On the Approximability of the Maximum Induced Matching Problem

W. Duckworth[1], D. Manlove[2*] and M. Zito[3†]

[1] *Department of Mathematics and Statistics, University of Melbourne, Australia*
billy@ms.unimelb.edu.au
[2] *Department of Computing Science, University of Glasgow, UK*
davidm@dcs.gla.ac.uk
[3] *Department of Computer Science, University of Liverpool, UK*
michele@csc.liv.ac.uk

Abstract

In this paper we consider the approximability of the maximum induced matching problem. We prove that there is some fixed constant c such that the problem of approximating a maximum induced matching in $3s$-regular graphs within a factor of c is NP-hard, for each $s \geq 1$. In addition we give an approximation algorithm with asymptotic performance ratio $d-1$ for the problem of finding a maximum induced matching in a d-regular graph, for each $d \geq 3$.

1 Introduction

For a given graph $G = (V, E)$, an *induced matching* M is a set $M \subseteq E$ such that (i) M is a matching in G and (ii) no two edges in M are joined by an edge of G. That is, the subgraph of G induced by $V(M)$ is exactly the set M. Let $\beta^*(G), \beta_0(G)$ and $\gamma(G)$ denote the size of a maximum induced matching, a maximum independent set and a minimum dominating set in G respectively. Define MIM, MIS (Garey and Johnson [11] problem GT20) and MDS (Garey and Johnson [11] problem GT2) to be the problems of determining $\beta^*(G), \beta_0(G)$ and $\gamma(G)$ respectively, for a given graph G. Let MIMD denote the decision version of MIM, i.e. the problem of deciding, for a given graph G and integer K, whether G admits an induced matching of size of least K.

Stockmeyer and Vazirani [20] introduced MIM as a variant of the maximum matching problem and motivated MIM as the "risk-free" marriage problem: find the maximum number of pairs such that each married person is compatible with no married person other than the one he (or she) is married to. Induced matchings have stimulated a great deal of interest in the discrete mathematics community,

*Supported by EPSRC grant GR/M13329
†Supported by EPSRC grant GR/L/77089

since finding large induced matchings is a subtask of finding a *strong edge-colouring* in a graph (see e.g. [8, 9, 15, 19]), a proper colouring of the edges such that no edge is adjacent to two edges of the same colour.

MIMD is NP-complete, even for bipartite graphs of maximum degree 4 [20]. Additionally Zito [21] shows that MIMD is NP-complete for $4k$-regular graphs for each $k \geq 1$. Ko and Shepherd [17] assert that there is a close relationship between the parameters β^* and γ: namely $\beta^*(S(G)) + \gamma(G) = n$ for any graph $G = (V, E)$, where $n = |V|$ and $S(G)$ denotes the subdivision graph of G. Thus, since MDS is NP-hard for planar cubic graphs [16], an immediate corollary of Ko and Shepherd's observation is that MIMD is NP-complete for planar bipartite graphs, where each vertex in one partite set has degree 2 and each vertex in the other partite set has degree at most 3. [1]

Cameron [5] shows that MIM is solvable in polynomial time for chordal graphs, and Golumbic and Laskar [12] give a polynomial-time algorithm for MIM in circular arc graphs. Recently Golumbic and Lewenstein [13] have constructed polynomial-time algorithms for MIM in trapezoid graphs, interval-dimension graphs and cocomparability graphs, and have given a linear-time algorithm for MIM in interval graphs. Fricke and Laskar [10] give a linear-time algorithm for MIM in trees. Independently, Zito [21], and Golumbic and Lewenstein [13], have given simpler linear-time algorithms for MIM in trees. Note that in [10, 12], induced matchings are referred to as as *strong matchings*.

Regarding the approximability of MIM, Zito [21] has shown that for every $k \geq 1$, there is a constant $c > 1$ such that the problem of approximating MIM within a factor of c on $4k$-regular graphs is NP-hard (so that MIM is APX-complete for $4k$-regular graphs). Additionally the transformation of Cameron [5], showing NP-completeness for MIMD in bipartite graphs, may be regarded as an L-reduction and hence gives APX-completeness for MIM in the same class of graphs. The transformation begins from MIS in arbitrary graphs; by starting from the APX-complete restriction of this problem to cubic graphs [1], the bipartite graph constructed by Cameron's reduction has vertices of degree 7.

