Network Constructors: A Model for Programmable Matter

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joint work with
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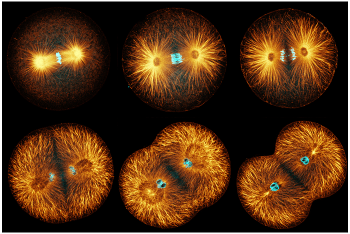
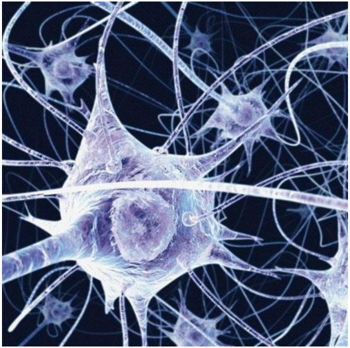
General Motivation

- A wide range of physical/biological systems are governed by algorithmic laws
- Usually collections of very large numbers of simple distributed entities
- Higher-level properties are the outcome of coexistence and constant interaction (cooperative and/or competing) of such entities
- Goal:
  - Reveal the algorithmic aspects of physical systems
  - Develop innovative artificial systems inspired by them





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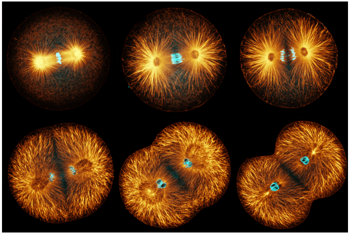

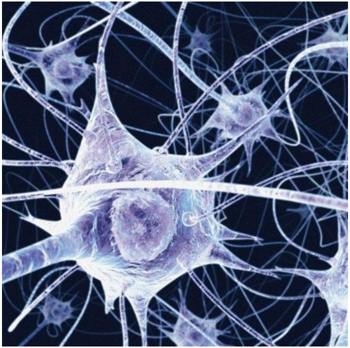

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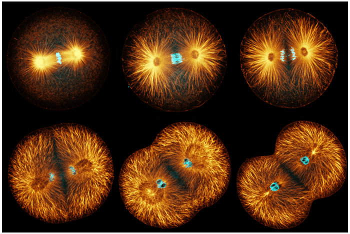

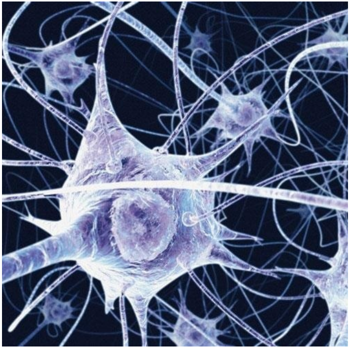

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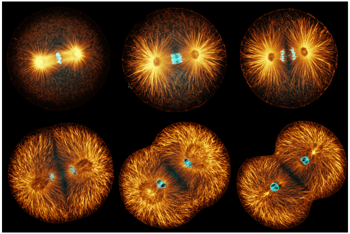

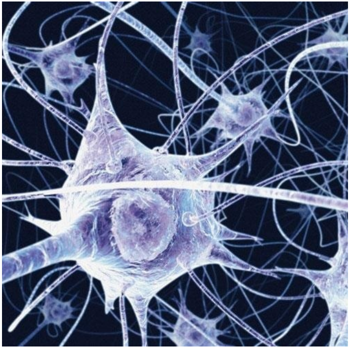

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- Cellular Automata model neural activity, self-replication, bacterial growth, ...
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- Network Constructors [Michail, Spirakis, PODC '14; Michail, PODC '15]: abstract and simple model of distributed network formation
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- Models of programmable matter equipped with active mobility/ actuation mechanisms

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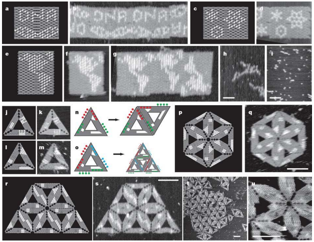

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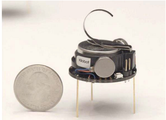

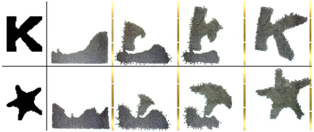

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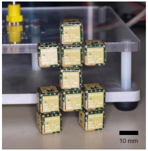

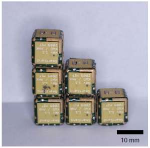

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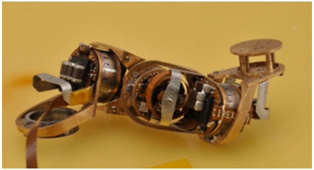

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Long-term Vision: Programmable Matter

- Manipulate matter via information-theoretic and computing mechanisms and principles
- Incorporation of information to the physical world
- Plausible future outcome of progress in high-volume nanoscale assembly
- Physical realization of any computer-generated object
- Profound implications for how we think about chemistry and materials
- Materials will become user-programmed, smart, and adaptive
- It will change the way we think about engineering and manufacturing

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Population Protocols

[Angluin, Aspnes, Diamadi, Fischer, and Peralta, PODC '04]

Distributed model. n computational entities, called nodes

1. interact in pairs
2. cannot control their interactions
   - passive mobility, like particles in a well-mixed solution
   - fair adversary or uniform random scheduler
3. have constant memory (uniformity)
4. do not have unique ids (anonymity)
5. $\delta : Q \times Q \rightarrow Q \times Q$: transition function

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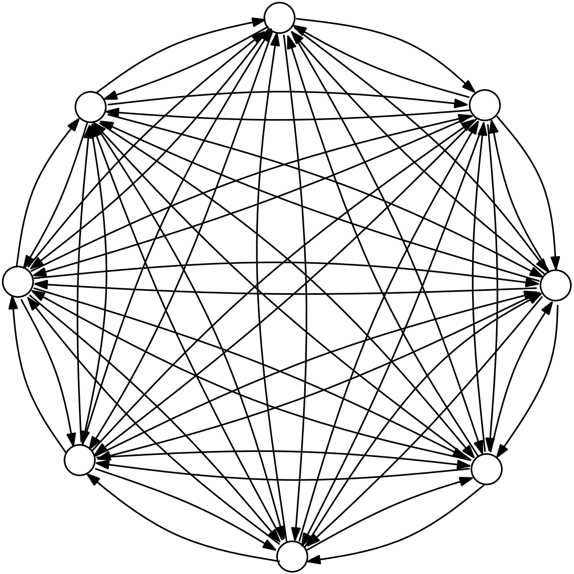

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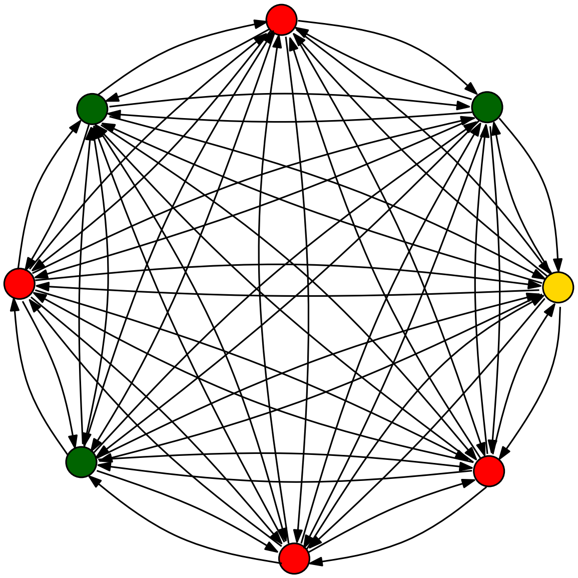

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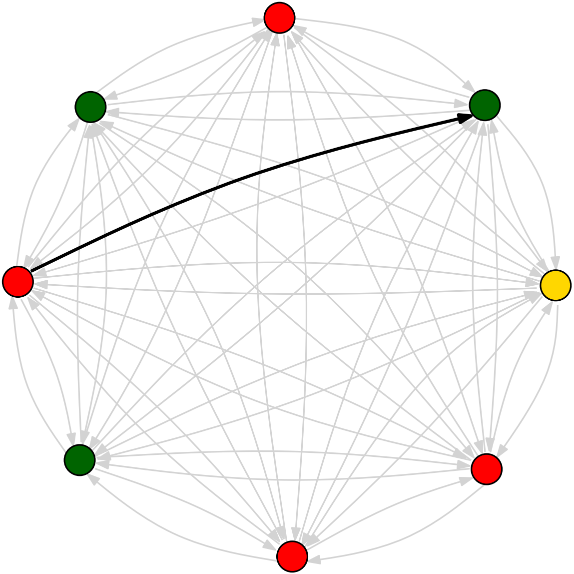

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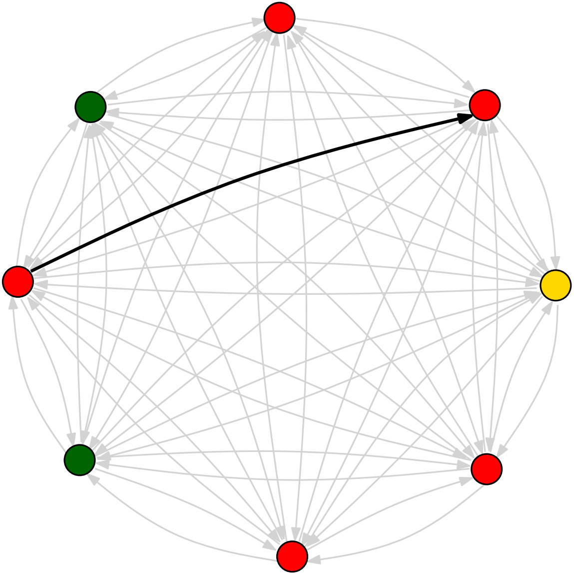

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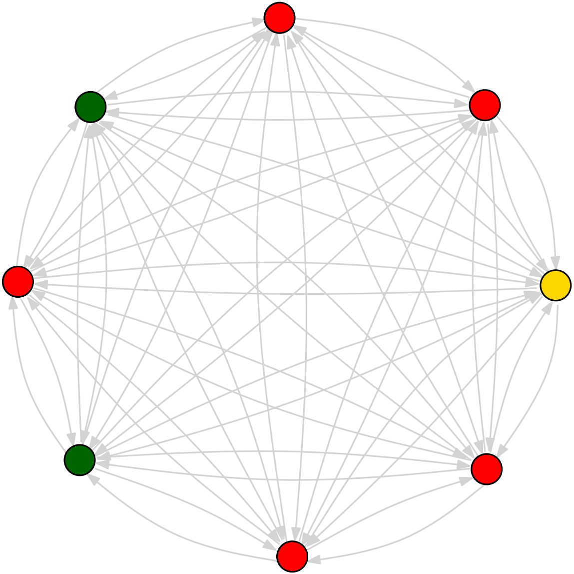

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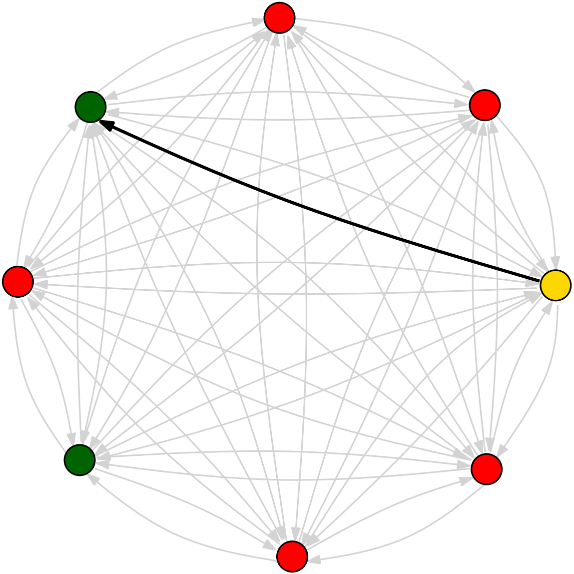

PPs and Natural Processes

- May be viewed as an abstraction of "fast-mixing" physical systems
- Chemical Reaction Networks
  - Finite sets of chemical reactions such as $A + B \to A + C$
  - Promising programming language for molecular control circuitry
  - Some CRNs can simulate space-bounded Turing machines
- PPs are formally equivalent to CRNs [Doty '14]
  - Consequence: bounds for PPs, usually translate to inherent properties of natural systems
  - e.g., time-consuming to generate exact quantities of molecular species quickly [DS15]
- Relations to self-assembly, gene regulatory networks, opinion spreading, and antagonism of species models
- A predicate is computable by PPs iff it is semilinear [AAER '07]

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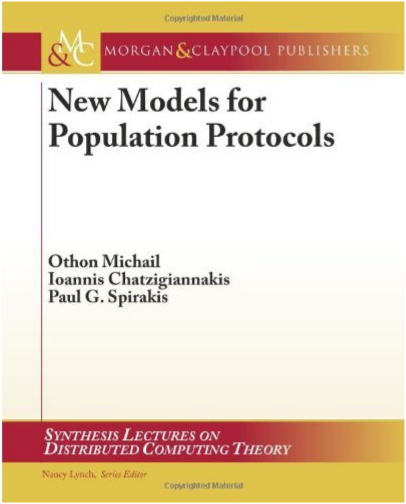
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Dynamic Environments: A Motivating Example

- n tiny computational devices
- Injected into a human circulatory system for monitoring/treating
- Move and interact passively (blood flow)
- Cooperation: can create bonds with each other
- Self-assemble into a desired global structure/network
- The artificial population evolves greater complexity, better storage capacity, and adapts and optimizes its performance to the specific task to be accomplished

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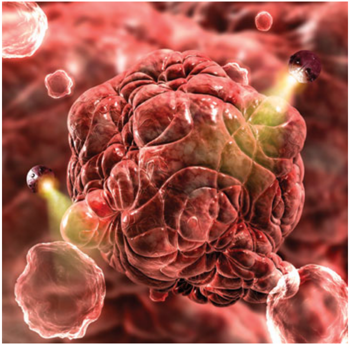

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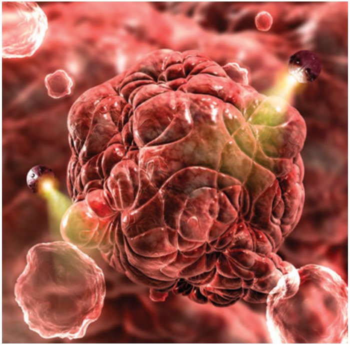

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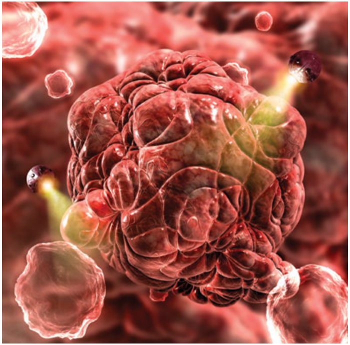

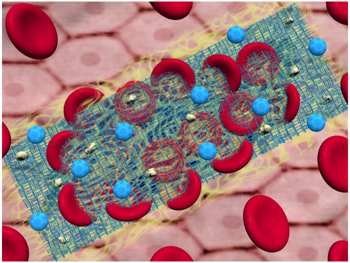

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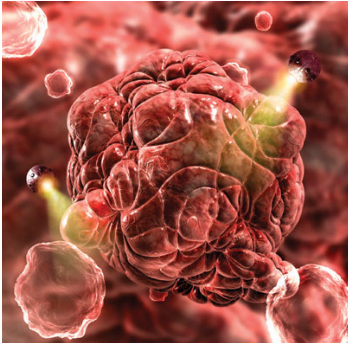

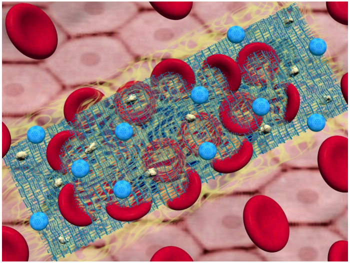

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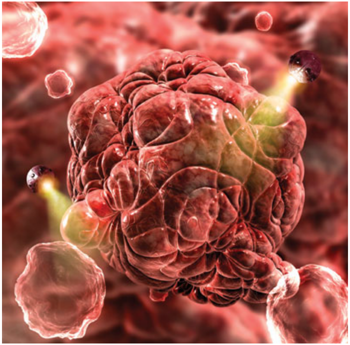
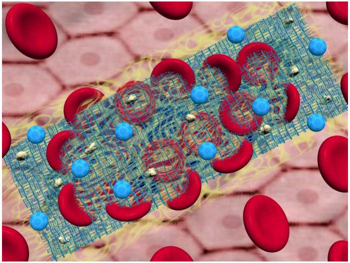
- The artificial population evolves greater complexity, better storage capacity, and adapts and optimizes its performance to the specific task to be accomplished





