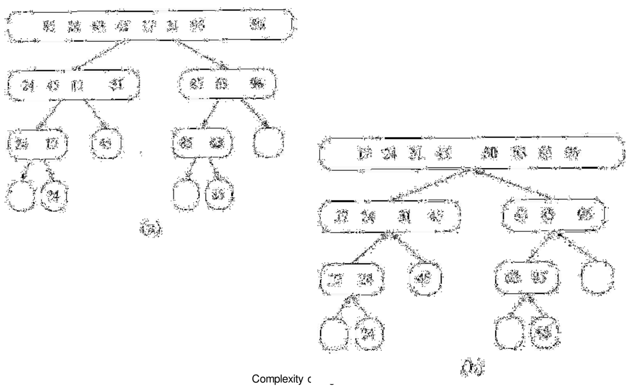
Quick-Sort

- Divide if ¦*S*¦>*1*, select a pivot value x in S and create three sequences: L, E and G, s.t.,
  - L stores elements in $S < x$
  - E stores elements in $S = x$
  - G stores elements in $S > x$
- Recur recursively sort sequences L & G
- Conquer put sorted elements from L, E and finally from G back to S.

Quick-Sort Tree



Quick-Sort (example)



Quick-Sort (worst case)

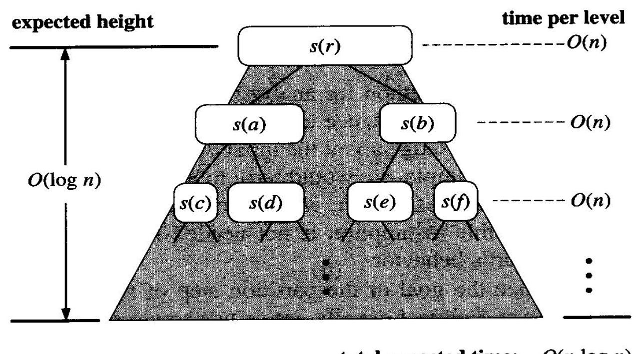
- Let s_i be the sum of the *input sizes* of the nodes at depth i in a quick sort tree T
- $s_i \leq n-i$ (and $s_i = n-i$ when use of pivots lead always to only one nonempty sequence: either L or G)
- The *worst-case complexity* is bounded by $O(n^2)$.

$$O\left(\sum_{i=0}^{n-1} s_i\right), \text{ which is } O\left(\sum_{i=0}^{n-1}(n-i)\right) \text{ that is, } O\left(\sum_{i=1}^{n} i\right)$$

Quick-Sort (randomised algorithm)

- Thm: the *expected running time* of randomised (pivot is chosen in random) quick-sort is *O(n log n)*
- Proof:
  - The *expected number* of times that a *fair coin* must be flipped until it shows heads *k* times is *2k*.
  - Randomly chosen pivot is right if neither of the groups L nor G is > ¾ ¦S¦
  - The probability of a success in choosing a *right pivot* is ½
  - A path in *quick-sort* tree can contain at most $\log_{4/3}$ n nodes with *right pivots*
  - Hence, the expected length of each path is $2\log_{4/3}$ n

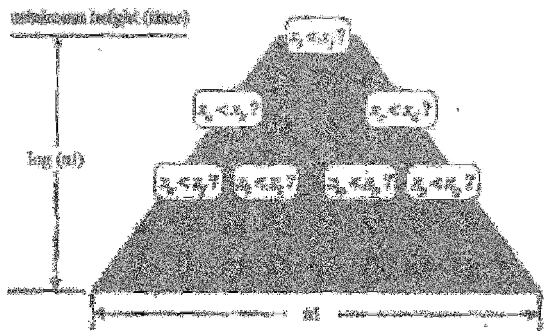
Quick-Sort (randomised algorithm)



Lower Bound (comparison-based model)

- In comparison-based model the input elements can be compared *only with themselves* and the result of each comparison $x_i \leq x_j$ is always *yes* or *no*
- Thm: the running time of any comparison-based sorting algorithm is **Ω**(n log n) in the worst case
- Proof:
  - Sorting of n elements can be identified with recognising a particular permutation of n elements
  - There is n!=n·(n-1) ·…·2·1 permutations of n elements
  - Each comparison splits a group of permutations into two groups (one that satisfies the inequality and one that doesn't)
  - In order to ensure that the size of each group of permutations is brought down to one we need $\log_2(n!) > \log (n/2)^{n/2} = n/2 \cdot \log n/2 = \Omega(n \log n)$ comparisons

Lower Bound (comparison-based model)



List ranking and prefix sums

- In the *link ranking problem* one is expected to compute for each element its distance to the front of the list
- In the *prefix sum problem* one is expected to compute for each prefix of the list the sum of the keys stored in this prefix
- Computing prefix sums with all keys of value 1 is equivalent to the *link ranking problem*.