We know of no explicit hardness results in the literature regarding the approximability of MIM in graphs of maximum degree 3. However, the above-mentioned relationship between γ and β^* due to Ko and Shepherd, combined with the APX-completeness of MDS in graphs of maximum degree 3 [1] gives APX-completeness for MIM in bipartite graphs of maximum degree 3. In Section 2 the resulting constant c for which it is NP-hard to approximate MIM within a factor of c in bipartite graphs of maximum degree 3 is computed. We give an alternative transformation in Section 2 which substantially improves on this constant c, yielding a tighter bound for the non-approximability of MIM in graphs of maximum degree 3. Additionally we extend our result to cubic graphs.

On the other hand, for d-regular graphs, MIM is approximable within d [21]. In Section 3 we improve this bound by presenting an approximation algorithm for MIM in d-regular graphs which has asymptotic performance ratio $d - 1$, for each $d \geq 3$.

[1] Ko and Shepherd [17] also assert that MIMD is NP-complete for planar cubic graphs; however their reasoning contains an error. In Section 2 we will prove this result as a corollary of establishing the APX-completeness of MIM in cubic graphs.

2 Non-approximability results

We firstly demonstrate that there is some fixed constant c such that the problem of approximating MIM within a factor of c is NP-hard in bipartite graphs of maximum degree 3. The proof makes use of an approximation preserving reduction. Although several notions of such reductions have been proposed (see for example [6]), the L-reduction defined in [18] is perhaps the easiest one to use.

Let P be an optimisation problem. For every instance x of P, and every solution y of x, let $c_P(x,y)$ be the *cost* of the solution y. Let $opt_P(x)$ be the cost of an optimal solution.

Definition 2.1 *Let P and Q be two optimisation problems. An* L-reduction *from P to Q is a four-tuple $(t_1, t_2, \alpha, \beta)$ where t_1 and t_2 are polynomial time computable functions and α and β are positive constants with the following properties:*

(1) t_1 maps instances of P to instances of Q and for every instance x of P,

$$opt_Q(t_1(x)) \leq \alpha \cdot opt_P(x).$$

(2) for every instance x of P, t_2 maps pairs $(t_1(x), y')$ (where y' is a solution of $t_1(x)$) to a solution y of x so that

$$|opt_P(x) - c_P(x, t_2(t_1(x), y'))| \leq \beta |opt_Q(t_1(x)) - c_Q(t_1(x), y')|.$$

Proposition 2.2 *For any $\varepsilon > 0$, the problem of approximating MIM in bipartite graphs of maximum degree 3 within a factor of $\frac{9570}{9569} - \varepsilon$ is NP-hard.*

Proof: Alimonti and Kann [1] prove that MDS is APX-complete for graphs of maximum degree 3. Their L-reduction begins from MVC (Garey and Johnson [11] problem GT1, this is the problem of fining a minimum vertex cover in a given graph) in graphs of maximum degree 3, and has constants $\alpha = 22$ and $\beta = 1$. As mentioned in Section 1, Ko and Shepherd [17] prove that $\beta^*(S(G)) + \gamma(G) = n$ for any graph $G = (V, E)$, where $n = |V|$. Now $\gamma(G) \geq \frac{n}{1+\Delta(G)}$ [14, p.50], where $\Delta(G)$ denotes the maximum degree of G. Hence $\beta^*(S(G)) \leq 3\gamma(G)$ so that there is an L-reduction from MDS in graphs of maximum degree 3 to MIM in bipartite graphs of maximum degree 3, with constants $\alpha = 3$ and $\beta = 1$. Finally Berman and Karpinski [4] prove that, for any $\varepsilon > 0$, the problem of approximating MVC in graphs of maximum degree 3 within a factor of $\frac{145}{144} - \varepsilon$ is NP-hard. Hence by combining these calculations, the result follows from [21, Theorem 5]. ■

For arbitrary graphs of maximum degree 3, it is possible to substantially improve on the lower bound computed in Proposition 2.2 by considering the following alternative transformation.