Distributed Network Construction

[Michail and Spirakis, PODC '14 and Distrib. Comput. '16]

Fundamental problem: Algorithmic distributed construction of an actual communication topology

- Processes can form/delete connections between them
- Physical or virtual connections depending on the application
- on/off case: a connection either exists (active) or does not exist (inactive)
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A Minimal Model

- Processes are finite automata
- Homogeneity: All begin from the same initial state and execute the same finite program
- Adversarial environment: Fair adversary scheduler choosing pairwise interactions (a la population protocols)
- Uniform random scheduler to estimate efficiency
- Complex global behaviour via simple, uniform, anonymous, homogeneous, and cooperative entities

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Related Work

- PPs/CRNs can only capture the dynamics of molecular counts and not of structure formation
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- We aim to capture the stable structures that may occur in a well-mixed solution

Our Goal: Determine what stable structures can result in such systems, how fast, and under what conditions (e.g. by what underlying codes/reaction-rules)

Applications

- Principles of algorithmic network formation
- Programmable matter
- Reconfigurable robotics
- Medicine (diagnosis/treatment)
- Smart material (that self-built, self-adjust, ...)
- Biomaterial manufacture
- Modeling and understanding network formation by physical/chemical/biological processes (e.g. molecules reacting in a well-mixed solution)

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The Model of Network Constructors

1. Q: finite set of node-states,
2. $q_0 \in Q$: initial node-state,
3. $Q_{out} \subseteq Q$: set of output node-states, and
4. $\delta : Q \times Q \times \{0,1\} \rightarrow Q \times Q \times \{0,1\}$: the transition function

- In every step, a pair uv is selected by the scheduler and u, v interact according to δ
- Fair scheduler: If a configuration is reachable infinitely often then it is eventually reached
- Output network: nodes that are in output states and edges between them that are active
- Stability: The output network cannot change in future steps

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An Illustration: Spanning Star

- 2 states: black and red
- Initially all black
- Construct a global star
- Program:

$$(b, b, 0) \rightarrow (b, r, 1)$$
$$(r, r, 1) \rightarrow (r, r, 0)$$
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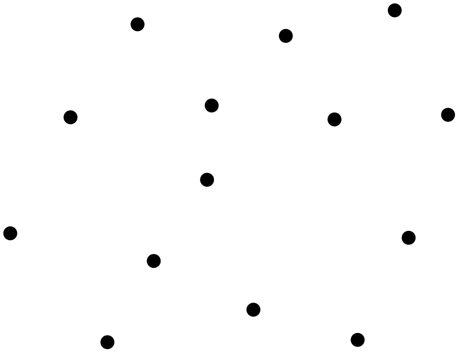

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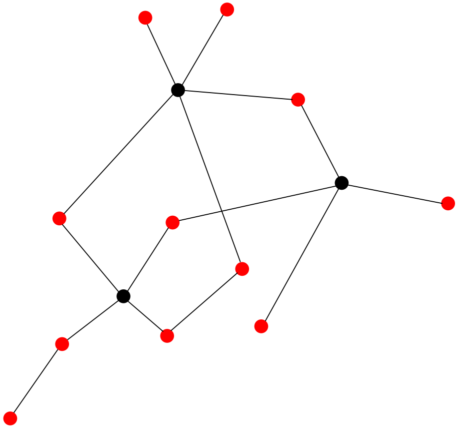

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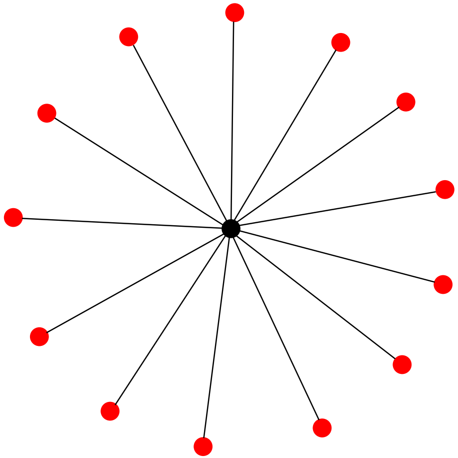
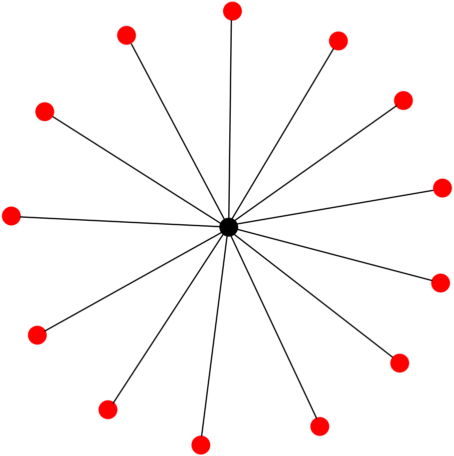


An Illustration: Spanning Star

- Size: 2 states
- Time: $O(n^2 \log n)$
- Optimal w.r.t. both
- Time = # interactions (sequencial)
  - Parallel time is on the average $\frac{1}{n} \cdot$ sequential time



Global Line

Global Line: For all n, the n processes must construct a spanning line

Theorem (Generic Lower Bound)

The expected time to convergence (always under the uniform random scheduler) of any protocol that constructs a spanning network is $\Omega(n \log n)$.

Proof.

Consider the time at which the last edge is activated. By that time, all nodes have some active edge incident to them, thus every node has interacted at least once. The latter can be shown to require an expected number of $\Theta(n \log n)$ steps. □

Theorem (Line Lower Bound)

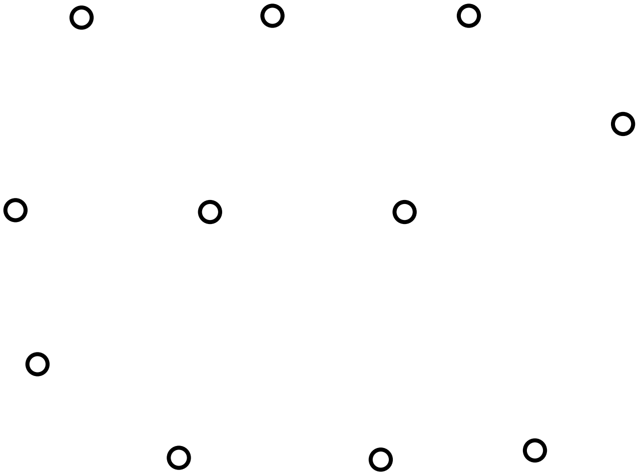
The expected time to convergence of any protocol that constructs a spanning line is $\Omega(n^2)$.

- Protocol *Simple-Global-Line*:

$$(q_0, q_0, 0) \rightarrow (q_1, l, 1)$$
$$(l, q_0, 0) \rightarrow (q_2, l, 1)$$
$$(l, l, 0) \rightarrow (q_2, w, 1)$$
$$(w, q_2, 1) \rightarrow (q_2, w, 1)$$
$$(w, q_1, 1) \rightarrow (q_2, l, 1)$$

  - Every node remembers its degree (q_0, q_1, q_2)
  - Every line has a unique leader (endpoint: l, internal: w)
  - Lines expand towards isolated nodes and merge to other lines via l
  - After merging, w performs a random walk to reach an endpoint

Simple-Global-Line Protocol



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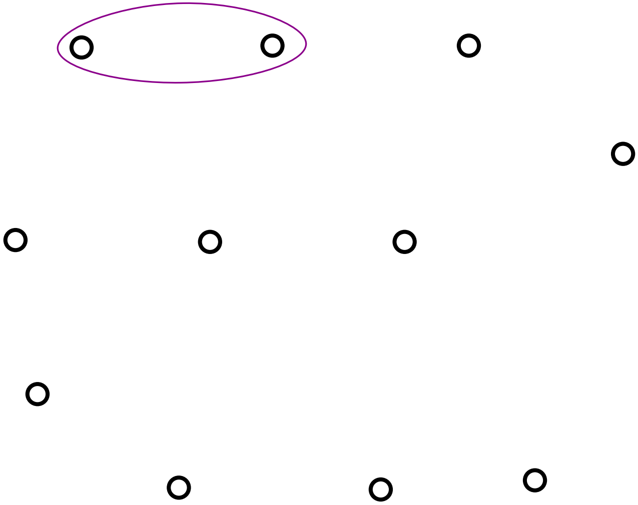

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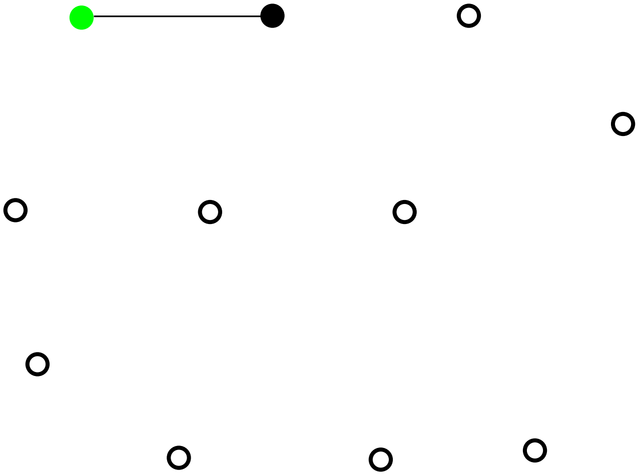

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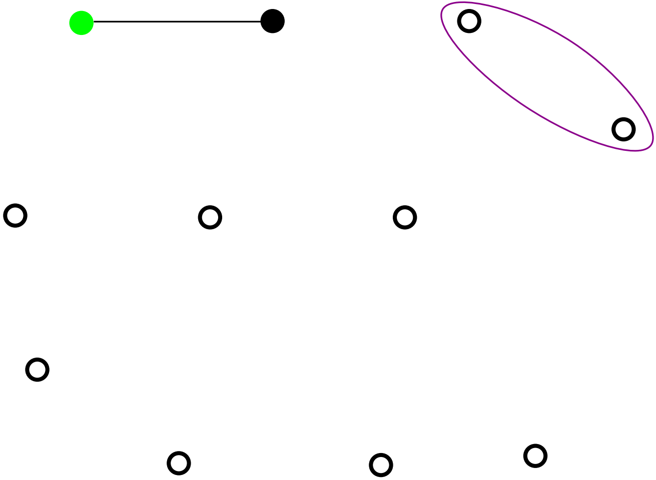

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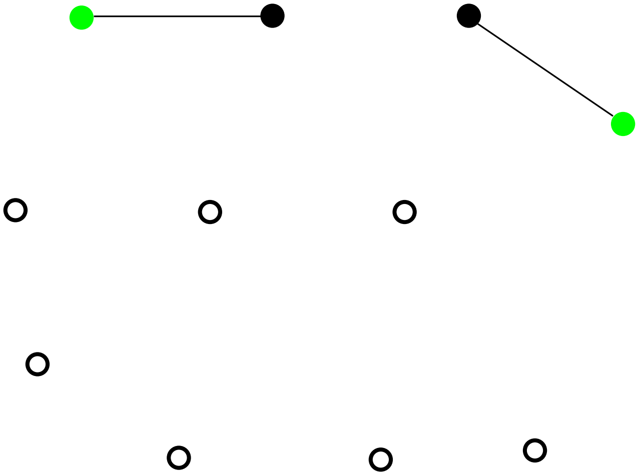

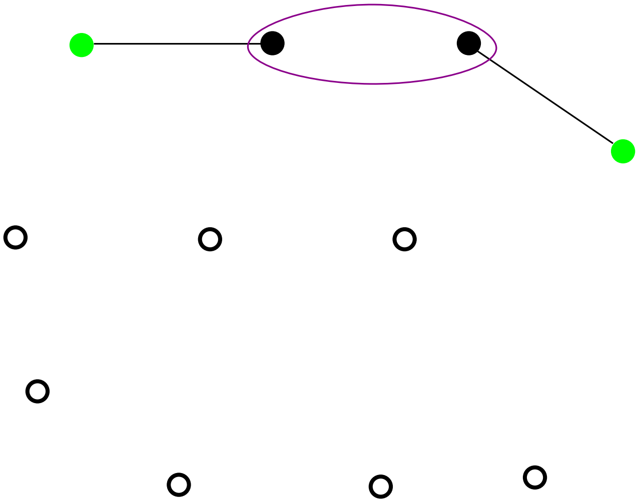

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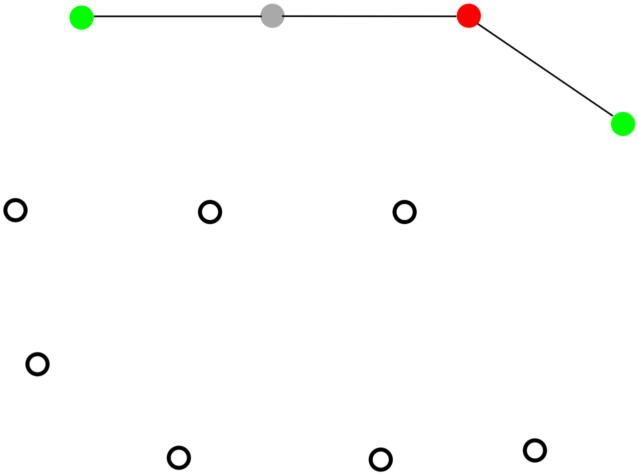

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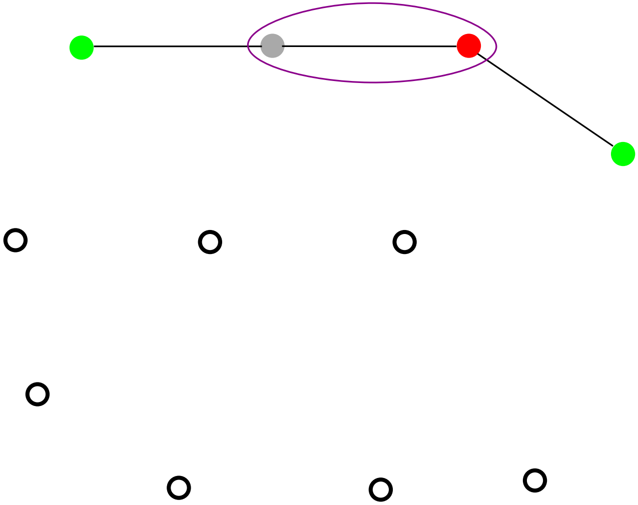

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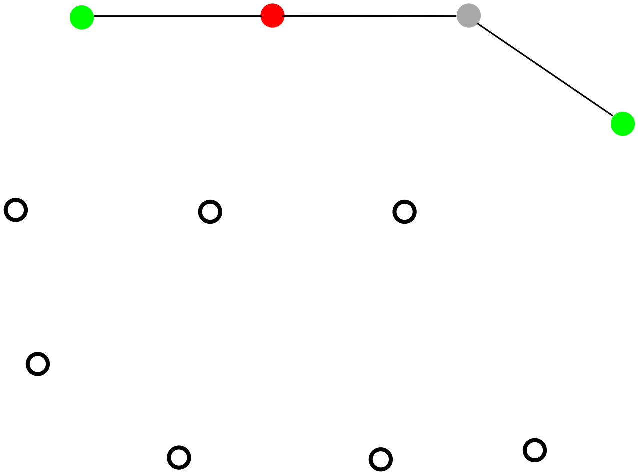

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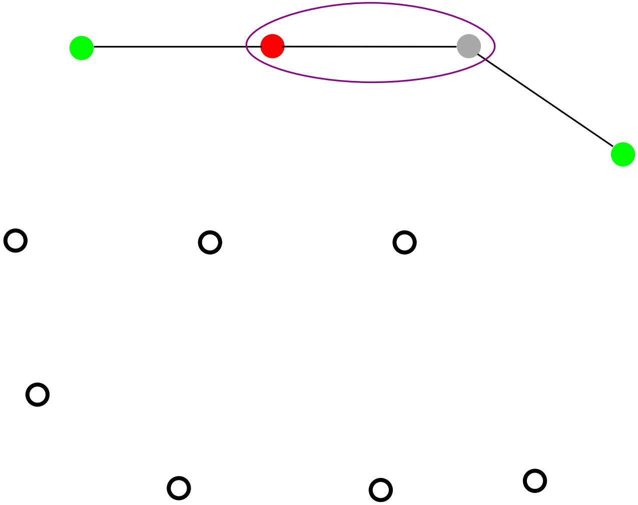

