We investigate the empire colouring problem (as defined by Heawood, in 1890) for maps whose dual planar graph is a tree. After noticing that if each empire has at most \( r \) countries then \( 2r \) colours are necessary and sufficient (in the worst-case) to solve the problem we concentrate on average-case analysis.

Let \( G_r(T_n) \) denote the probability space induced by the process of selecting a random labelled tree \( T_n \) on vertex set \( V = \{1, \ldots, n\} \) and (independently) a random partition of the set \( V \) into \( \frac{n}{r} \) blocks (or \emph{empires}) of size \( r \). We call a typical element of the space \( G_r(T_n) \) a \emph{random} \( r \)-empire tree.

In our work we study assignments of colours to \( V \) that give the same colour to all vertices in the same block and different colours to vertices in blocks connected by (at least) one edge of \( T_n \). We first prove that, for each fixed \( r \geq 1 \), there exists a positive integer \( s_r \) such that, for large \( n \), almost all \( n \) country empire trees with empires of size \( r \) cannot be coloured in such a way with at most \( s_r \) colours. The values of \( s_r \) for the first few values of \( r \) are given in the table below.

| \( r \) | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | \ldots | 20 | \ldots | 50 |
|-------|---|---|---|---|---|---|---|---|---|------|----|------|
| \( s_r \) | 2 | 3 | 3 | 4 | 4 | 4 | 5 | 5 | 5 | \ldots | 9 | \ldots | 17 |

(we also find that, for large \( r \), \( s_r = \lceil \frac{r}{\log r} \rceil (1 + O(\frac{1}{\log \log r})) \)). Furthermore, our main result shows that, by counting the spanning trees of a particular class of graphs, it is possible to find all moments of \( Z_{s,r} \), a random variable for the number of balanced \( s \)-colourings of \( G_r(T_n) \) and, for each integer \( s \), and \( r \) greater than one and \( k \geq 1 \), there exist constants \( C_{s,r,k} > 0 \) such that

\[
\mathbb{E} Z_{s,r}^k \sim C_{s,r,k} (a_n)^k
\]

(here \( a_n = n^{-\frac{r-1}{2}} (s^{\frac{1}{r}-1} (s-1))^n \)). A consequence of our analysis is the following result

\textbf{Theorem 1} For any fixed integer \( r \geq 2 \) and \( s > s_r \), a random \( r \)-empire tree is \( s \)-colourable with (at least) constant probability.

MSC2000: 05C80, 05C15.

Keywords: random trees, empire colouring, probability.