Traveling Salesman Problems in Temporal Graphs

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joint work with
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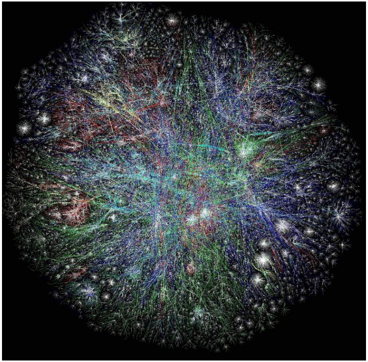
Motivation

A great variety of systems are dynamic:

- Modern communication networks: inherently dynamic, dynamicity may be of high rate
  - mobile ad hoc, sensor, peer-to-peer, opportunistic, and delay-tolerant networks
- Social networks: social relationships between individuals change, existing individuals leave, new individuals enter

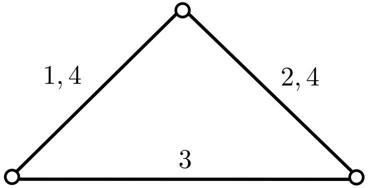


- Transportation networks: transportation units change their positions in the network as time passes
- Physical systems: e.g. systems of interacting particles

Definition (Temporal Graph)

A temporal graph (or dynamic graph) D is an ordered pair of disjoint sets (V, A) such that $A \subseteq \binom{V}{2} \times \mathbb{N}$. The set V is the set of nodes and the set A is the set of time-edges.

- Loosely speaking a graph that changes with time
- Labels indicate availability times of edges



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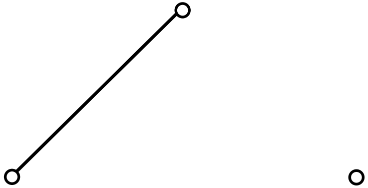

$t = 1$

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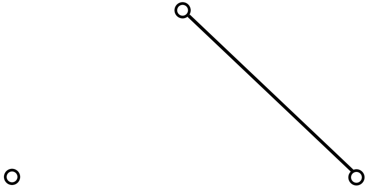

$t = 2$

Temporal Graphs

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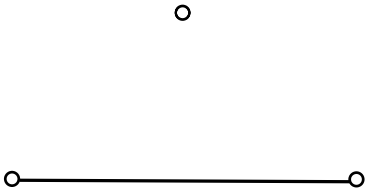

$t = 3$

Definition (Temporal Graph)

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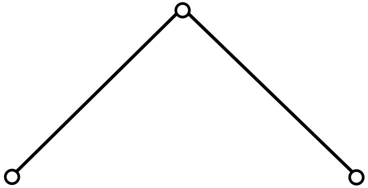
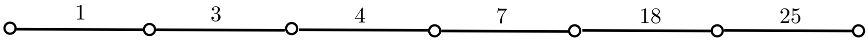


$t = 4$

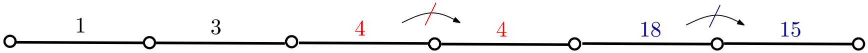
Time-Respecting Paths

- Paths with strictly increasing labels
- a.k.a. journeys
- A journey:



Time-Respecting Paths

- Paths with strictly increasing labels
- a.k.a. journeys
- A non-journey:



State of the Art

- The structural and algorithmic properties of temporal graphs are not well understood yet
- Many dynamic languages derived from **NP**-complete languages can be shown to be **PSPACE**-complete [Orlin, STOC '81]
- max-flow min-cut holds with unit capacities [Berman, Networks '96]
- Classical Menger's theorem is violated [KKK, STOC '00]
- Reformulation of Menger's theorem valid for all temporal graphs & parameters for optimal temporal network design [MMCS, ICALP '13]
- Distributed Computing on Dynamic Networks
    - Worst-case dynamicity [KLO, STOC '10], [MCS, JPDC '14]
    - Population Protocols (interacting automata) [AADFP, Distr. Comp. '06], [MCS, Book, '11], [MS, PODC '14]
    - Randomly Dynamic Networks [CMMPS, PODC '08]