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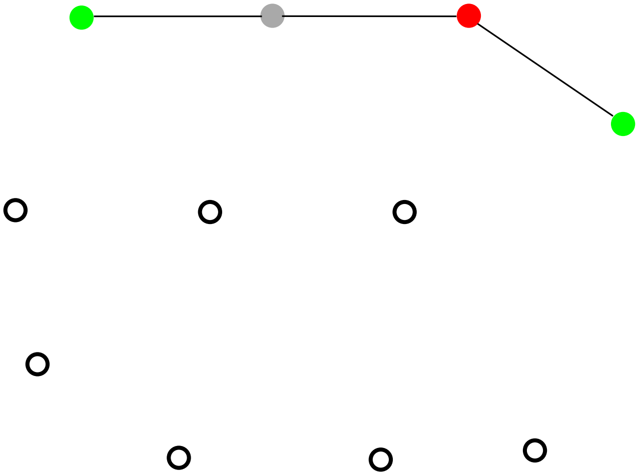

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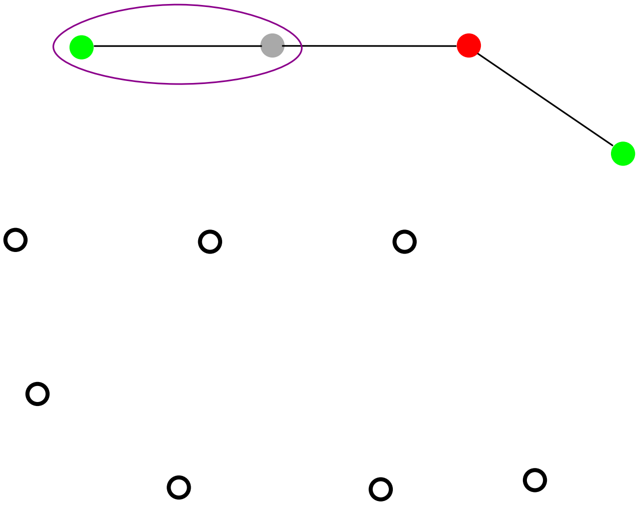

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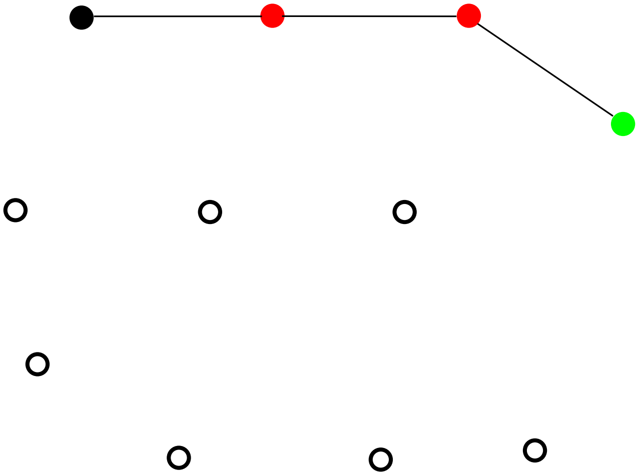

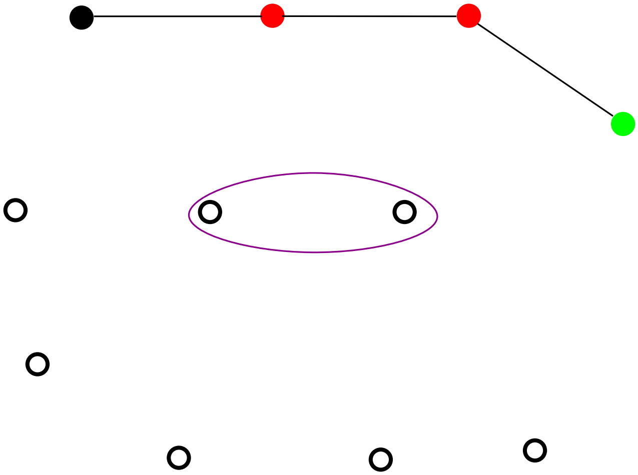

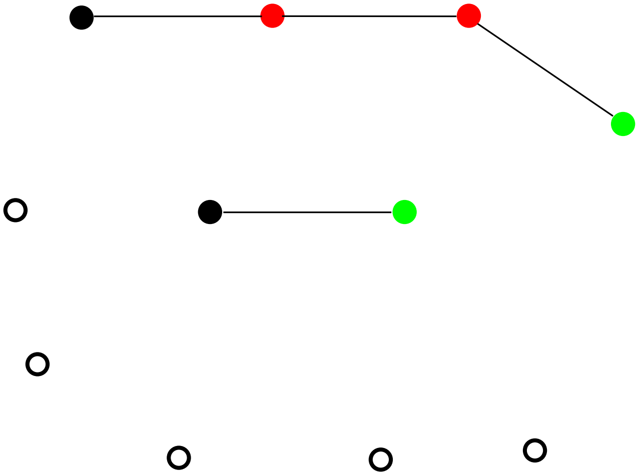

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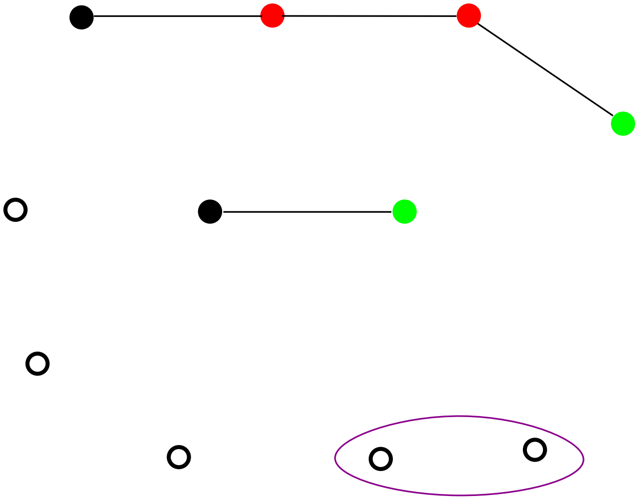

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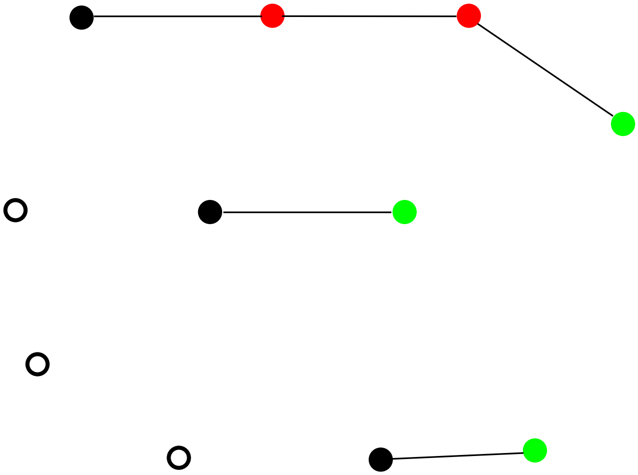

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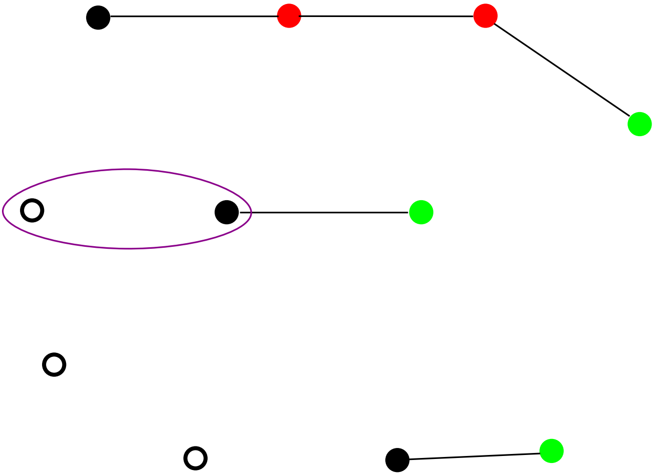

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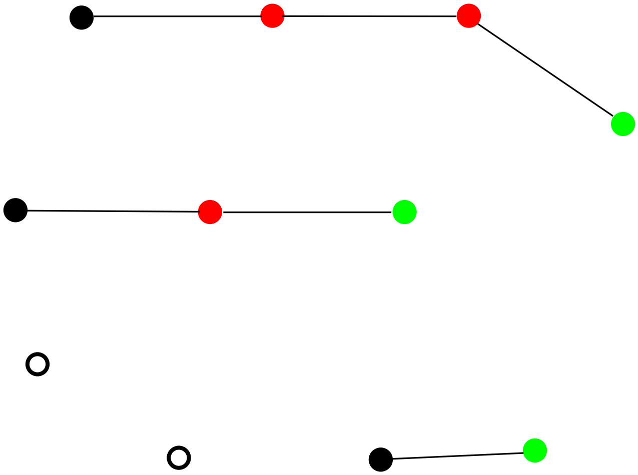

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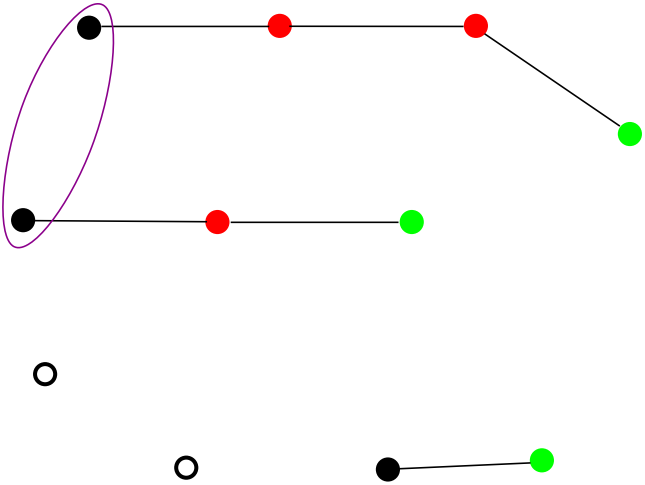

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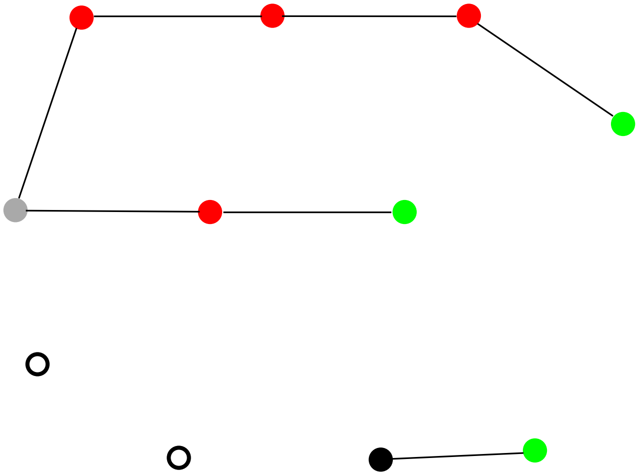

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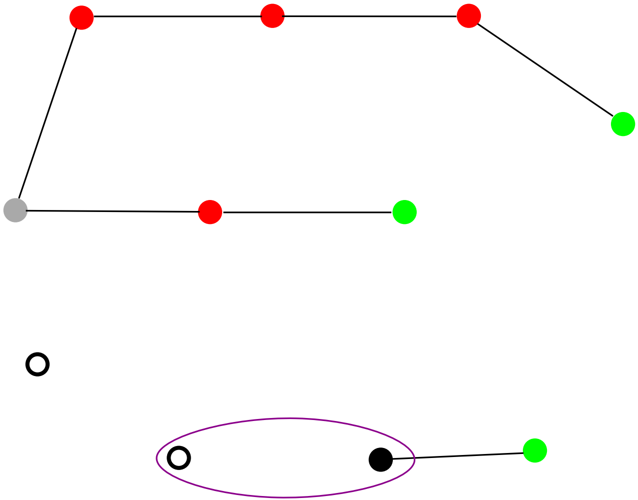

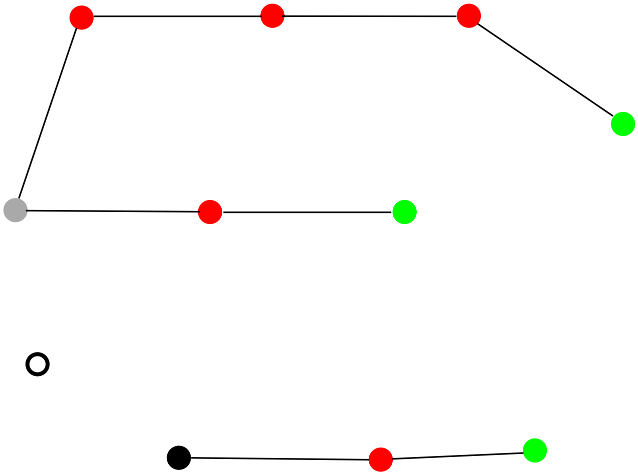

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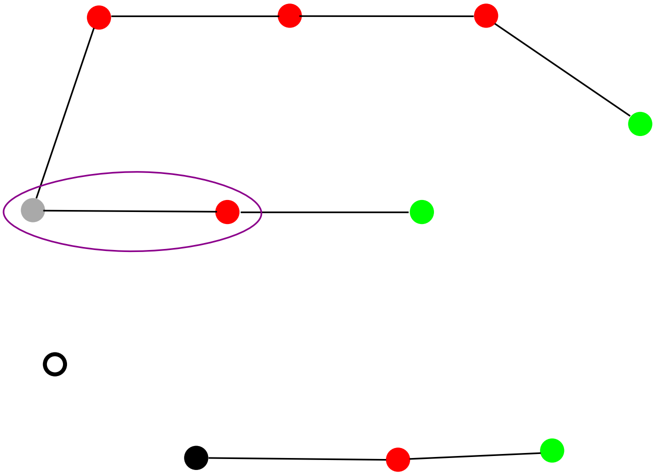

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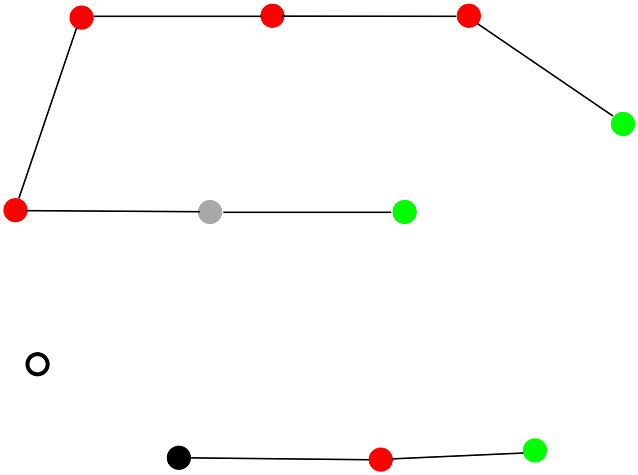

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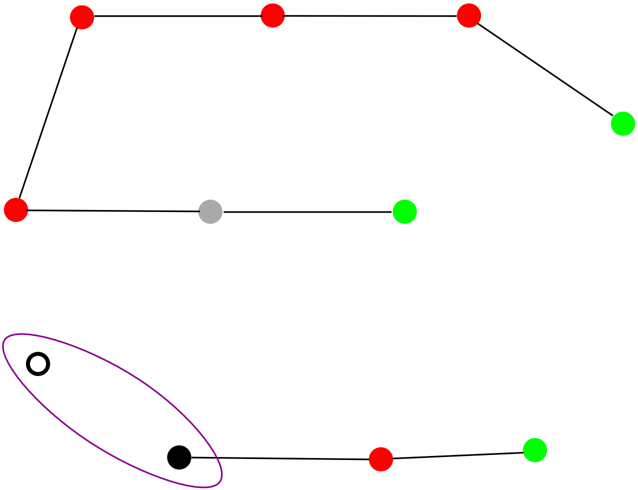

