Simple and Fast Approximate Counting and Leader Election in Populations

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Population Protocols

- **Distributed computing model** formed by $n$ resource-limited mobile agents
  - Interact in pairs
  - Cannot control their interactions
    - Passive mobility, like particles in a well-mixed solution.
    - A pair of agents interacts in every discrete step under a *uniform random scheduler*.
  - Complete communication graph $G = (V, E)$

- **Anonymous** (i.e., do not have unique IDs) agents.

- **Finite set of states** $Q$.

- **Transition function** $\delta: Q \times Q \rightarrow Q \times Q$.

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Figure 1: Complete Interaction graph. Each colour represents a different state.

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Related Work

➤ **Approximate Counting:** *not much is known*
  - [Michail, PODC, ‘15] computes an approximate count between \(n/2\) and \(n\) in \(O(n \log n)\) parallel time w.h.p.

➤ **Leader Election:**
  - Simple pairwise-elimination protocol requires linear parallel time [AADFP, Distr. Comp., ’06].
  - Any *standard* population protocol requires linear parallel time to solve leader election [DS, DISC, ’15].
  - If we strengthen the PP model, we can develop sub-linear time protocols.
    - [AG, ICALP, ‘15] stabilises in \(O(\log^3 n)\) parallel time, assuming \(O(\log^3 n)\) states at each agent.
    - [GSU, arXiv, ‘18] \(O(\log n \log \log n)\) parallel time, assuming \(O(\log \log n)\) states.
    - [GS, SODA ’18] \(O(\log^2 n)\) parallel time, assuming \(O(\log \log n)\) states.
    - Mediated PP model: [MOKY, Distr. Comp. ‘12], [DLFSV, TAMC, ‘17]
Problems and Definitions

Approximate Counting Problem.
We define as Approximate Counting the problem in which a leader must determine an estimation $\hat{n}$ of the population size, where $\frac{n}{a} < n < \hat{n}$.

Leader Election Problem.
Each node eventually decides whether it is a leader or not, subject to only one node decides that it is the leader.

One-way epidemic.
There is a single node in state $a$, $n-1$ nodes in state $q$ and the only effective transition is $(a, q) \rightarrow (a, a)$. The expected number of steps until all nodes become $a$ is $\Theta(\log n)$ parallel time.
Approximate Counting Protocol

- Requires a **unique leader**.
- Runtime: $\Theta(\log n)$ parallel time.
- $O(1)$ number of states, except for the unique leader ($O(\log^2 n)$ states).

**Protocol 1 Population Size Estimation (PSE)**

\[
\begin{align*}
Q &= \{q, a, l_{c_0,c_1}\} \\
\delta : \\
(l_0,0,q) &\rightarrow (l_1,0,a) \\
(a,q) &\rightarrow (a,a) \\
(l_{c_0,c_1},q) &\rightarrow (l_{c_0+1,c_1},q), \text{ if } c_0 > c_1 \\
(l_{c_0,c_1},a) &\rightarrow (l_{c_0,c_1+1},a), \text{ if } c_0 > c_1 \\
(l_{c_0,c_1},\cdot) &\rightarrow (\text{halt},\cdot), \text{ if } c_0 = c_1
\end{align*}
\]

- The leader is in state $l$, and the rest are in state $q$.
- The leader initiates the epidemic (state $a$).
- The leader stores two counters $c_q$ and $c_a$.
- Counts the number of interactions with $q$ and $a$ nodes.
- Termination condition: $c_q = c_a$
Approximate Counting Protocol

THEOREM 1.

Our Approximate Counting protocol gives a constant-factor approximation of $\log n$ in $O(\log n)$ parallel time w.h.p..

➢ **First Phase** – Number of infected nodes $[1, \frac{n}{2}]$:
  - $c_q$ reaches $O(\log n)$ w.h.p.
  - $c_a$ is increased by a small constant number ($\approx \log n$) (w.h.p. less than $O(\sqrt{\log n})$)
  - The protocol does not terminate w.h.p. until more than half of the population has been infected.

➢ **Second Phase** – Number of infected nodes $(\frac{n}{2}, n]$:
  - $c_q$ is increased by a small constant number ($\approx \log n$) (w.h.p. less than $O(\sqrt{\log n})$)
  - $c_a$ eventually catches up $c_q$.

➢ $2^{c_q}$ is an upper bound on the population size.

➢ First phase: $\Theta(\log n)$ parallel time. Second phase: $\Theta(\log n)$ parallel time.
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Leader Election Protocol

- Assumes that the nodes know an upper bound of $\log(n)$.
- They produce random numbers in $[1, m]$.
- Elects a unique leader in $O(\log^2 n / \log m)$ parallel time w.h.p..
- Initially all nodes are potential leaders (i.e., all nodes start from the same state).
- The protocol proceeds in rounds by monotonously reducing the set of possible leaders, until only one survives.

- All nodes store three variables: Round, Random Number and Counter.
- The Counter is increased by one in every interaction, as long as the node remains leader.
- When Counter reaches $blogn$, it resets it to zero and increases the Round by one.
- They try to spread their tuple $(Round, Random Number)$ throughout the population.
Leader Election Protocol

- We say that a node with the tuple \((e_1, r_1)\) wins during an interaction with \((e_2, r_2)\) if:
  1. \(e_1 > e_2\), or
  2. \(e_1 = e_2\) and \(r_1 > r_2\)

- We call \((e_1, r_1)\) the *dominant tuple* during the round \(e_1\), if it wins every other tuple.
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Leader Election Protocol

THEOREM 2.

Our Leader Election protocol elects a unique leader in $O\left(\frac{\log^2 n}{\log m}\right)$ parallel time w.h.p..

- During a round $e$, the dominant tuple is spread as an epidemic (in $\Theta(\log n)$ parallel time).
- No leader can enter the next round if its tuple has not been spread throughout the whole population before.
- At least one leader will always exist in the population.
- After $O\left(\frac{\log n}{\log m}\right)$ rounds, a unique leader exists in the population w.h.p..
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➢ After $O\left(\frac{\log n}{\log m}\right)$ rounds, a unique leader exists in the population w.h.p.
Composition of our protocols

➢ Our leader election protocol requires a rough estimate on the size of the population

➢ Our approximate counting protocol requires a unique leader

✓ We now combine our protocols in order to construct a leaderless and size-oblivious protocol which elects a unique leader and gives a constant factor approximation of $\log(n)$. 
Composition of our protocols

➢ Leader Election and Approximate Counting protocol:

A. In the Leader Election protocol, instead of using one counter, use two counters $c_q$ and $c_a$

1. $c_q$: Counts the non-followers
2. $c_a$: Counts the followers

B. When $c_q = c_a$

1. Move to next round
2. Reset counters ($c_a = 0$ and $c_q = 1$)
3. Update the value of $s$ as follows: $s' = \frac{s(e_1-1)+c_q}{e_1}$
4. When $e_1 = \left\lfloor\frac{as}{\log m}\right\rfloor$ stop increasing the round.
Composition of our protocols
Conclusions

- Approximate Counting protocol
- Leader Election protocol
- Composition of these protocols

- Can we design a \textit{polylogarithmic} time population protocol that solves the leader election problem, if we allow the agents to communicate only constant amount of bits during an interaction?
Thank You