Theorem 2.3 *For any $\varepsilon > 0$, the problem of approximating MIM in graphs of maximum degree 3 within a factor of $\frac{1260}{1259} - \varepsilon$ is NP-hard.*

Proof: We give a transformation from MIS in cubic graphs. Hence let $G = (V, E)$ (a graph of maximum degree 3) be an instance of this problem. Assume that $V = \{v_1, v_2, \ldots, v_n\}$. We form an instance $G' = (V', E')$ (graph of maximum degree 3)

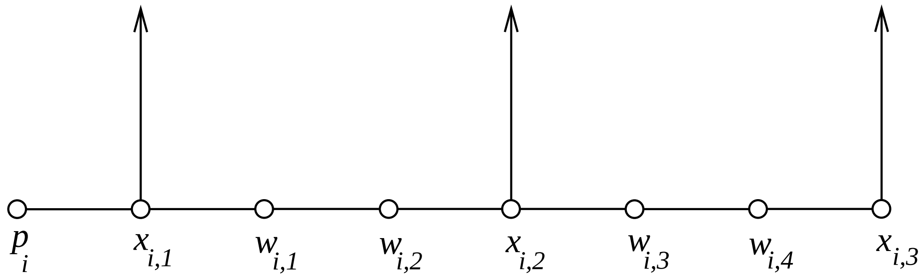

Figure 1: A typical vertex component from the constructed instance of MIM (the arrows denote cross-edges).

of MIM. Corresponding to every vertex $v_i \in V$ $(1 \leq i \leq n)$, construct a *vertex component* C_i of G' as follows. Let the vertices in C_i be V_i, where

$$V_i = \{p_i\} \cup \{x_{i,j} : 1 \leq j \leq 3\} \cup \{w_{i,j} : 1 \leq j \leq 4\}.$$

Join all vertices in V_i to form a P_8 in the order $p_i, x_{i,1}, w_{i,1}, w_{i,2}, x_{i,2}, w_{i,3}, w_{i,4}, x_{i,3}$ (let E_i denote these seven edges). Let $\{v_{i,1}, v_{i,2}, v_{i,3}\}$ be the set of vertices adjacent to v_i in G. For each j $(1 \leq j \leq 3)$, the vertex $x_{i,j}$ of C_i is joined in G' to exactly one of the $x_{r,s}$ vertices in the vertex component C_r corresponding to $v_{i,j}$ respectively, where $v_r = v_{i,j}$ and $1 \leq s \leq 3$ (call such an edge of G' a *cross-edge of* G'). There is obviously a degree of freedom involved in making such attachments; however the actual choice of assignment does not affect the remainder of the proof. Add to E_i the cross-edges of G' incident to vertices in C_i. Let T_i denote the edges $\{p_i, x_{i,1}\}, \{w_{i,2}, x_{i,2}\}, \{w_{i,4}, x_{i,3}\}$ of C_i, and let $T = \bigcup_{i=1}^{i=n} T_i$.

Let $V' = \bigcup_{i=1}^{i=n} V_i$ and let $E' = \bigcup_{i=1}^{i=n} E_i$. It is clear that the graph G' constructed has maximum degree 3. A typical vertex component C_i of G' is illustrated in Figure 1. We now demonstrate that $\beta^*(G') = 2n + \beta_0(G)$.

For, suppose that I is an independent set of G, and let $k = |I|$. We construct a set S as follows. For each i $(1 \leq i \leq n)$, if $v_i \in I$ then add the edges in T_i to S. If $v_i \notin I$ then add the edges $\{w_{i,1}, w_{i,2}\}, \{w_{i,3}, w_{i,4}\}$ to S. It may be verified that S is an induced matching in G', and $|S| = 3k + 2(n - k) = 2n + k$, so that $\beta^*(G') \geq 2n + \beta_0(G)$.

Conversely, let $\mathcal{S}$ denote the set of induced matchings in G', each of size $\beta^*(G')$. Choose S to be a member of $\mathcal{S}$ which maximises $|S \cap T|$. Clearly $|S \cap E_i| \leq 3$ for each i $(1 \leq i \leq n)$, and it may be verified that $|S \cap E_i| = 3$ if and only if each of the vertices $x_{i,1}, x_{i,2}, x_{i,3}$ is incident to some edge of S. In fact, by the choice of S, $|S \cap E_i| = 3$ if and only if $S \cap E_i = T_i$. For otherwise, $S' = (S \backslash E_i) \cup T_i$ is an induced matching in G' at least as big as S, but $|S' \cap T| > |S \cap T|$, a contradiction. Define