Overview-Contribution

- Introduce the notion of time to well-known combinatorial optimization problems
- Main focus on temporal analogues of traveling salesman problems
- Exploring the nodes of a temporal graph as soon as possible
  - Cannot be approximated within cn, for some constant $c > 0$, in general temporal graphs
  - and within $(2 - \varepsilon)$, for every constant $\varepsilon > 0$, in the special case in which $D(t)$ is connected for all $1 \leq t \leq l$
- TTSP(1,2) (best approximations):
  - $(1.7 + \varepsilon)$-factor for the generic TTSP(1,2)
  - $(13/8 + \varepsilon)$-factor when the lifetime is restricted to n
- In the way, we introduce and study temporal versions of other fundamental combinatorial optimization problems

Temporal (Node) Exploration

Problem (TEXP)

Given a temporal graph $D = (V, A)$ and a source node $s \in V$, find a temporal walk that begins from s and visits all nodes minimizing the arrival time.

- Temporal version of the well-known Graphic TSP
- In the static case, there is a $(3/2 - \varepsilon)$-approximation for undirected graphs [GSS, FOCS '11] and a $O(\log n / \log \log n)$ for directed [AGMGS, SODA '10]. In contrast:

Theorem

There exists some constant $c > 0$ such that TEXP cannot be approximated within cn unless $\mathbf{P} = \mathbf{NP}$.

Theorem

For every constant $\varepsilon > 0$, there is no $(2-\varepsilon)$-approximation for TEXP in continuously (strongly) connected temporal graphs unless $\mathbf{P} = \mathbf{NP}$.

Proof.

- Reduction from HAMPATH (input graph G, source s)
- D consists of 3 strongly connected static graphs T_1, T_2, T_3 persisting for the intervals $[1, n_1 - 1]$, $[n_1, n_2 - 1]$, $[n_2, 2n_2 + 1]$, resp.
- Restrict attention to instances of HAMPATH of order at least $2/\varepsilon$
- Set $n_2 = n_1^2 + n_1$
- If G is hamiltonian, then $\mathrm{OPT} = n_1 + n_2 - 1 = n_1^2 + 2n_1 - 1$ while if G is not hamiltonian, then $\mathrm{OPT} \geq 2n_2 + 1 = 2(n_1^2 + n_1) + 1 > 2(n_1^2 + n_1)$ which can be shown to introduce the desired $(2-\varepsilon)$ gap

□

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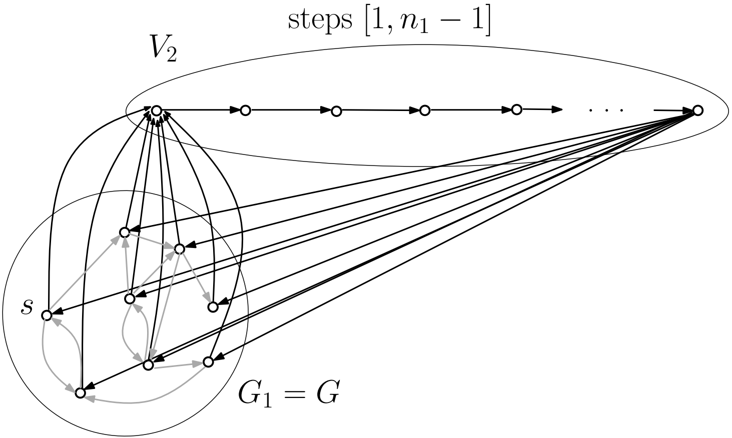
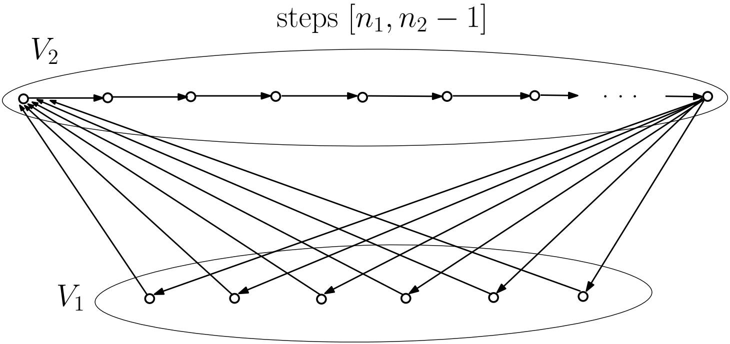
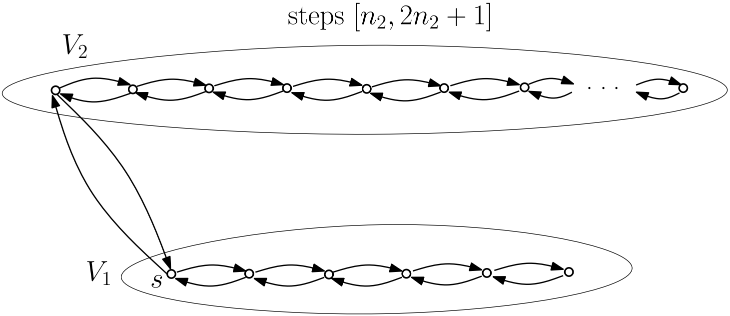
Temporal Exploration: Graph T_1



