On the Complexity of the Empire Colouring Problem

Michele Zito
University of Liverpool
M.Zito@liverpool.ac.uk

Coauthors: Andrew R. A. McGrae (Liverpool)

Let r and s be fixed positive integers. Assume that the n vertices of a planar graph are partitioned into blocks (or *empires*) each containing exactly r vertices. The (s, r)-*colouring* problem (s-COL_r) asks for a colouring of the vertices of the graph that uses at most s colours, never assigns the same colour to adjacent vertices in different empires and, conversely, assigns the same colour to all vertices in the same empire, disregarding adjacencies. For $r = 1$ the problem coincides with the classical vertex colouring problem on planar graphs. The generalization for $r \geq 2$ was defined by Percy Heawood in 1890 in the same paper in which he refuted a previous "proof" of the famous Four Colour Theorem.

When $r = 2$ it is well-known that twelve colours are enough to solve any instance of such problem, and in fact four colours suffice if the input graph is a tree. We show that the problem is NP-hard when only three colours are allowed even if the input graph is a forest of paths. We also prove that the problem can be solved in polynomial time on graphs with no induced subgraph of average degree larger than 3/2. The results generalize to larger values of r and s.

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