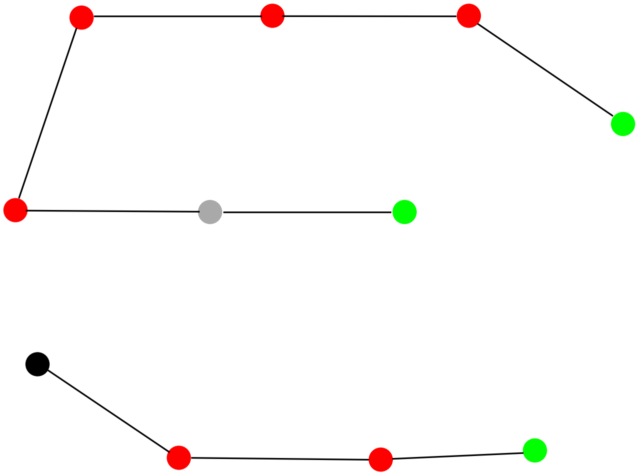

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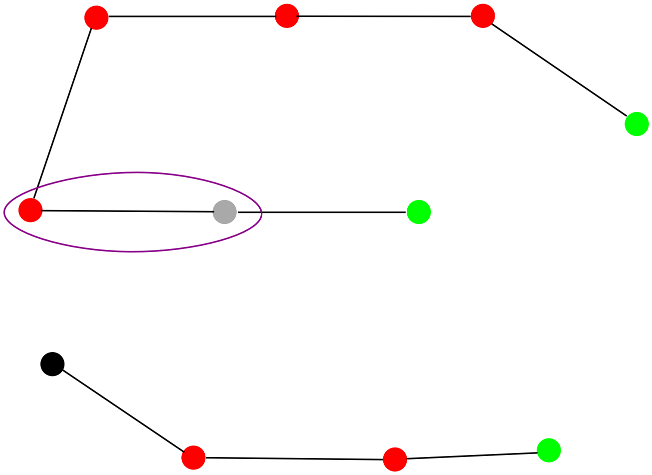

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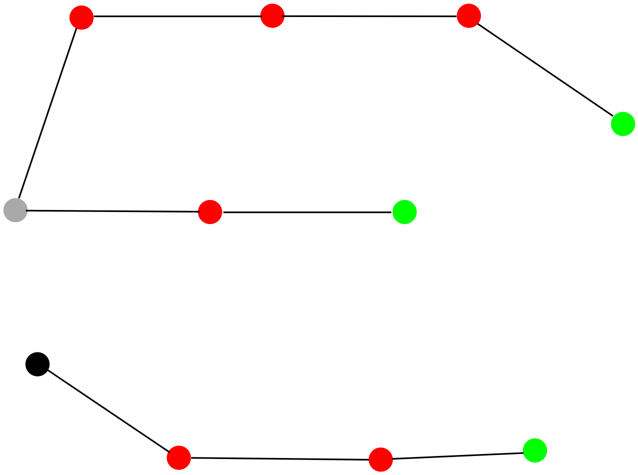

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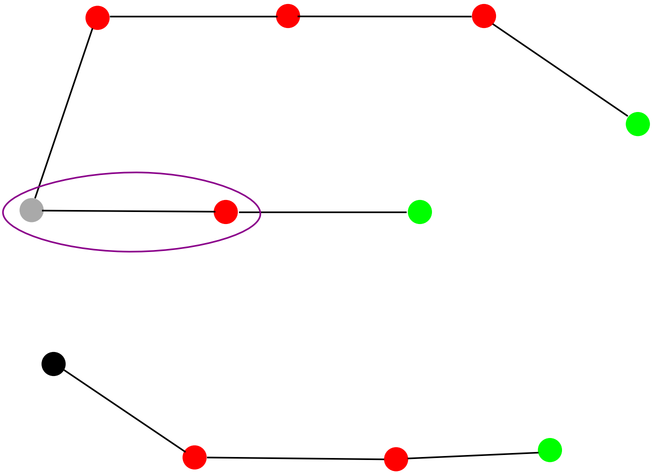

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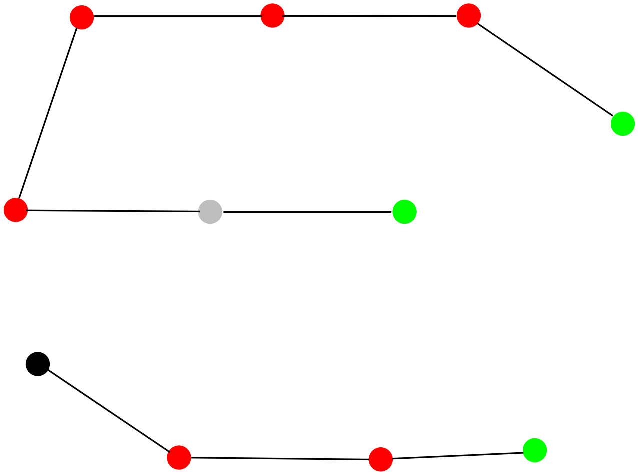

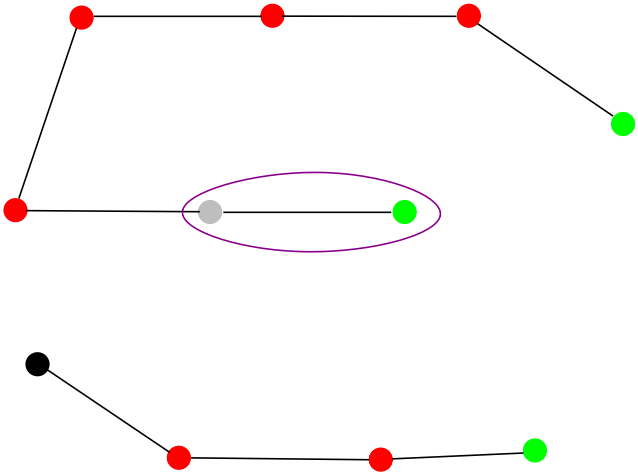

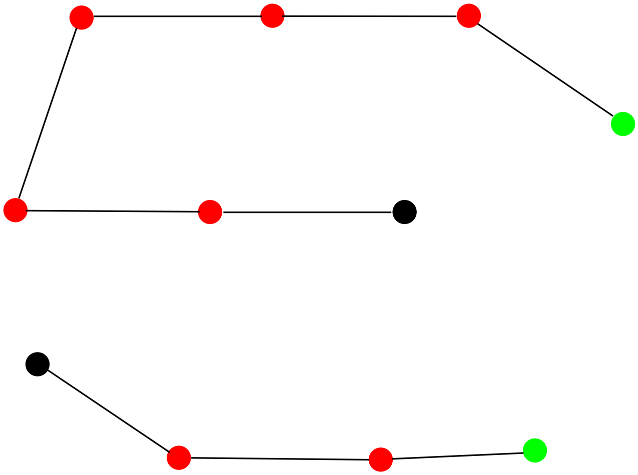

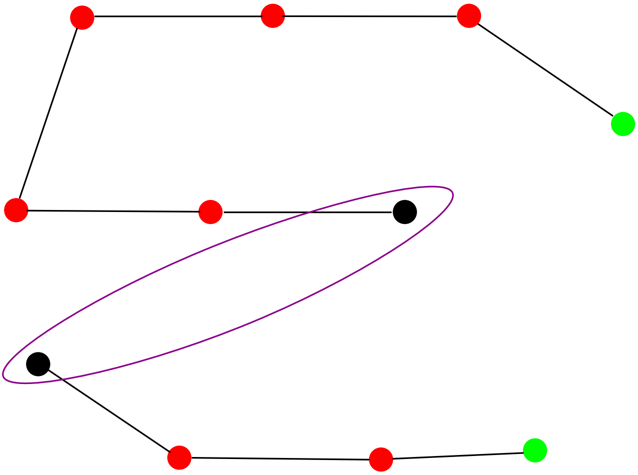

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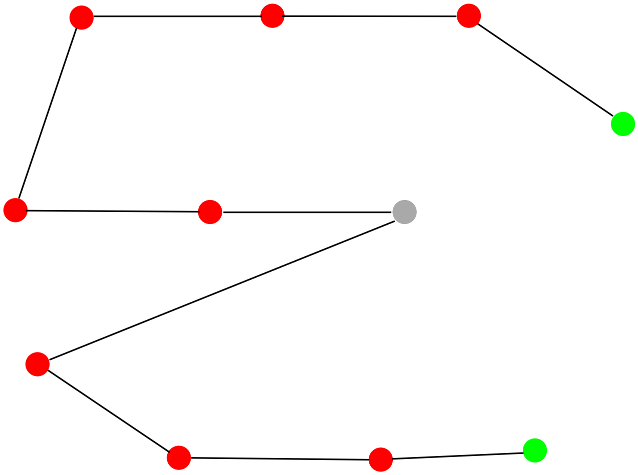

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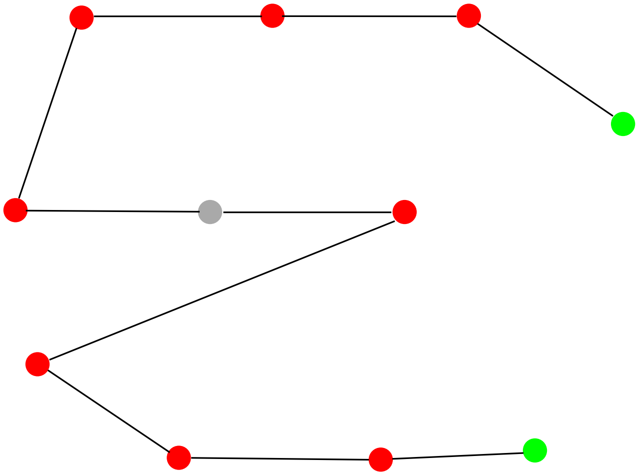
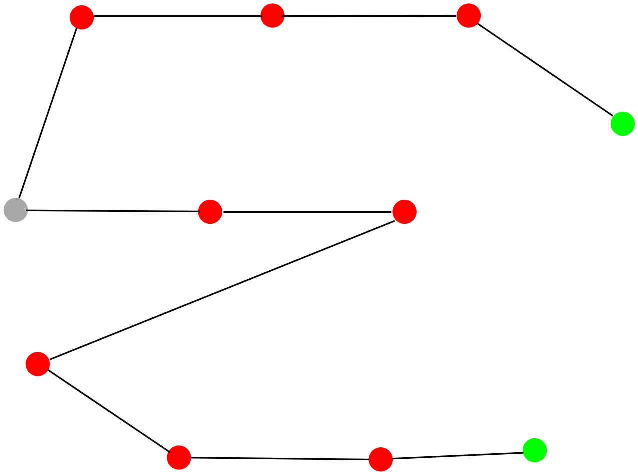
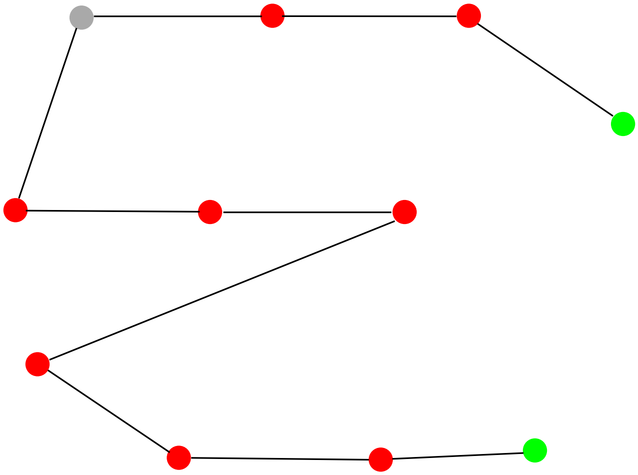


Simple-Global-Line Protocol





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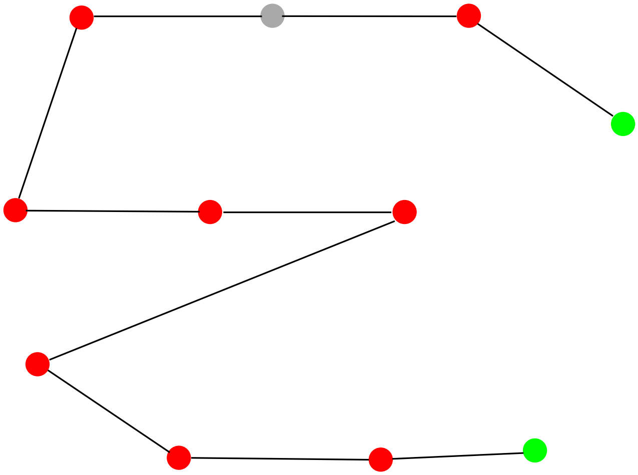

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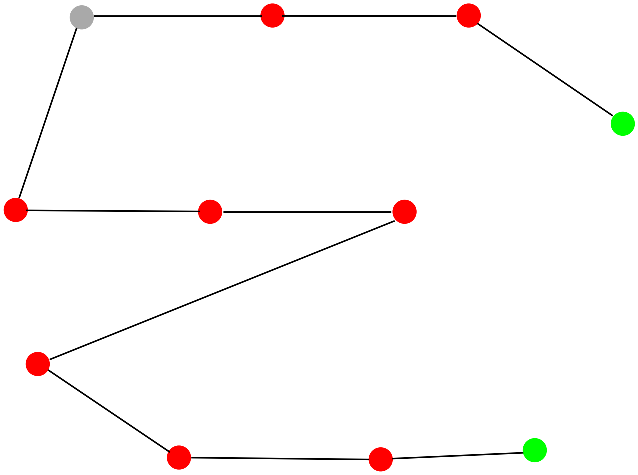

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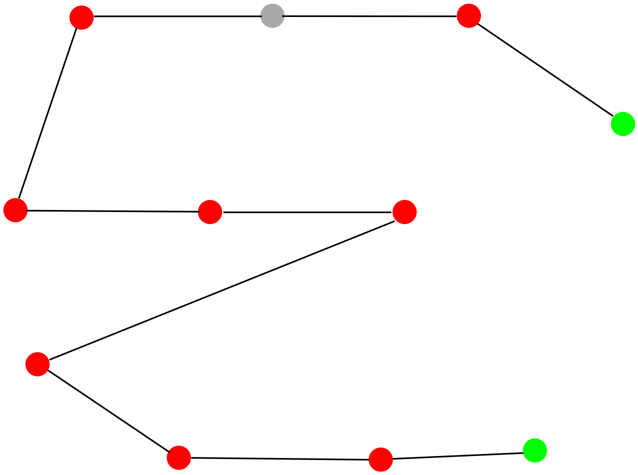

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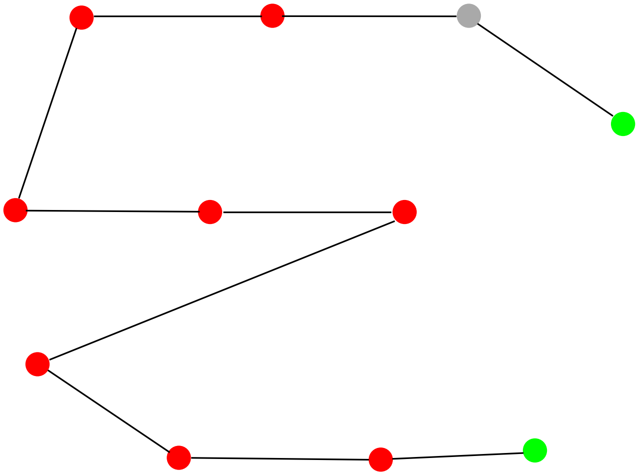

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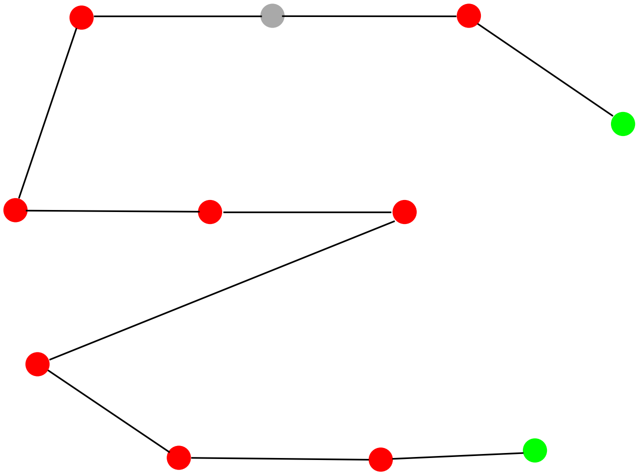

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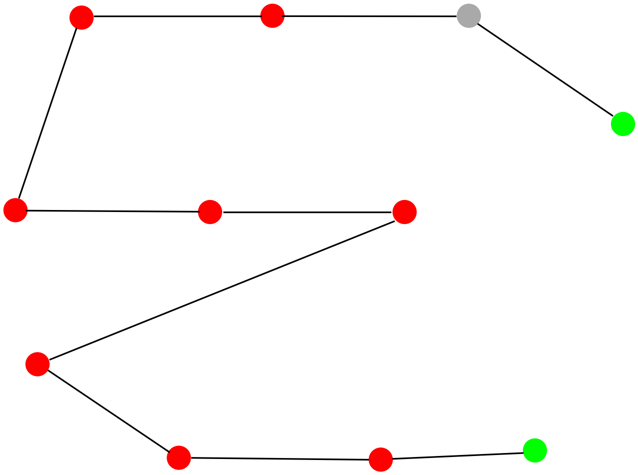

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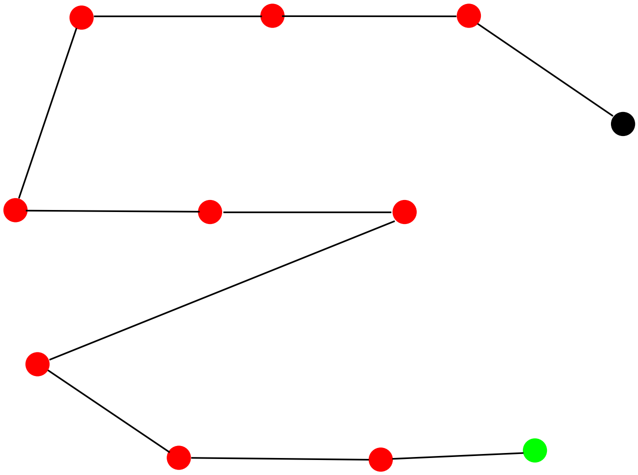

Simple-Global-Line Protocol

Theorem

Protocol Simple-Global-Line constructs a spanning line. It uses 5 states and its expected running time is $\Omega(n^4)$ *and* $O(n^5)$.