Temporal Exploration: Graph T_2



Temporal Exploration: Graph T_3



On the positive side:

Theorem

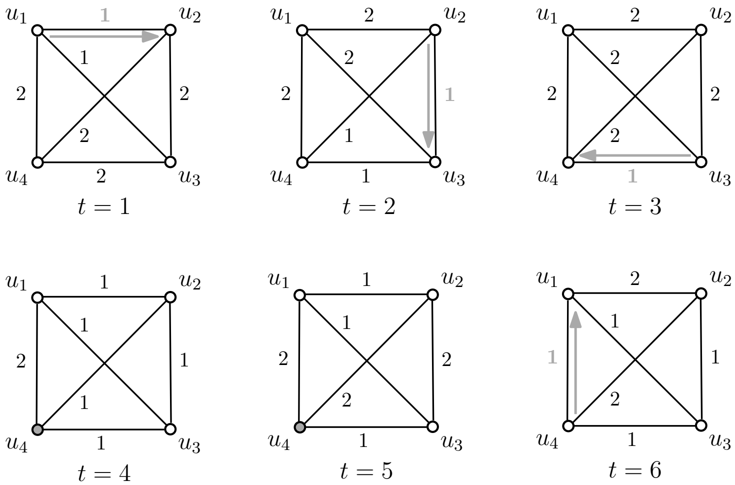
We provide a d-approximation algorithm for TEXP restricted to temporal graphs with dynamic diameter $\leq d$ and lifetime $\geq (n-1)d$.

Problem (TTSP(1,2))

Given a complete temporal graph $D = (V, A)$ and a cost function $c : A \to \{1, 2\}$ find a temporal TSP tour of minimum total cost. A TSP tour $(u_1, t_1, u_2, t_2, \ldots, t_{n-1}, u_n, t_n, u_1)$ is temporal if $t_i < t_{i+1}$ for all $1 \leq i \leq n-1$. The cost of such a TSP tour is $\sum_{1 \leq i \leq n} c((u_i, u_{i+1}), t_i)$, where $u_{n+1} = u_1$.

- **APX**-hard as a generalization of the well known (A)TSP(1,2) to weighted temporal graphs [PY, Math. Oper. Res. '93]
- Cannot be approximated within any factor less than 207/206 [KS, CATS '13]

TTSP(1,2): Example



Approximating TTSP(1,2): Our Approach

- Compute a temporal matching M using as many 1s as possible and then patch the edges of M in a time-respecting way to obtain a TTSP tour
- This approach gives a $(3/2)$-factor approximation for ATSP(1,2) (the best currently known is $5/4$ [Bläser '04])

However:

- Computing temporal matchings is **NP**-hard.

Max-TEM($\geq k$) is **NP**-hard

Definition (Max-TEM($\geq k$))

Given a temporal graph $D = (V, A)$ find a maximum cardinality temporal matching $M = \{(e_1, t_1), (e_2, t_2), \ldots, (e_h, t_h)\}$ satisfying that there is a permutation $t_{i_1}, t_{i_2}, \ldots, t_{i_h}$ of the t_js s.t. $t_{i_{(l+1)}} \geq t_{i_l} + k$ for all $1 \leq l \leq h - 1$.

Theorem

Max-TEM($\geq k$) is **NP***-hard for every independent of the lifetime polynomial-time computable $k \geq 1$.*

Proof.