$$I = \{v_i \in V : S \cap E_i = T_i\}.$$

We firstly claim that I is independent in G. For if $\{v_i, v_j\} \in E$, then $\{x_{i,r}, x_{j,s}\} \in E'$, where $1 \leq r, s \leq 3$. If $v_i \in I$ then $x_{i,r}$ is incident to some edge of T_i. But S is an induced matching in G', so that $x_{j,s}$ cannot be incident to some edge of S. Thus $S \cap E_j \neq T_j$, so that $v_j \notin I$ as required. Now let $k = |I|$ and suppose for a contradiction that $k < \beta_0(G)$. Then $\beta^*(G') = |S| \leq 3k + 2(n - k) = 2n + k <$

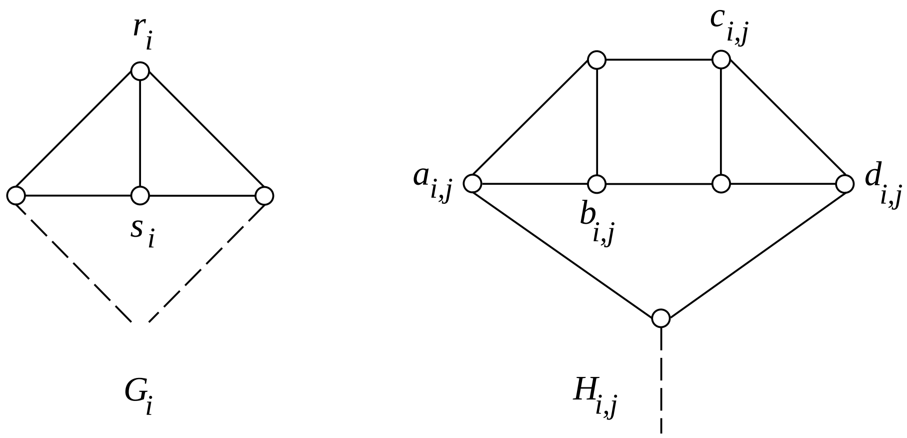

Figure 2: Typical gadgets in the constructed instance of MIM.

$2n + \beta_0(G)$, a contradiction. Hence $k = \beta_0(G)$ so that $\beta^*(G') = 2n + \beta_0(G)$ as required.

Note that any maximal independent set is a dominating set, so $\beta_0(G) \geq \gamma(G)$. Also $\gamma(G) \geq \frac{n}{1+\Delta(G)}$ as mentioned in the proof of Proposition 2.2. Hence $\beta_0(G) \geq \frac{n}{4}$, so that $\beta^*(G') \leq 9\beta_0(G)$. Thus our transformation is an L-reduction, from MIS in graphs of maximum degree 3 to MIM in the same class of graphs, with parameters $\alpha = 9$ and $\beta = 1$. Now Berman and Karpinski [4] prove that, for any $\varepsilon > 0$, the problem of approximating MIS in cubic graphs within a factor of $\frac{140}{139} - \varepsilon$ is NP-hard. Hence by combining these calculations, the result follows from [21, Theorem 5]. ■

We now extend the result of Theorem 2.3 to cubic graphs by considering suitable degree attachments.

Corollary 2.4 *For any $\varepsilon > 0$, the problem of approximating MIM in cubic graphs within a factor of $\frac{7420}{7419} - \varepsilon$ is NP-hard.*

Proof: We use the same transformation as in Theorem 2.3, together with copies of the gadgets shown in Figure 2. Let i ($1 \leq i \leq n$) be given, and consider the vertex component C_i. Using the dashed edges, attach the graph G_i to p_i and attach the graph $H_{i,j}$ to $w_{i,j}$ ($1 \leq j \leq 4$). Then the graph G' so obtained is cubic. It may be verified that any induced matching in G' has at most one edge from G_i and at most two edges from $H_{i,j}$, for any i ($1 \leq i \leq n$) and j ($1 \leq j \leq 4$). Furthermore these upper bounds can be attained by selecting, for example, the edges $\{r_i, s_i\}, \{a_{i,j}, b_{i,j}\}, \{c_{i,j}, d_{i,j}\}$ as shown in Figure 2. Hence we have that $\beta^*(G') = 13n + \beta_0(G)$, so that this revised transformation is an L-reduction with parameters $\alpha = 53$ and $\beta = 1$. Again, using Berman and Karpinski's lower bound for MIS in cubic graphs, the result follows from [21, Theorem 5]. ■