Proof

We have to prove two things:

1. *There is a set S of output-stable configurations whose active network is a spanning line*
2. *For every reachable configuration C it holds that $C \rightsquigarrow C_s$ for some $C_s \in S$*

1. *Spanning line, non-leader endpoints in state q_1, non-leader internal nodes in q_2, and unique leader either in l (endpoint) or in w (internal)*
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- *Upper bound:* $(n-2)[O(n^2)+O(n^4)] = O(n^5)$
- *Lower Bound:*
    - *w.h.p. constructs $\Theta(n)$ different lines of length 1 during its course*
    - *$k = \Theta(n)$ disjoint lines $\Rightarrow$ $k-1 = \Theta(n)$ distinct merging processes*

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□

Fast-Global-Line Protocol

$$
\begin{aligned}
(q_0, q_0, 0) &\to (q_1, l, 1) \\
(l, q_0, 0) &\to (q_2, l, 1) \\
(l, l, 0) &\to (q_2', l', 1) \\
(l', q_2, 1) &\to (l'', f_1, 0) \\
(l', q_1, 1) &\to (l'', f_0, 0) \\
(l'', q_2', 1) &\to (l, q_2, 1) \\
(l, f_0, 0) &\to (q_2, l, 1) \\
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\end{aligned}
$$

Fast-Global-Line Protocol

- Avoid mergings as they seem to consume much time
  - even deterministic merging takes time $\Omega(n^3)$ and $O(n^4)$
- When leaders of lines interact, they play a pairwise game
- The winner grows by one towards the other line and the loser sleeps
- A sleeping line cannot increase any more and only loses nodes by lines that are still awake
- A single leader is guaranteed to always win and eventually remain unique and this occurs quite fast
- The leader makes progress (by one) in most interactions and every such progress is in turn quite fast
- Running-time: $O(n^3)$

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All Established Bounds

Protocol	# states	Expected Time	Lower Bound
Simple-Global-Line	5	$\Omega(n^4)$ and $O(n^5)$	$\Omega(n^2)$
Intermediate-Global-Line	8	$\Omega(n^3)$ and $O(n^4)$	$\Omega(n^2)$
Fast-Global-Line	9	$O(n^3)$	$\Omega(n^2)$
Cycle-Cover	3	$\Theta(n^2)$ (optimal)	$\Omega(n^2)$
Global-Star	2 (optimal)	$\Theta(n^2 \log n)$ (optimal)	$\Omega(n^2 \log n)$
Global-Ring	9		$\Omega(n^2)$
2RC	6		$\Omega(n \log n)$
kRC	$2(k+1)$		$\Omega(n \log n)$
c-Cliques	$5c-3$		$\Omega(n \log n)$
Graph-Replication	12	$\Theta(n^4 \log n)$	

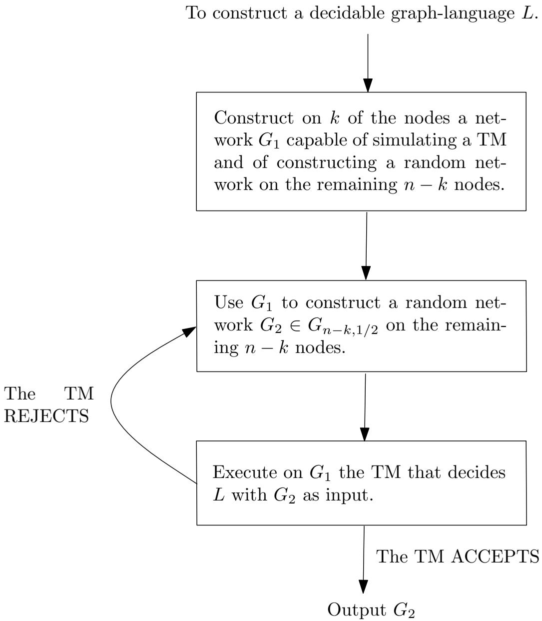
Generic Constructors

Question: Is there a generic constructor capable of constructing a large class of networks?

1. constructors that simulate a Turing Machine
2. a constructor that simulates a distributed system with names and logarithmic local memories

- l: the binary length of the input of a TM
  - In what follows, $l = \Theta(n^2)$
- **DGS**$(f(l))$: the class of graph languages decidable by a TM of (binary) space $f(l)$ (input graph in adjacency matrix encoding)
- **REL**$(g(n))$: the class of graph languages constructible with useful space $g(n)$ (relation or on/off class)
- **PREL**$(g(n))$: (i) allow transitions that with probability $1/2$ give one outcome and with probability $1/2$ another (ii) all graphs must be constructed equiprobably

Generic Constructors: Main Mechanism



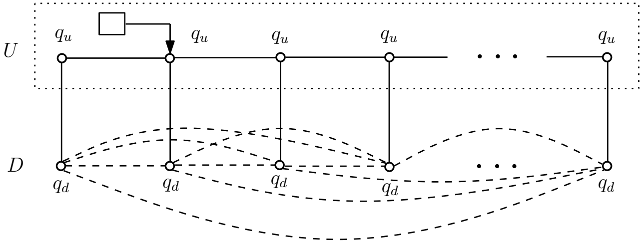
Theorem (Linear Waste-Half)

DGS$(O(n)) \subseteq$ **PREL**$(\lfloor n/2 \rfloor)$. *In words, for every graph language L that is decidable by a $O(n)$-space TM, there is a protocol that constructs L equiprobably with useful space $\lfloor n/2 \rfloor$.*

Proof

- *Partition the population into equal sets U and D and construct an active perfect matching between U and D*
- *Construct a spanning line in U (e.g. Fast-Global-Line)*
- *Organize the line into a TM M*
- *M must compute a graph from L* and *construct it on D*
  - *uniquely identify the nodes of D by their distance from one endpoint*
  - *to modify edge (i, j) mark appropriately the D-nodes at distances i and j from one endpoint*

Linear Waste-Half



- *M computes a graph G from L equiprobably: activate/deactivate each edge of D equiprobably and independently of other edges*
- *Then it simulates on input G the TM that decides L in $\sqrt{l}$ space to determine whether $G \in L$*
- *The TM rejects: M repeats the random experiment to produce a new random graph G'*
- *The TM accepts: release the network, set D-nodes to q_{out}*
- *Reinitialize whenever the global line protocol makes progress* □

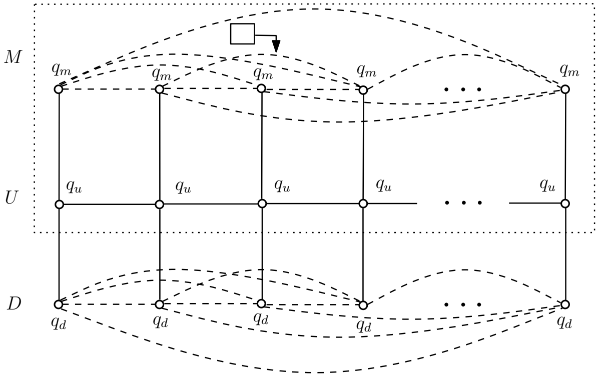
Reinitialization

- Nodes cannot detect termination and cannot sequentially compose subroutines
  - Instead, all subroutines must be executed in parallel and become "sequentialized" by perpetual reinitializations
- Executed whenever a line on U-nodes expands
- The protocol "assumes" that no further expansions will occur, restores the components of the simulation to their original values, ensures that each U-node has a D-neighbor, and starts drawing a new random graph
- The assumption may be wrong several times
- However, eventually the assumption of the protocol will be correct
  - The simulation will be reinitialized and executed for the last time on the correct sets U and D

Later we shall discuss terminating protocols that are correct w.h.p.

Theorem (Linear Waste-Two Thirds)

DGS$(O(n^2) + O(n)) \subseteq$ **PREL**$(\lfloor n/3 \rfloor)$. *That is, for every graph language L that is decidable by a $O(n^2)$-space TM, there is a protocol that constructs L equiprobably with useful space $\lfloor n/3 \rfloor$.*



Logarithmic Waste

Theorem (Logarithmic Waste)

$\mathbf{DGS}(O(\log n)) \subseteq \mathbf{PREL}(n - \log n)$.

Proof.

- Construct a line to count n in binary, i.e. to occupy $\log n$ cells
- Release the constructed counter and reinitialize the other (free) nodes
- So, there is a logarithmic memory with a leader, knowing a good estimate of $n - \log n$
- Construct a random graph on the free nodes:
  - For every free node, let it toss a coin on one after the other its edges to other free nodes
  - To know when to stop, count on the line up to $n - \log n$ (known) □

Supernodes

- A population consisting of n nodes can be partitioned into k *supernodes* each consisting of $\log k$ nodes, for the largest such k
- The internal structure of each supernode is a line, thus it can be operated as a TM of memory logarithmic in the total number of supernodes
- This amount of storage is sufficient for the supernodes to obtain unique names and exploit their names and their internal storage to realize nontrivial constructions
- We are interested in the networks that can be constructed at the supernode abstraction layer

Supernodes

Theorem (Partitioning into Supernodes)

For every network G that can be constructed by k nodes having local memories $\lceil \log k \rceil$ and unique names there is a NET that constructs G on $n = k\lceil \log k \rceil$ nodes.

Network Transformations

[Michail and Spirakis, TCS '16]

- Minimal strengthenings of NETs that can maximize computational power
  - Also gain termination
- Initial configuration: any connected graph spanning V
- Ability to detect small local degrees
  - e.g., a node can detect if its active degree is 0
  - we can now simulate any constructor that assumes an empty initial network
- Pre-elected leader or pairs of nodes able to tell whether they have a neighbor in common

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Characterization

- Gives the maximum computational power that one can hope for in this family of models
- Can compute with termination any symmetric predicate computable by a TM of space $\Theta(n^2)$, and no more than this, i.e., it is an exact characterization
- Symmetricity can only be dropped by UIDs or other means of maintaining an ordering of the nodes' inputs
- This power is maximal because the distributed space of the system is $\Theta(n^2)$
- Universal computations are now terminating and not just eventually stabilizing
- The additional mechanisms are minimal. Removing each one leads to
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Approach

- Develop protocols exploiting initial connectivity to
  - transform the network to a less symmetric and detectable
  - without ever breaking its connectivity
- Still, very symmetric components cannot be exploited for powerful computations
- Symmetric target-networks may not be detectable for termination
- Our protocols always transform to a spanning line
  - Detectable
  - Can serve as a linear memory
  - Exploit it to count n and to simulate TMs of space $\Theta(n)$ or even $\Theta(n^2)$ (with an additional transformation)

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- Unique leader + detect local degree 1
  - Time-optimal transformation to spanning line, with running time $\Theta(n^2 \log n)$
  - Implies a full-power TM simulation
- Identical nodes
  - Impossibility: no protocol to transform to an acyclic topology without breaking connectivity
- Common neighbor detection
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Line-Around-a-Star

- Unique leader in state l and all other nodes in state q_0
- The leader (center) starts connecting with the q_0s converting them to p_0s (peripherals)
- p_0s repel by disconnecting edges between them
- A p_0 with degree 1 becomes p (normal peripheral)
- The leader constructs a line with its left endpoint on the center of the star and expanding over the peripherals until it covers them all
- Terminates when the degree of the center becomes 1 for the first time

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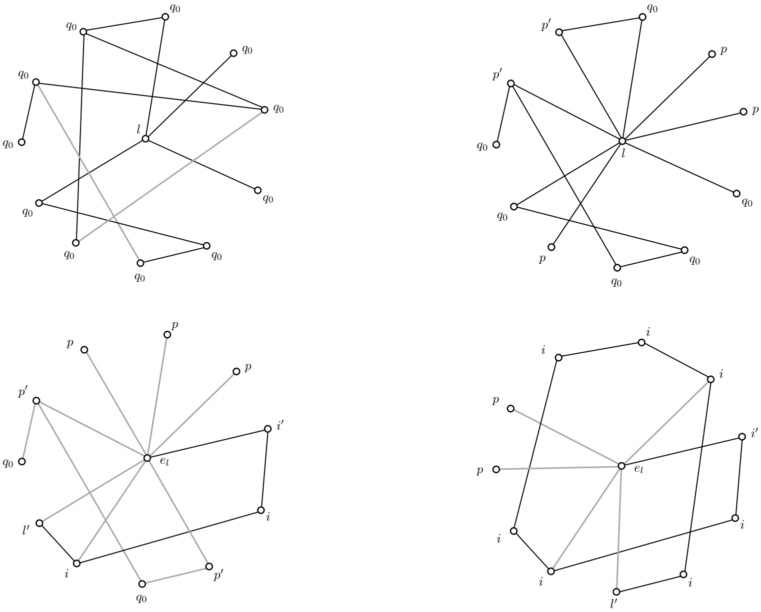
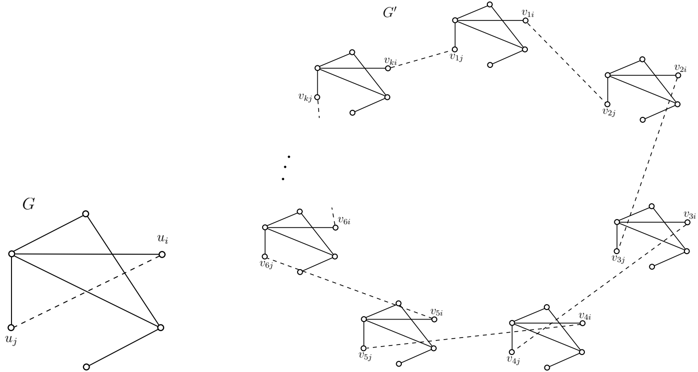
An Example Execution



Theorem (Strong Impossibility)

For every connected graph G with at least one cycle, there is an infinite family of graphs $\mathcal{G}$ such that for every $G' \in \mathcal{G}$ every protocol (beginning from identical states on all nodes) that makes G acyclic may disconnect G' in some executions.



Theorem

By assuming that nodes are equipped with a common neighbor detection mechanism and have the ability to detect local degrees 1 and 2, Protocol Line-Transformer solves the Terminating Line Transformation problem in the setting in which all nodes are initially identical. Its running time is $O(n^3)$.