Reduction from BALANCED 3SAT in which every variable x_i appears n_i times negated and n_i times non-negated. □

Approximating TTSP(1,2): Our Approach

- We approximate TTSP(1,2) by approximating temporal matchings. We follow 2 approaches:
  1. Via independent sets in k-claw free graphs
  2. Via k-Set Packing

Via Independent Sets

Theorem

There is a (3/5)-approximation algorithm for Max-TEM(≥ 1).

Proof

- *Consider the static expansion $H = (S, E)$ of D and an edge $e = (u_{(i-1)j}, u_{ij'}) \in E$*
- *Conflicts (edges that cannot be taken together in the matching):*
  1. *Edges of the same row as e*
  2. *Edges of the same column as $u_{(i-1)j}$*
  3. *Edges of the same column as $u_{ij'}$*
- *Consider the graph $G = (E, K)$ where $(e_1, e_2) \in K$ iff e_1 and e_2 satisfy some of the above constraints*
- *Temporal matchings of D are now equivalent to independent sets of G*

Via Independent Sets

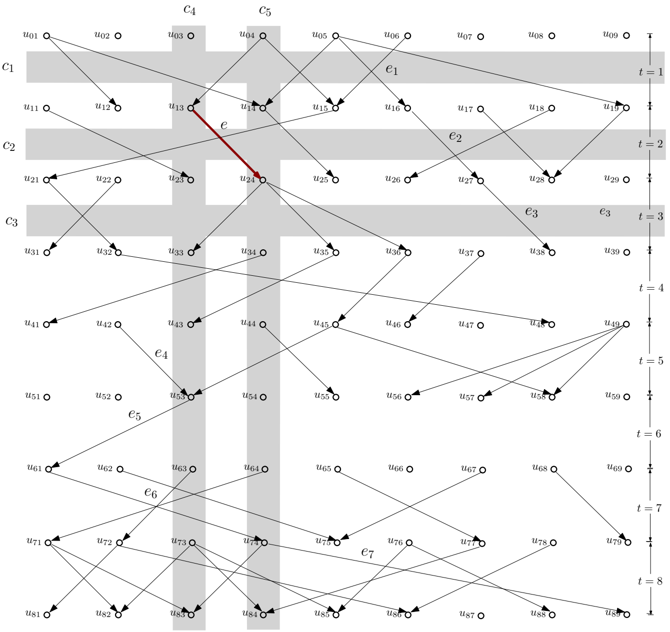
- *G is 4-claw free $\Rightarrow$ there is no 4-independent set in the neighborhood of any node*
  - *Take any $e \in E$ and any set $\{e_1, e_2, e_3, e_4\}$ of four neighbors of e in G*
  - *There are only 3 constraints thus at least two of the neighbors, say e_i and e_j, must be connected to e by the same constraint*
  - *But then e_i and e_j must also satisfy the same constraint with each other $\Rightarrow$ they are also connected by an edge in G*
- *From [Halldórsson, SODA '95] we have a factor of $3/5$ for MIS in 4-claw free graphs ($1/(h/2+\varepsilon)$ in $(h+1)$-claw free graphs, $h \geq 4$)*

$\square$

Lemma

There is a $\frac{1}{2+\varepsilon}$-approximation algorithm for Max-TEM(≥ 2).

Via Independent Sets



Via Independent Sets

Lemma

An $(1/c)$-factor approximation for Max-TEM(≥ 2) implies a $(2 - \frac{1}{2c})$-factor approximation for TTSP(1,2).

Theorem

There is a $(7/4 + \varepsilon)$-approximation algorithm for TTSP$(1, 2)$.

Theorem

There is a $(12/7 + \varepsilon)$-factor approximation algorithm for TTSP$(1, 2)$ when $\alpha(D) = n$.

Proof.

Approximate TEMPORAL PATH PACKING via reduction to MIS in 8-claw free graphs. □

Via Set Packing

Theorem

There is a $(1.7 + \varepsilon)$-approximation algorithm for $TTSP(1, 2)$.