Corollary 2.5 *MIMD is NP-complete, even for planar cubic graphs.*

Proof: Clearly MIMD is in NP. The reduction described by Theorem 2.3 and Corollary 2.4 preserves the planarity of the input graph G. By reducing from the NP-hard restriction of MIS to planar cubic graphs [11, problem GT20], we obtain NP-hardness for MIMD in planar cubic graphs also. ■

Corollary 2.5 raises the question of whether MIM might be hard to approximate within a certain constant even if the input is a planar cubic graph. We remark that MIM has a polynomial-time approximation scheme (PTAS) for planar cubic graphs. To demonstrate this we introduce an additional graph problem. Given a graph $G = (V, E)$, define a set $S \subseteq V$ to be *2-independent* if, for any two vertices $u, v \in S$, $d_G(u, v) \geq 3$. Let M2IS denote the problem of finding a maximum 2-independent set in a given graph G, and let $\beta_0^2(G)$ denote the size of such a set. Also let $L(G)$ denote the line graph of a given graph G. Duckworth et al. [7] observe that $\beta^*(G) = \beta_0^2(L(G))$ and furthermore, there is a 1-1 correspondence between the induced matchings of G and the 2-independent sets of $L(G)$. Additionally, by using the method of Baker [2], they show that M2IS admits a PTAS for planar graphs. Now suppose that G is any planar cubic graph. Then $L(G)$ is planar (c.f. [3, Theorem 10.4]), so that the PTAS for M2IS in planar graphs, together with the simple reduction from MIM to M2IS, gives a PTAS for MIM in planar cubic graphs.

Additionally, note that all hardness results presented so far can be extended to more general classes of graphs showing some regularity. Define a (δ, Δ)-graph to be a graph with minimum degree δ and maximum degree Δ.

Definition 2.6 *Given any integer $s \geq 1$, the* s-padding *of a graph G, denoted by G_s, is obtained by replacing every vertex u by a distinct set of vertices $u_1, \dots, u_s$, with $\{u_i, v_j\} \in E(G_s)$ if and only if $\{u, v\} \in E(G)$ ($i, j \in \{1, 2, \dots, s\}$).*

The following results are simple consequences of Definition 2.6.

Lemma 2.7 *If G is a (δ, Δ)-graph with n vertices and m edges then G_s is a $(s \cdot \delta, s \cdot \Delta)$-graph with sn vertices and $s^2 m$ edges, for any integer $s \geq 1$.*

Lemma 2.8 *For any graph G and any integer $s \geq 1$, $\beta^*(G) = \beta^*(G_s)$.*

The following theorem follows from Proposition 2.2, Corollary 2.4, and Lemmas 2.7 and 2.8.

Theorem 2.9 *For any integer $s \geq 1$ and any $\varepsilon > 0$,*

1. *the problem of approximating MIM in bipartite $(2s, 3s)$-graphs within a factor of $\frac{9570}{9569} - \varepsilon$ is NP-hard.*
2. *the problem of approximating MIM in $3s$-regular graphs within a factor of $\frac{7420}{7419} - \varepsilon$ is NP-hard.*

3 Approximation algorithm

In this section we present a greedy algorithm for approximating MIM in regular graphs. At each step of the algorithm, an edge is added to the induced matching and a number of vertices and edges are deleted in order to guarantee that the matching constructed is indeed induced. The choice of edge to be added to the induced matching is based on selecting vertices of minimum degree.

Algorithm MinGreedy shown in Figure 3 takes a d-regular graph $G = (V, E)$ as input and returns an induced matching $M \subseteq E$, where $d \geq 3$. In what follows, V_i

denotes the set of vertices of degree i and we let Y_i represent $|V_i|$. The set of neighbours of a vertex $v \in V$ is represented by $N(v)$ and for $S \subseteq V$, $adj(S)$ denotes the set of edges incident with vertices in S. We assume the input graph to be connected, otherwise the algorithm may be applied to each connected component. Then, for each step of the algorithm, after the first and before its completion, $\sum_{i=1}^{d-1} Y_i > 0$. The first step of the algorithm involves adding an arbitrary edge to the induced