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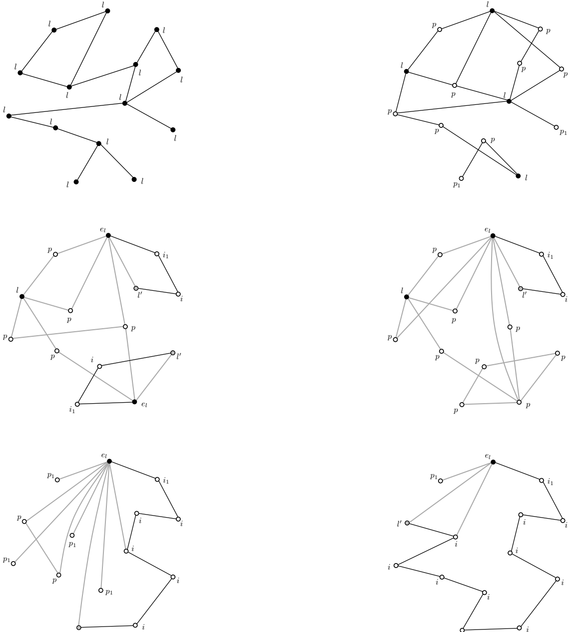

All Established Bounds

Protocol	Leader	DD	CND	Expected Time	Lower Bound
Online-Cycle-Elimination	Yes	1	No	$\Theta(n^4)$	$\Omega(n^2 \log n)$
Line-Around-a-Star	Yes	1	No	$\Theta(n^2 \log n)$ (opt)	$\Omega(n^2 \log n)$
Line-Transformer	No	1,2	Yes	$O(n^3)$	$\Omega(n^2 \log n)$

- Leader: Whether it makes use of a pre-elected unique leader
- DD: what local degree detection is used
- CND: whether it uses common neighbor detection
- Expected Time: Expected running time under the uniform random scheduler
- Lower Bound: Best known lower bound

A Geometric Version

[Michail, PODC '15]

- Adjust some of the abstract parameters of NETs
- Allowable interactions are geometrically constrained (by already formed structures)
- Each device can connect to other devices only via a limited number of ports (4 in 2D and 6 in 3D)
  - Connections are made at unit distance and are perpendicular to neighboring connections
  - Known universal constructors do not apply in this case

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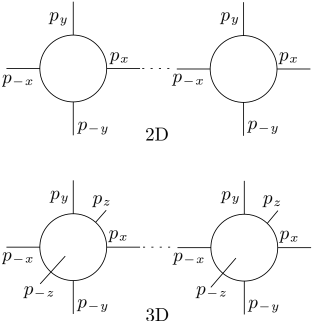

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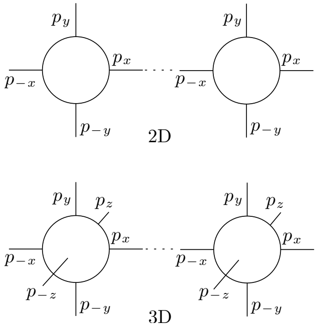

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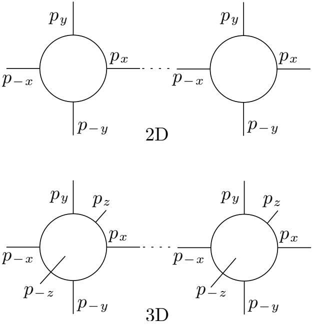

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The Model

- The transition function is now:
  $\delta : (Q \times P) \times (Q \times P) \times \{0, 1\} \to Q \times Q \times \{0, 1\}$,
  $P = \{u, r, d, l\}$ is the set of ports in 2D
- In every step, a pair $(v_1, p_1)(v_2, p_2)$ is selected by the scheduler and v_1, v_2 interact via their p_1, p_2 ports according to δ
- Valid configuration: its connected components are subnetworks of the 2D grid network
- Uniform random scheduler: selects independently and uniformly at random from the permitted interactions (leading to valid config.)
- Output shape: nodes that are in output (or halting) states and edges between them that are active

An Illustration: Spanning Square

- δ:

$$
\begin{aligned}
(L_u, u), (q_0, d), 0 &\rightarrow (q_1, L_r, 1) \\
(L_r, r), (q_0, l), 0 &\rightarrow (q_1, L_d, 1) \\
(L_d, d), (q_0, u), 0 &\rightarrow (q_1, L_l, 1) \\
(L_l, l), (q_0, r), 0 &\rightarrow (q_1, L_u, 1) \\
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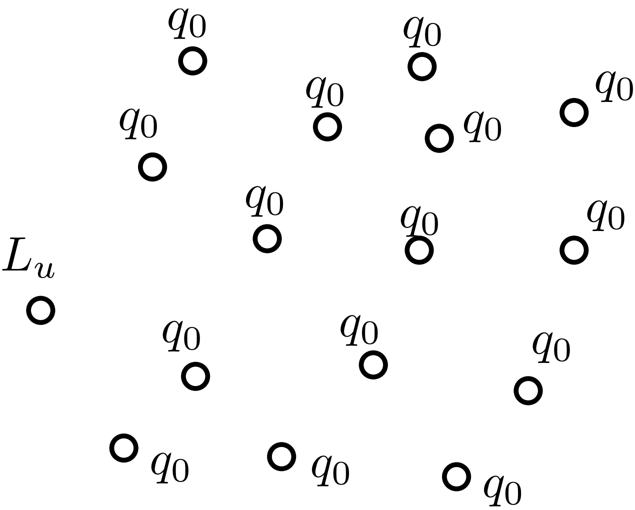
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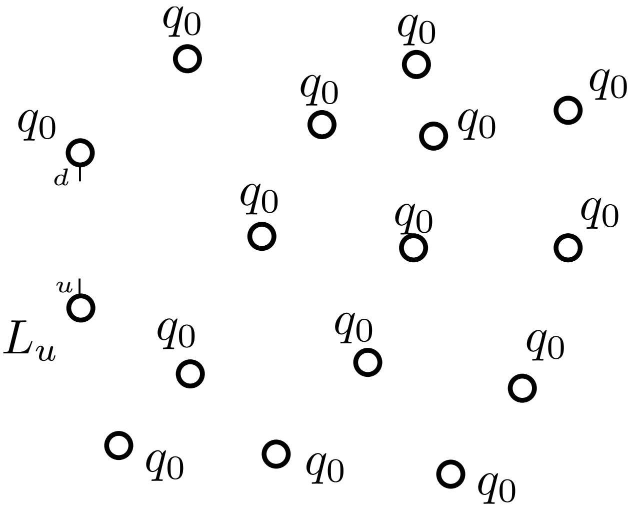
Spanning-Square Protocol



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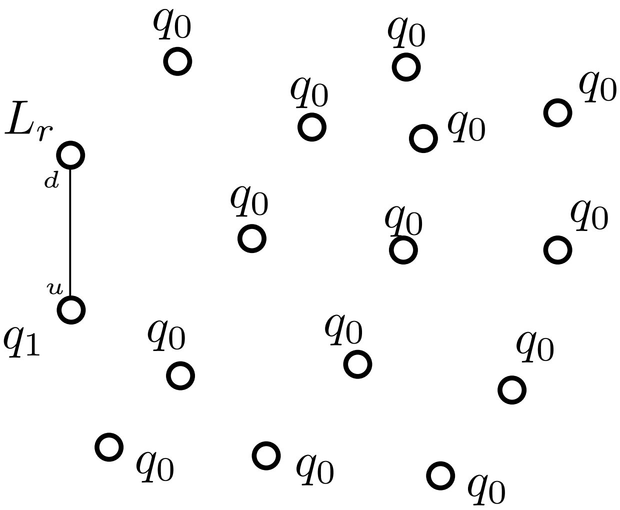

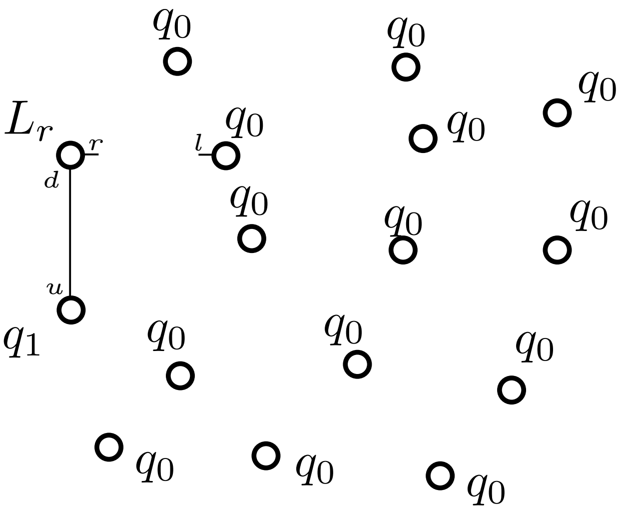

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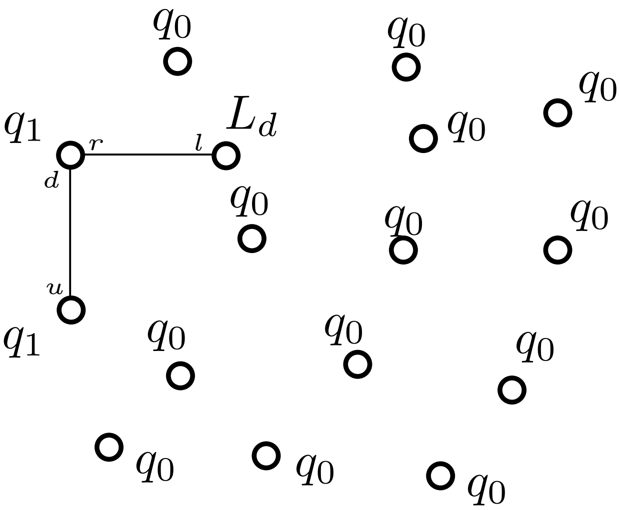

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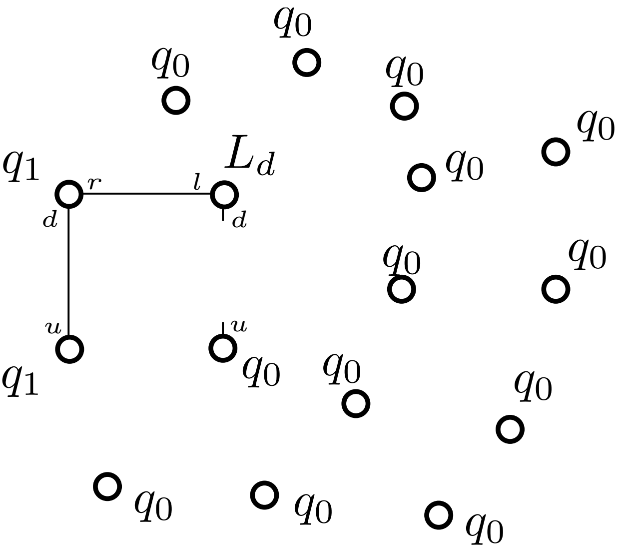

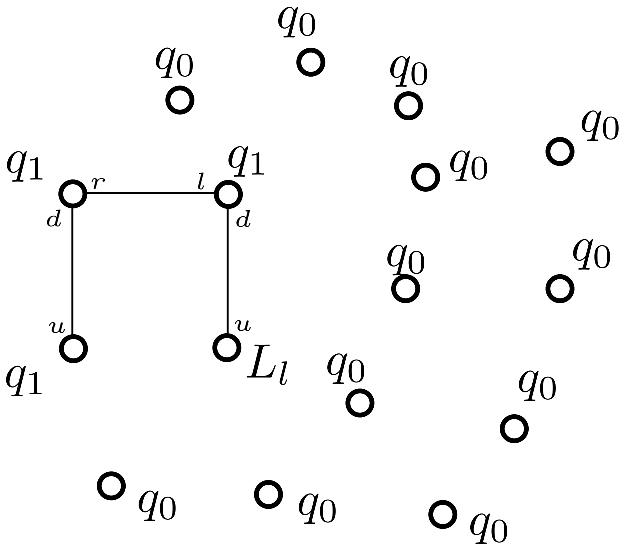
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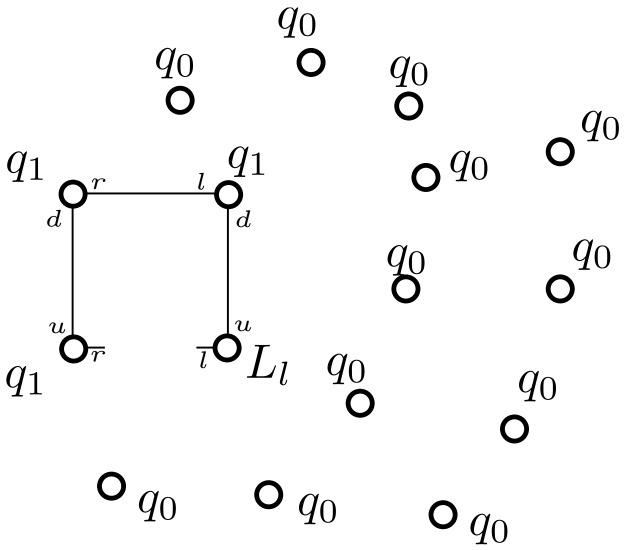

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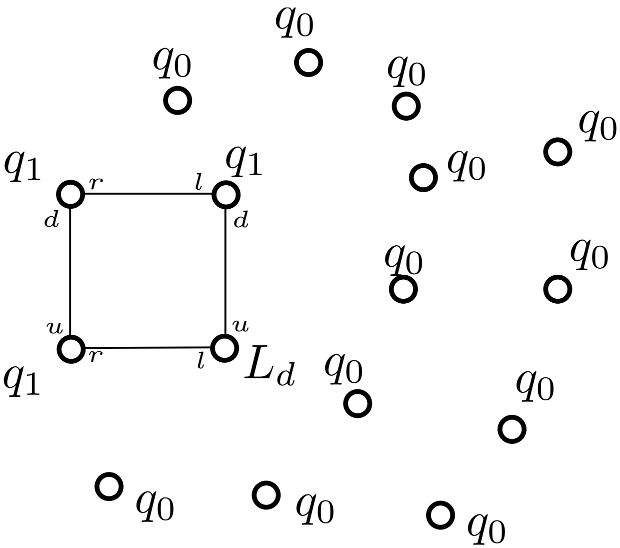

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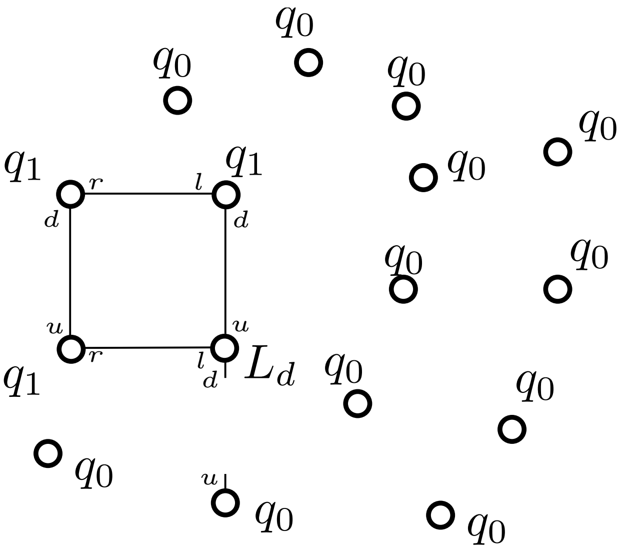

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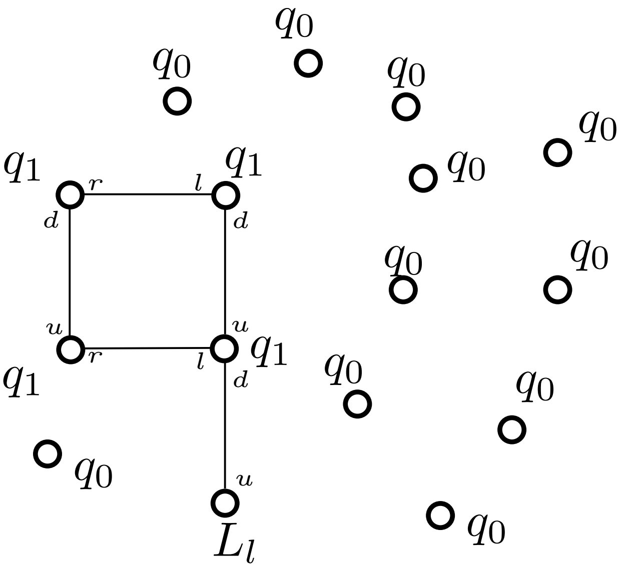

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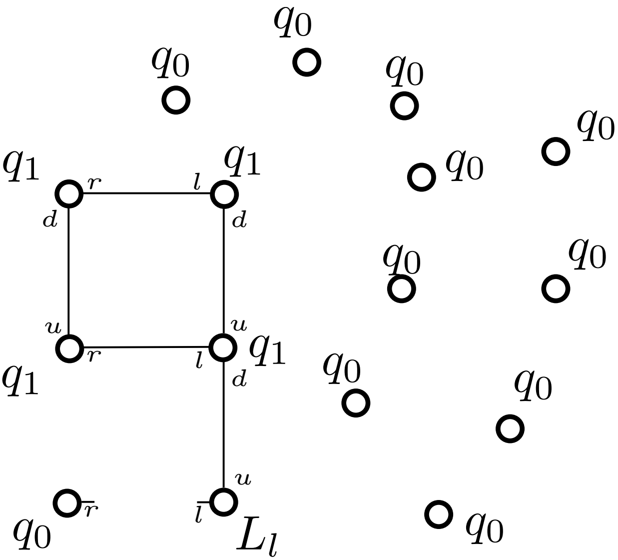

