Colouring random empires in random trees

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(joint work with Andrew McGrae)

We investigate the empire colouring problem (as defined by Heawood, in 1890) for maps whose dual planar graph is a tree. After noticing that if each empire has at most r countries then $2r$ colours are necessary and sufficient (in the worst-case) to solve the problem we concentrate on average-case analysis.

Let $G_r(T_n)$ denote the probability space induced by the process of selecting a random labelled tree T_n on vertex set $V = \{1, \ldots, n\}$ and (independently) a random partition of the set V into $\frac{n}{r}$ blocks (or *empires*) of size r. We call a typical element of the space $G_r(T_n)$ a *random r-empire tree.* In our work we study assignments of colours to V that give the same colour to all vertices in the same block and different colours to vertices in blocks connected by (at least) one edge of T_n. We first prove that, for each fixed $r \geq 1$, there exists a positive integer s_r such that, for large n, almost all n country empire trees with empires of size r cannot be coloured in such a way with at most s_r colours. The values of s_r for the first few values of r are given in the table below.

r	2	3	4	5	6	7	8	9	10	...	20	...	50
s_r	2	3	3	4	4	4	5	5	6	...	9	...	17

(we also find that, for large r, $s_r = \lceil \frac{r}{\log r} \rceil (1 + O(\frac{1}{\log \log r})))$. Furthermore, our main result shows that, by counting the spanning trees of a particular class of graphs, it is possible to find all moments of $Z_{s,r}$, a random variable for the number of balanced s-colourings of $G_r(T_n)$ and, for each integer s, and r greater than one and $k \geq 1$, there exist constants $C_{s,r,k} > 0$ such that

$$\mathrm{E} Z_{s,r}^k \sim C_{s,r,k} (a_n)^k$$

(here $a_n = n^{-\frac{s-1}{2}} (s^{\frac{1}{r}-1}(s-1))^n$). A consequence of our analysis is the following result

Theorem 1 *For any fixed integer $r \geq 2$ and $s > s_r$ a random r-empire tree is s-colourable with (at least) constant probability.*

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