select $e = \{u, v\}$ from E
$M \leftarrow \{e\}$
$E \leftarrow E \setminus (adj(N(u)) \cup adj(N(v)))$
$V \leftarrow V \setminus (N(u) \cup N(v))$
while $\left(\sum_{i=1}^{d-1} Y_i > 0 \right)$
{
  $j \leftarrow \min_{u \in V} \{ \, deg(u) \, \}$
  select u from V_j
  $k \leftarrow \min_{v \in N(u)} \{ \, deg(v) \, \}$
  select v from $V_k \cap N(u)$
  $M \leftarrow M \cup \{\{u, v\}\}$
  $E \leftarrow E \setminus (adj(N(u)) \cup adj(N(v)))$
  $V \leftarrow V \setminus (N(u) \cup N(v))$
}

Figure 3: Algorithm MinGreedy.

matching and deleting the appropriate edges and vertices. At each subsequent step, a vertex u is chosen from those of minimum degree and a vertex v is chosen from the vertices of minimum degree in $N(u)$. The edge $\{u, v\}$ is added to the induced matching and the appropriate edges and vertices are deleted. We now establish a lower bound on the size of the induced matching returned by Algorithm MinGreedy.

Theorem 3.1 *Given a connected d-regular graph on n vertices, Algorithm MinGreedy returns an induced matching of size at least* $d(n-2)/2(2d-1)(d-1)$*, for each* $d \geq 3$.

Proof: Let $\alpha = (2d-1)(d-1)$. In the first step, the total number of edges deleted is at most

$$1 + 2(d-1) + 2(d-1)^2 = 2d(d-1) + 1 = \alpha + d.$$

Subsequently, at each step, the total number of edges deleted is at most

$$1 + (d-1) + (d-2) + (d-1)^2 + (d-2)(d-1) = (2d-1)(d-1) = \alpha.$$

Assuming the worst case in each step, we have

$$|M| \geq 1 + \frac{\frac{dn}{2} - \alpha - d}{\alpha} = \frac{d(n-2)}{2\alpha} = \frac{d(n-2)}{2(2d-1)(d-1)}. \tag{1}$$

■

Corollary 3.2 *Algorithm MinGreedy approximates MIM for d-regular graphs with asymptotic approximation ratio d-1, for each* $d \geq 3$.

Proof: Zito [21] showed that the size of an optimum induced matching M^* of a d-regular graph on n vertices satisfies the inequality

$$|M^*| \leq \frac{dn}{2(2d-1)}. \tag{2}$$

Using the bounds given in (1) and (2), we derive an upper bound on the approximability of finding a maximum induced matching of a d-regular graph and we have

$$\frac{|M^*|}{|M|} \leq \frac{\frac{dn}{2(2d-1)}}{\frac{d(n-2)}{2(2d-1)(d-1)}} = \frac{n}{n-2}(d-1).$$

■

It may be shown that for each $d \geq 3$, there exists infinitely many d-regular graphs for which Algorithm MinGreedy only realises the lower bound given in (1). In order to demonstrate this, we consider the *operations* performed by the algorithm (an operation being the process of selecting an edge for inclusion into the induced matching and the subsequent deletion of the necessary vertices and edges). An operation may be described using formulae that denotes its effect on the sizes of the sets V_i. For example, the operation shown in Figure 4 adds the edge $\{u, v\}$ to the induced matching and may be represented by the formulae $\{Y_3 \leftarrow Y_3 - 4; Y_2 \leftarrow Y_2 - 4; Y_2 \leftarrow Y_2 + 3;\}$ since vertices labelled u, v, a, b and c are deleted and vertices d, e and f are changed from vertices of degree 3 to vertices of degree 2.

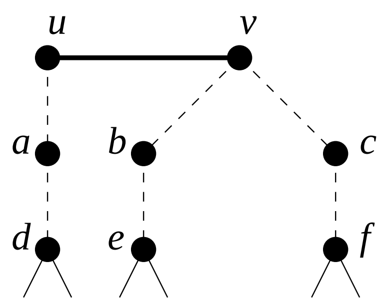

Figure 4: An example operation.