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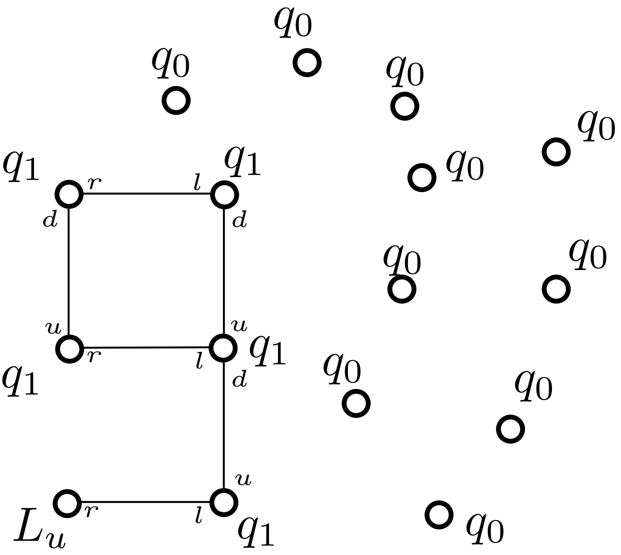

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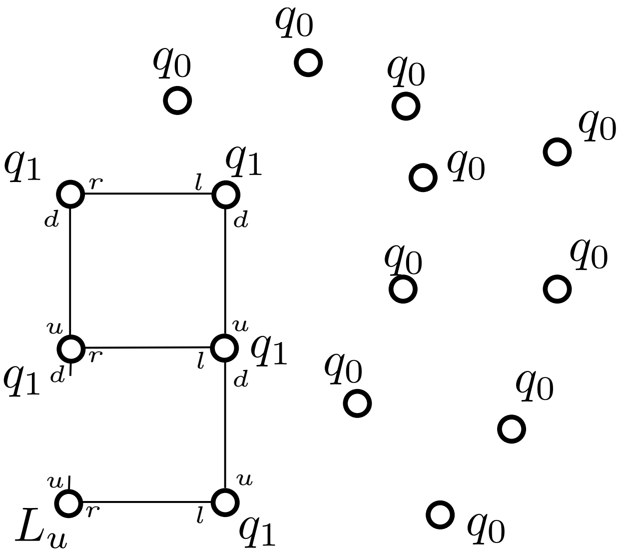

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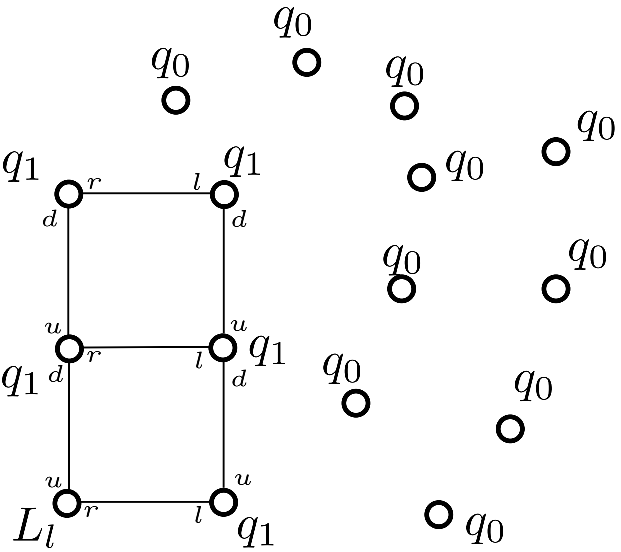

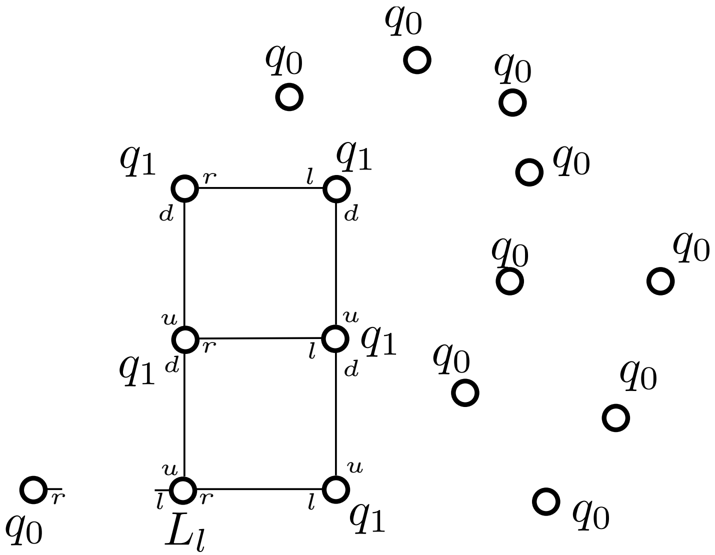
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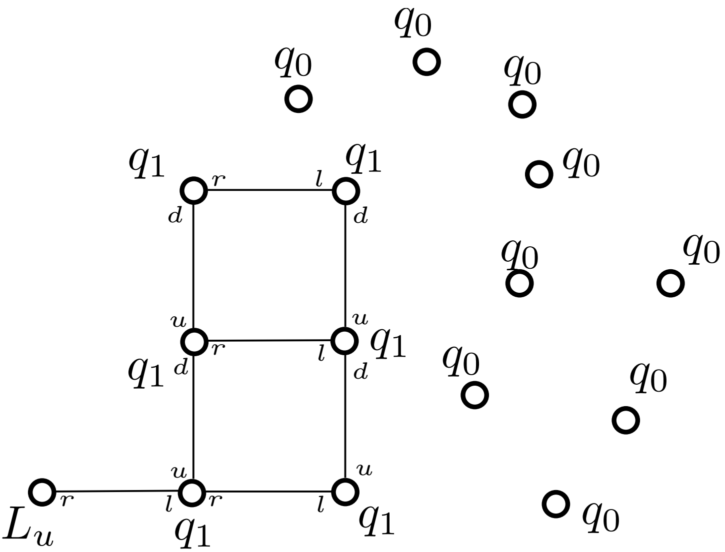
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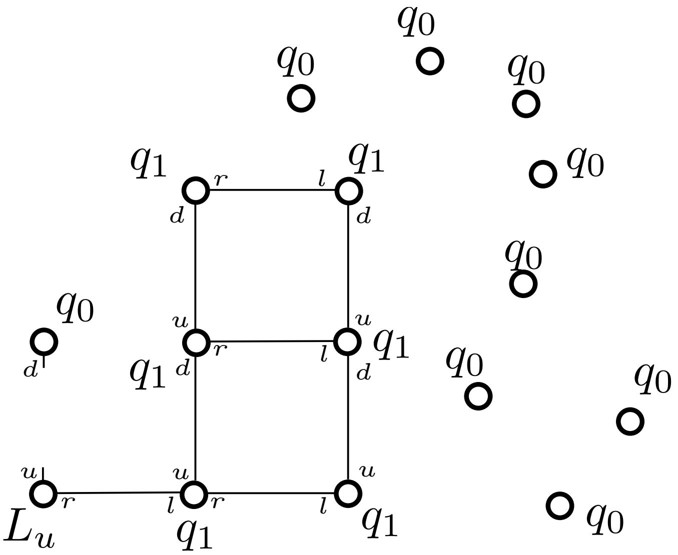

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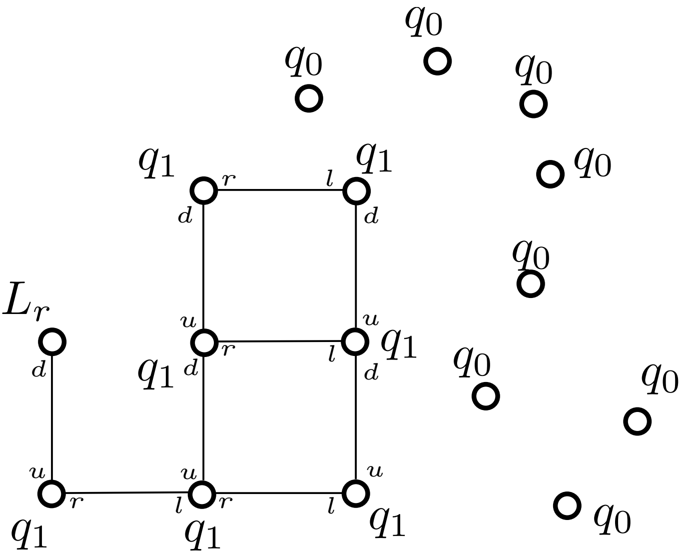
Spanning-Square Protocol



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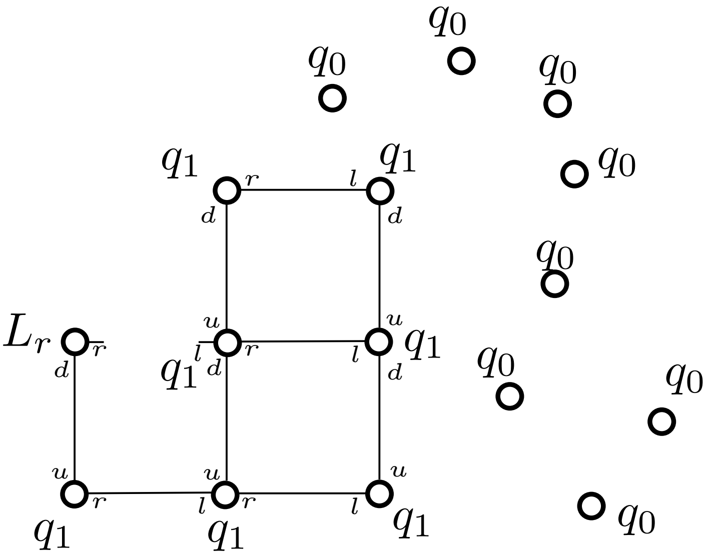

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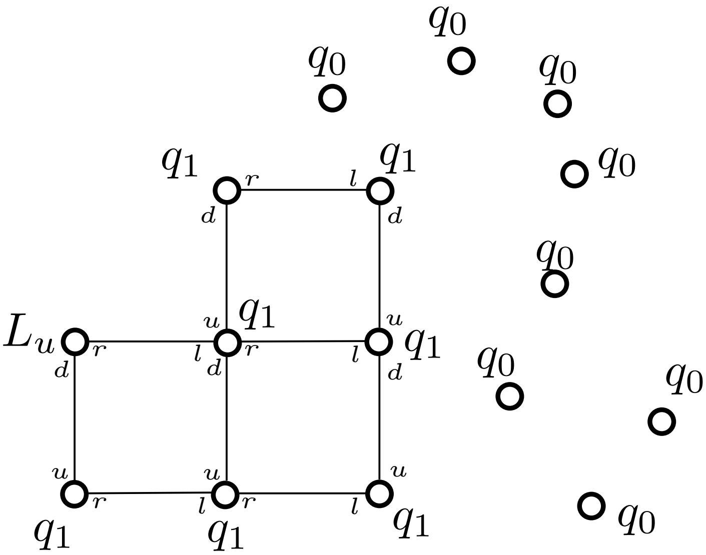

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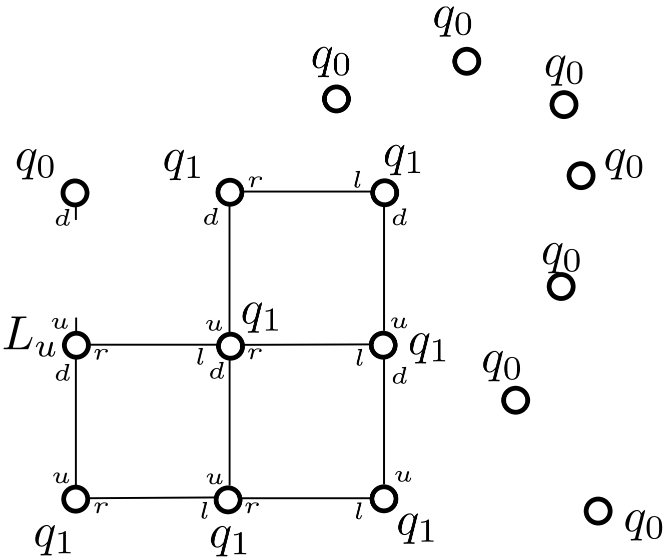

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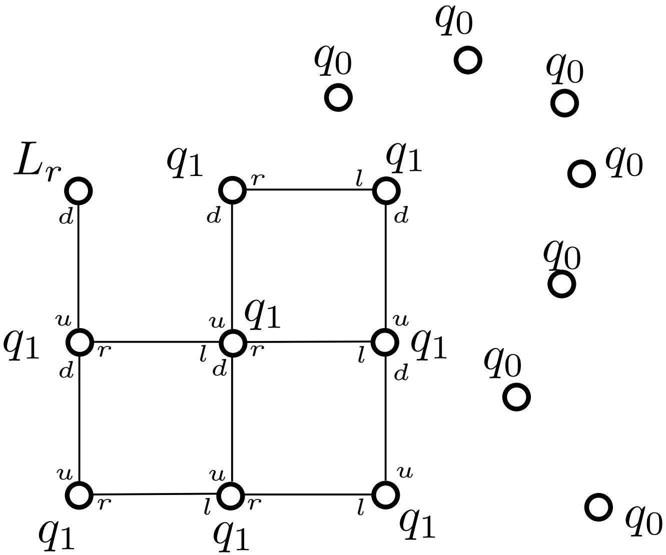

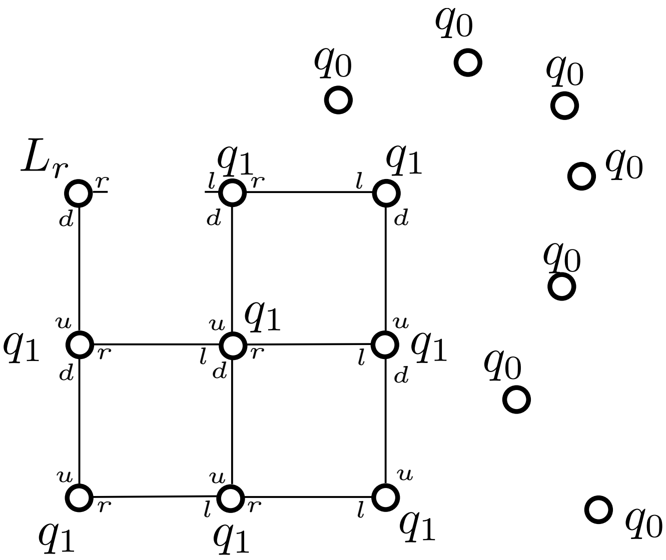

q_0
q_0
q_0
q_0
q_0
q_0
q_0
L_r
q_1
q_1
q_1
q_1
q_1
q_1
q_1
q_1
r
l
d
u

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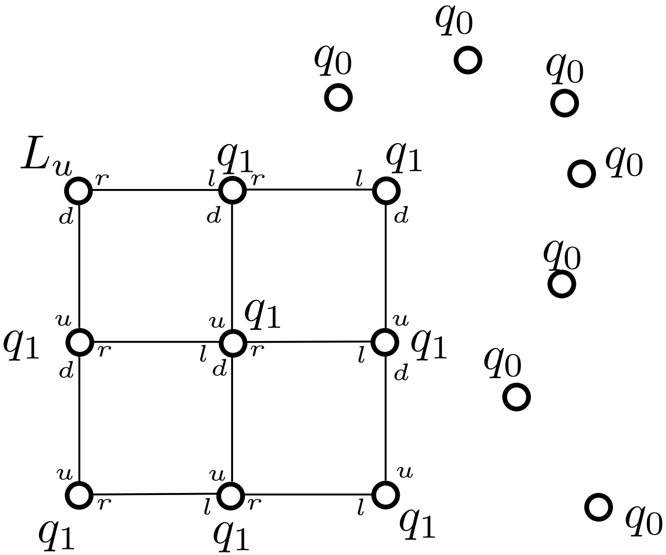