Proof

- *Suffices to give a $\frac{3}{5+\varepsilon}$-approximation algorithm for Max-TEM(≥ 2)*
- *Express the temporal matching problem as a 4-Set Packing*
- *k-Set Packing can be approximated within $3/(k + 1 + \varepsilon)$ [Cygan, FOCS '13] yielding $3/(5 + \varepsilon)$ for $k = 4$*
- *k-Set Packing: Given family $F \subseteq 2^U$ of sets of size at most k, (U is some universe) find a maximum size subfamily of F of pairwise disjoint sets*
- *Given $D = (V, A)$, set $U = V \cup \{1, 2, \ldots, \alpha(D)\}$*

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Via Set Packing

- *Let $H = (S, E)$ be the static expansion of D*
- *Construct F: For every $(u_{ij}, u_{(i+1)j'}) \in E$ set $F \leftarrow F \cup \{\{u_j, u_{j'}, i-1, i\}\}$*
- $\{u_j, u_{j'}, (i-1), i\} \in 2^U$ *because u_j, $u_{j'}$, $i-1$, and i are pairwise distinct elements, thus $F \subseteq 2^U$*
- *Also every set contains 4 elements, thus we have an instance of* 4-Set Packing
- *Observe now that there is a temporal matching of size h in D iff there is a packing of F of size h*

$\square$

Via Set Packing

Theorem

There is a $(13/8+\varepsilon)$-factor approximation algorithm for $TTSP(1,2)$ when $\alpha(D)=n$.

Proof

- *Every TTSP tour must use precisely the time-labels $1,2,\ldots,n$, otherwise it cannot cover all nodes in n steps*
- *So, the optimum TTSP tour can be partitioned into two temporal matchings M_O (odd) and M_E (even) both with time differences ≥ 2 between consecutive labels*
- *$o(D')$: number of edges of cost one of a single-label subgraph D' of the temporal graph D. We have $o(M_O)+o(M_E)=2n-OPT_{TTSP}$*
- *We now approximate the maximum odd (OPT_O) and maximum even (OPT_E) matchings of D by expressing it as a 3-SET PACKING*

Via Set Packing

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- *We now approximate the maximum odd* (OPT_O) *and maximum even* (OPT_E) *matchings of* D *by expressing it as a* 3-Set Packing

Via Set Packing

- *We get:* $ALG_O \geq \frac{3}{4}OPT_O$ *and* $ALG_E \geq \frac{3}{4}OPT_E$*. From the two computed matchings we keep the one with maximum cardinality. Denote its cardinality by* ALG_M*. Clearly,* $2ALG_M \geq ALG_O + ALG_E$*, so we have*

$$ALG_M \geq \frac{1}{2}(ALG_O + ALG_E) \geq \frac{1}{2} \cdot \frac{3}{4}(OPT_O + OPT_E) = \frac{3}{8}(OPT_O + OPT_E)$$
$$\geq \frac{3}{8}[o(M_O) + o(M_E)] = \frac{3}{8}(2n - OPT_{TTSP}) = \frac{6}{8}n - \frac{3}{8}OPT_{TTSP}$$

- *Complete the matching arbitrarily with the missing edges to obtain a TTSP tour.* ALG_{TTSP}*: cost of the produced TTSP tour.*

$$ALG_{TTSP} \leq 2n - ALG_M \leq 2n - \frac{6}{8}n + \frac{3}{8}OPT_{TTSP} = \frac{10}{8}n + \frac{3}{8}OPT_{TTSP}$$
$$\leq \frac{10}{8}OPT_{TTSP} + \frac{3}{8}OPT_{TTSP} = \frac{13}{8}OPT_{TTSP}. \qquad \square$$

Open Problems

- Find a (3/2)-approximation for the general TTSP(1,2) or for the special case with lifetime restricted to n
- Approximations for temporal path packings and temporal cycle covers
  - Have proved very useful for approximating TSP in the static case
- How does the generic metric TSP behave in temporal graphs?
  - Is there some temporal analogue of triangle inequality or some other computationally equivalent natural assumption?
- Temporal graphs defined by the mobility patterns of mobile wireless entities modeled by a sequence of unit disk graphs
  - Well-motivated as a natural source of temporal graphs
  - May allow for better approximations
- Our results are a first step towards answering the following fundamental question:

  To what extent can algorithmic and structural results of graph theory be carried over to temporal graphs?

Thank You!