List ranking and prefix sums

P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}
3	0	0	**5**	**7**	0	0	**2**	0	0	0	**4**	0	**8**	0	**1**
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

3	0	0	**5**	**7**	0	0	**2**	0	0	0	**4**	0	**8**	0	**1**
1	0	0	1	1	0	0	1	0	0	0	1	0	1	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

All read at distance 2^0 & add to their own values

3	0	0	**5**	**7**	0	0	**2**	0	0	0	**4**	0	**8**	0	**1**
1	0	0	1	1	0	0	1	0	0	0	1	0	1	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

3	3	0	**5**	**12**	7	0	**2**	2	0	0	**4**	4	**8**	8	**1**
1	1	0	1	2	1	0	1	1	0	0	1	1	1	1	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

All read at distance 2^1 & add to their own values

3	3	0	**5**	**12**	7	0	**2**	2	0	0	**4**	4	**8**	8	**1**
1	1	0	1	2	1	0	1	1	0	0	1	1	1	1	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

3	3	3	**8**	**12**	12	12	**9**	2	2	2	**4**	4	**12**	12	**9**
1	1	1	2	2	2	2	2	1	1	1	1	1	2	2	2
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

All read at distance 2^2 & add to their own values

3	3	3	8	12	12	12	9	2	2	2	4	4	12	12	9
1	1	1	2	2	2	2	2	1	1	1	1	1	2	2	2
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

3	3	3	8	15	15	15	17	14	14	14	13	6	14	14	13
1	1	1	2	3	3	3	4	3	3	3	3	2	3	3	3
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

All read at distance 2^3 & add to their own values

3	3	3	8	15	15	15	17	14	14	14	13	6	14	14	13
1	1	1	2	3	3	3	4	3	3	3	3	2	3	3	3
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

3	3	3	8	15	15	15	17	17	17	17	21	21	29	29	30
1	1	1	2	3	3	3	4	4	4	4	5	5	6	6	7
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

List ranking and prefix sums

P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}
3	0	0	5	7	0	0	2	0	0	0	4	0	8	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

3	3	3	8	15	15	15	17	17	17	17	21	21	29	29	30
1	1	1	2	3	3	3	4	4	4	4	5	5	6	6	7
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

List ranking and prefix sums

- List ranking and prefix sums can be computed in *O(log n)* time when n is the size of the input
  - During every single round we increase knowledge about preceding block of 2^i positions in *O(1)* time.
  - After O(log n) rounds od doubling the job is done
- We need also another tool that will allow us to *collect* and *distribute* information to all processors also in *O(log n)* time

Information dissemination

- P_0 informs neighbour at distance 2^0

P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}
m	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

m	m	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Information dissemination

- P_0 , P_1 inform neighbours at distance 2^1

P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}
m	m	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

m	m	m	m	0	0	0	0	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Information dissemination

- P_0, P_1 P_2, P_3 inform neighbours at distance 2^2

P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}
m	m	m	m	0	0	0	0	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

m	m	m	m	m	m	m	m	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Information dissemination

- P_0, …, P_6, P_7 inform neighbours at distance 2^3

P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}
m	m	m	m	m	m	m	m	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

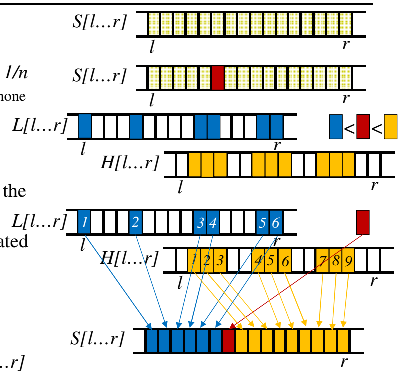
m	m	m	m	m	m	m	m	m	m	m	m	m	m	m	m
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Information collection/dissemination

- The process of collection of information is done by reversing communication (direction of arrows) used during information dissemination
- Both processes take time $O(\log n)$.
- This means that processors can all agree on simple decisions (via exchanging small messages), e.g., *"is there any work left to do?"* in time $O(\log n)$.

Parallel Quick-Sort

- Sequence $S[l…r]$ is being sorted
  - The local size of the input $n= r-l+1$
- Each P_i $(i=l,…,r)$ picks value $S[i]$ with prob. $1/n$
  - A unique pivot value p is communicated to all (if none or more values are picked the process is repeated)
- The values from S[l...r] are distributed to $L[l…r]$ & $H[l…r]$
- Using list ranking and prefix sums compute the ranks of values in L and H
- The number of values #L in L is communicated
- The values are copied back to S as follows
  - Value with rank α in L is moved to S[l+α-1]
  - The pivot p is moved to S[l+#L]
  - Value with rank β in H is moved to S[l+#L+β-1]
- Sort recursively $S[l…l+\#L-1]$ & $S[l+\#L+1…r]$

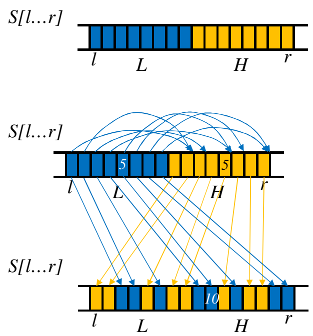


Parallel Quick-Sort

- The complexity analysis of parallel quick-sort
  - Every stage takes at most time $O(\log n)$
  - Expected number of stages is $O(\log n)$
  - The total computation time is $O(\log^2 n)$
  - The number of processors needed is n
  - The total work is $O(n\log^2 n)$
  - One can reduce work to optimal using $n/\log n$ processors

Parallel Merge-Sort

- Assume two halves L & H of $S[l…r]$ are already (recursively) sorted
  - The local size of the input $n= r-l+1$
- Using binary search compute a rank of each L value in the other half H, and vice versa
- Combine (add) the two ranks (from L and H) to find the new position in the sorted sequence



Parallel Merge-Sort

- The complexity analysis of parallel merge-sort
  - Each stage (binary search) takes at most time $O(\log n)$
  - The number of recursive stages is $O(\log n)$
  - The total computation time is $O(\log^2 n)$
  - The number of processors needed is n
  - The total work is $O(n\log^2 n)$
  - One can reduce work to optimal using $n/\log n$ processors