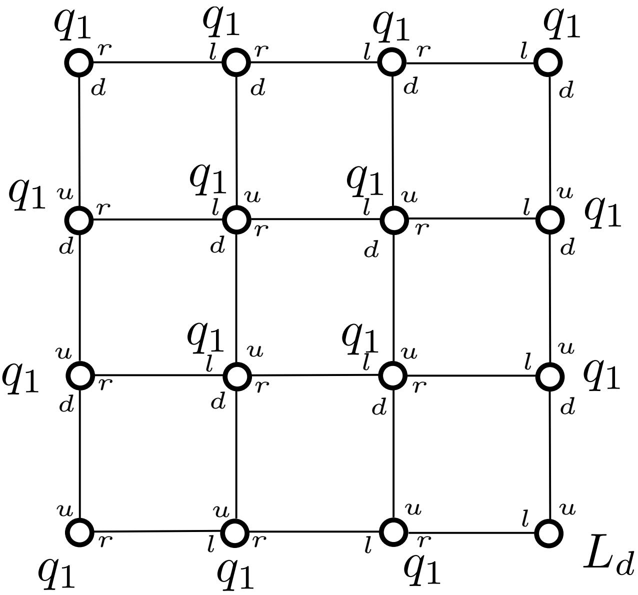
q_1
r
l
d
u
L_d

Probabilistic Counting

- Uniform random scheduler
- Terminating protocol that counts n (or a large fraction of n) w.h.p.
- The best that we can hope for
  - There is no protocol that always terminates and is always correct
- Main subroutine of generic constructors
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Fast Probabilistic Counting with a Leader

- Unique leader l, all other in q_0 initially
- Disregard for a while ports, geometry, and link activations
- The scheduler selects in every step equiprobably one of the $n(n-1)/2$ possible node pairs
- Assume for simplicity that the leader can store two n-counters in its memory
- Classical PP with an additional leader with linear memory
- Then, the protocol is adjusted to work in the present model
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Counting-Upper-Bound Protocol

- $I(r_0, r_1)$: The state of I, where r_0, r_1 are the values of the two counters, $0 \leq r_0, r_1 \leq n$
- Rules:

$$(I(r_0, r_1), q_0) \rightarrow (I(r_0 + 1, r_1), q_1)$$
$$(I(r_0, r_1), q_1) \rightarrow (I(r_0, r_1 + 1), q_2), \text{ and}$$
$$(I(r_0, r_1), \cdot) \rightarrow (halt, \cdot) \text{ if } r_0 = r_1$$

- r_0 counts the number of q_0s in the population
- r_1 counts the number of q_1s in the population
- When a q_0 (q_1) is counted it is converted to q_1 (q_2)
- Terminates when $r_0 = r_1$ for the first time

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Counting-Upper-Bound Protocol

Theorem

Counting-Upper-Bound halts in every execution. Moreover, if the scheduler is a uniform random one, when this occurs, w.h.p. it holds that $r_0 \geq n/2$.

Proof

- $p_{ij} = i/(i+j)$: *probability that an effective interaction is an* (l, q_0)
- $q_{ij} = 1 - p_{ij} = j/(i+j)$: *probability that it is an* (l, q_1)
- *r.w. on a line with* $n+1$ *positions* $0, 1, \ldots, n$
- *a particle begins from position* b, *absorbing barrier at* 0, *and reflecting at* n, *position corresponds to* $r_0 - r_1 = j$

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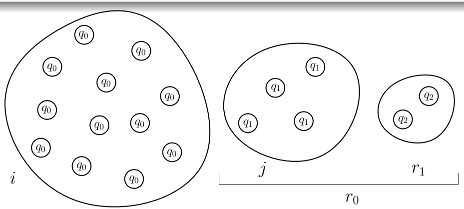

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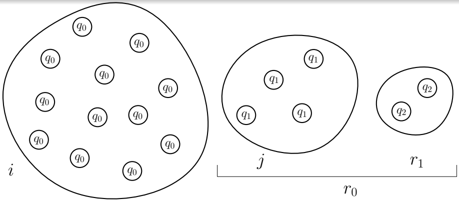

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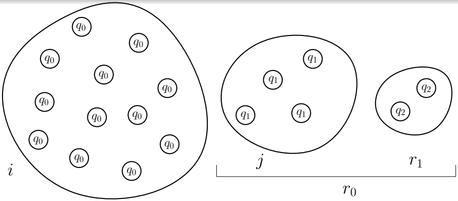

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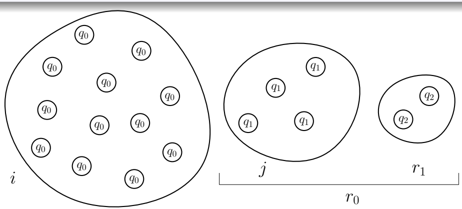

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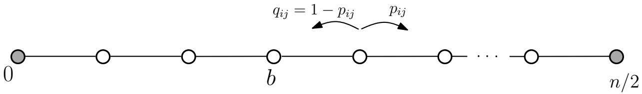

Proof

- *"difficult" r.w.: the transition probabilities depend on the position j and also on $i + j$ which decreases in time*
- *upper bound* $\mathrm{P}[$*failure*$] = \mathrm{P}[$*reach 0 before* $r_0 \geq n/2$ *holds*$] \leq$
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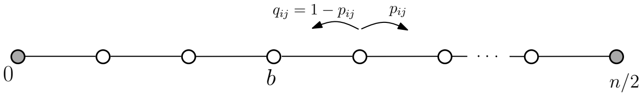

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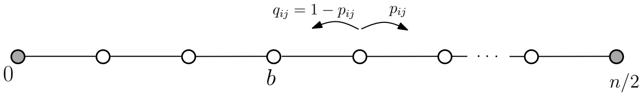

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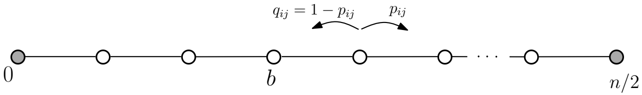

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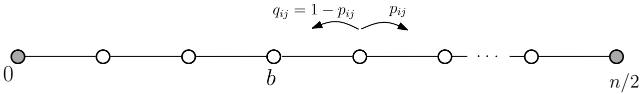

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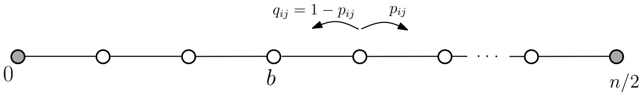

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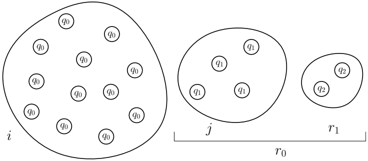

Proof

- *If we set $n' = (n/2) - 1$ we have $i + j \geq n'$*
- *When $r_0 + r_1 = n + 1$ we have $n + 1 = r_0 + r_1 \leq 2r_0 \Rightarrow r_0 \geq n/2$*
- *During the first n effective interactions: $i + j \geq n' = (n/2) - 1$*
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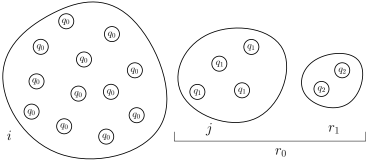

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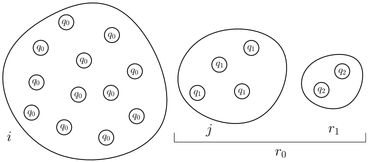

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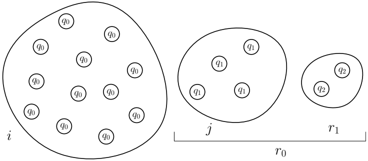

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- *When $r_0 + r_1 = n + 1$ we have $n + 1 = r_0 + r_1 \leq 2r_0 \Rightarrow r_0 \geq n/2$*
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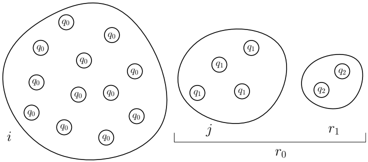
Counting-Upper-Bound Protocol



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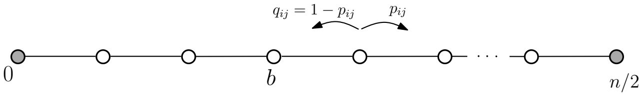

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- $i + j \geq n'$ *implies that* $p_j \geq (n' - j)/n'$ *and* $q_j \leq j/n'$
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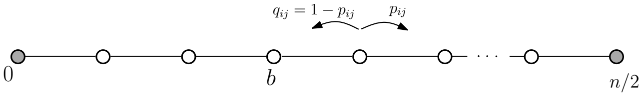

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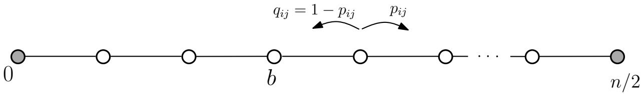

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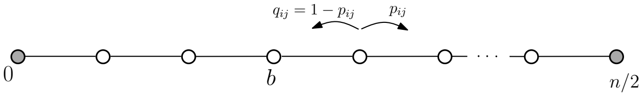

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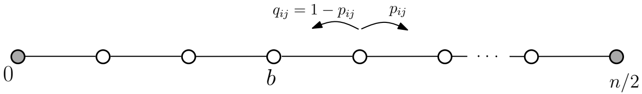

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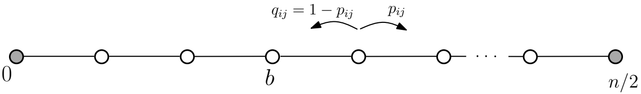

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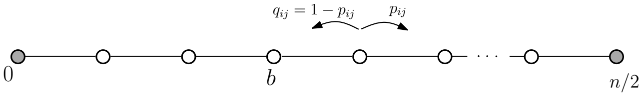

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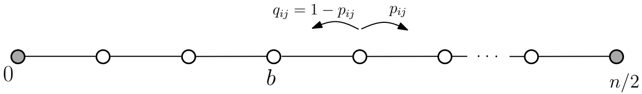

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- *It holds that $j \leq b$, implying that $p \geq (n' - b)/n'$ and $q \leq b/n'$*
- *Set $p = (n' - b)/n'$ and $q = b/n'$*
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- *Imagine now an absorbing barrier at 0 and another one at b*
- *Whenever the r.w. is on $b - 1$ it will either return to b before reaching 0 or it will reach 0 (and fail) before returning to b*
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- *Bound* P[*reach b before 0 (when beginning from position 1)*]
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- *If we set* $x = q'/p' = p/q = (n' - b)/b$ *we have that:*

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  - *With probability $\geq 1 - 1/n^c$ (for constant $c = b - 2$) the protocol has not terminated after at least n effective interactions*
  - *Implies that the leader has counted at least half of the nodes* □

Further Remarks

- Expected running time: $O(n^2 \log n)$ interactions
- Experiments show that in most cases the estimation is closer to $(9/10)n$
- Exact value of n: l waits an additional large polynomial of r_0
- The unique leader seems to be necessary (also some experimental evidence)
  - very interesting open problem
  - w.h.p. all states coexist with $\Theta(n)$ cardinalities [Doty, SODA '14]
  - a node may observe the same as in a fixed population and terminate after meeting a few nodes
- If there is no leader but UIDs we can solve the problem
  - The maximum id can be made to simulate the behavior of a leader

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Generic Constructors

- Characterization for the class of constructible 2D shape languages
- Simulate shape-constructing TMs to realize their output-shape in the distributed system

1. Counting Protocol: constructs w.h.p. a line of length $\Theta(\log n)$, containing n in binary
2. The leader exploits knowledge of n to construct a $\sqrt{n} \times \sqrt{n}$ square
3. Simulate the TM on the square n distinct times, one for each pixel
    - input: index of pixel and $\sqrt{n}$, in binary
    - output: on or off (decision for the corresponding pixel)
4. Release the connected shape consisting of the on pixels

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Characterization

Theorem

Let $\mathcal{L} = (S_1, S_2, \ldots)$ be a connected 2D shape language, such that $\mathcal{L}$ is TM-computable in space d^2. Then there is a protocol that w.h.p. constructs $\mathcal{L}$. In particular, for all $d \geq 1$, whenever the protocol is executed on a population of size $n = d^2$, w.h.p. it constructs S_d and terminates. In the worst case, when G_d (that is, the shape of S_d) is a line of length d, the waste is $(d-1)d = O(d^2) = O(n)$.

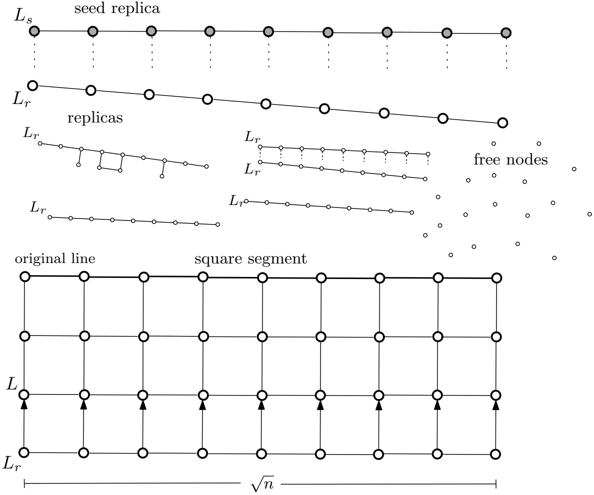
Counting-on-a-line Protocol

- Adapt Counting-Upper-Bound to work in the present model
- The same probabilistic process
- The leader constructs a line that stores the two counters in binary
  - The line grows whenever more space is required

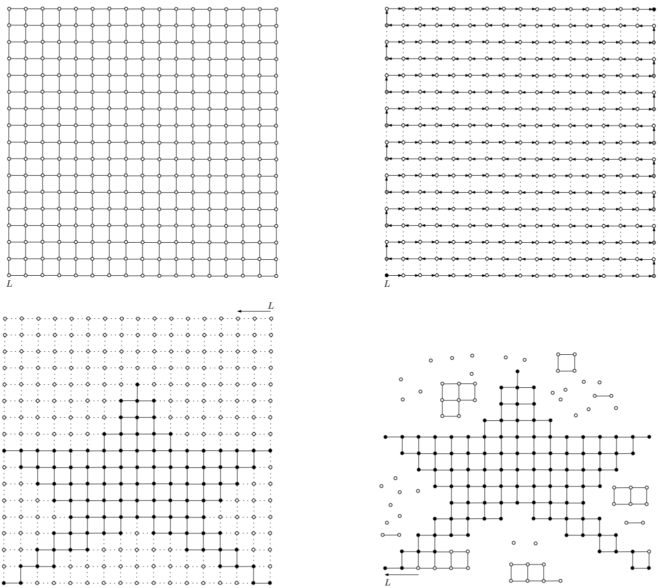
Lemma

Counting-on-a-Line protocol terminates in every execution. Moreover, when the leader terminates, w.h.p. it has formed an active line of length $\log n$ containing n in binary in the r_0 components of the nodes of the line (each node storing one bit).

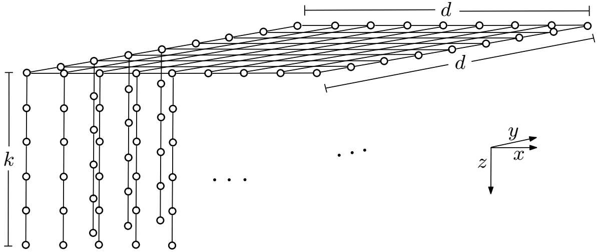
Constructing a $\sqrt{n} \times \sqrt{n}$ Square



Simulating and Releasing



Parallelizing the Simulations



Theorem

Let $\mathcal{L} = (S_1, S_2, \ldots)$ be a TM-computable connected 2D shape language, such that S_d is computable in space $k = f(d)$ and k is computable in space $O(k \cdot d^2)$. Then there is a protocol that w.h.p. constructs $\mathcal{L}$. In particular, for all $d \geq 1$, whenever the protocol is executed on a population of size $n = k \cdot d^2$, w.h.p. it constructs S_d and terminates, by executing d^2 simulations in parallel each with space $O(k)$.

Open Problems

- Complete characterization of constructed networks
- Give a faster than $O(n^3)$ protocol for global line (e.g. $O(n^2 \log n)$)
- Count w.h.p. and terminate if all nodes are initially identical?
- Models of active mobility/actuation (or hybrid active-passive)
- Take other physical considerations into account
  - mass, strength of bonds, rigid and elastic structure, collisions
- Structures that optimize some global property or that achieve a behavior/functionality
- Protocols that efficiently reconstruct broken parts of the structure
- Draw connections to natural processes and to self-assembly and programmable matter models
- We need more real systems-collectives of large numbers of simple interacting devices

Thank You!