Note that at the start of the algorithm all vertices have degree d and at the end of the algorithm all vertices have degree 0. The change in Y_d over the execution of the algorithm is therefore $-n$. For $1 \leq i \leq d-1$ the *net* change in Y_i over the execution of the algorithm is 0. This is due to the fact that for $1 \leq i \leq d-1$, although Y_i may increase and decrease throughout the execution of the algorithm, $Y_i = 0$ at both the start and the end of the algorithm.

For an arbitrary $d \geq 3$, we construct an infinite family of d-regular graphs for which Algorithm MinGreedy has its poorest worst case performance. For any $k \geq 1$, such a graph $G_{d,k}$ consists of two parts. Firstly we have a subgraph that will be processed by the first step of the algorithm. Secondly the remaining part of the graph will consist of a chain of repeating subgraphs that are processed by the main body of the algorithm.

It may be verified that there exists an operation that may be performed as the first step of the algorithm that destroys vertices of degree d and generates two vertices of degree $d-1$. As an example, Figure 5 shows one such operation for $d = 3$.

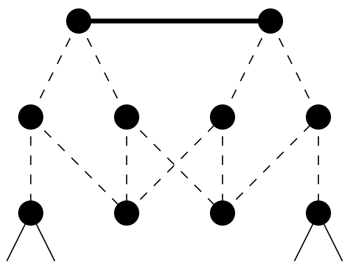

Figure 5: An example first step operation for a cubic graph.

A repeating component is formed by connecting a number of copies of two different subgraphs which we will call Type 1 and Type 2. Type 1 consists of an edge whose endpoints have degree d and $d - 1$. All neighbours of these endpoints have degree d and are distinct. All neighbours of these neighbours are also of degree d and distinct. The subgraph therefore forms a tree structure. Type 2 consists of an edge whose endpoints have degree d and $d - 1$. All neighbours of these endpoints have degree d and are distinct. The set of neighbours form one half of the vertices of a (d-1)-regular bipartite graph on two sets of $2d - 3$ vertices. As an example, Type 1 and Type 2 operations for $d = 3$ are given in Figure 6.

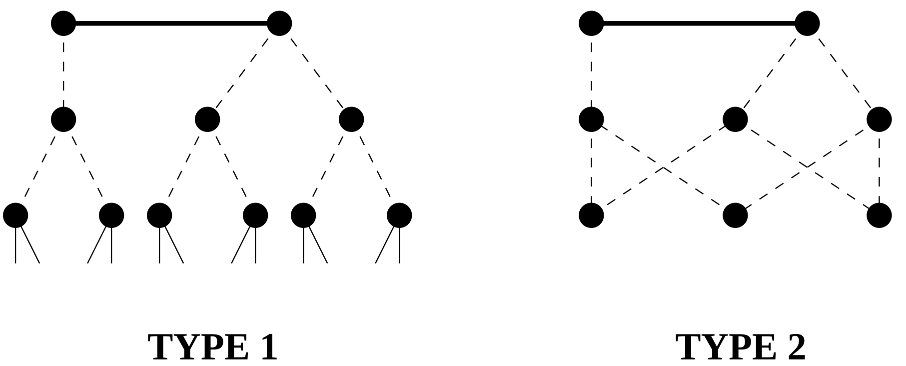

Figure 6: Type 1 and Type 2 operations for Algorithm MinGreedy on a cubic graph.

In each instance the bold edges in Figure 6 are added by Algorithm MinGreedy to the induced matching. The formulae describing the operations of Type 1 and Type 2 are given by $\{Y_{d-1} \leftarrow Y_{d-1}-1; Y_d \leftarrow Y_d-(2d-1)(d-1); Y_{d-1} \leftarrow Y_{d-1}+(2d-3)(d-1)\}$ and $\{Y_{d-1} \leftarrow Y_{d-1}-2(d-1); Y_d \leftarrow Y_d-2(d-1)\}$ respectively. Since for each operation of Type 1, the increase in Y_{d-1} is $(2d - 1)(d - 2)$ and for each operation of Type 2, the decrease in Y_{d-1} is $2(d - 1)$, we may form a *repeating component* by suitably attaching $2(d - 1)$ copies of Type 1 to $(2d - 1)(d - 2)$ copies of Type 2. Again, an example for $d = 3$ is given in Figure 7.

Next we form a *chain* of k repeating components by connecting successive pairs of repeating components by an edge between the vertex labelled y in the first of the

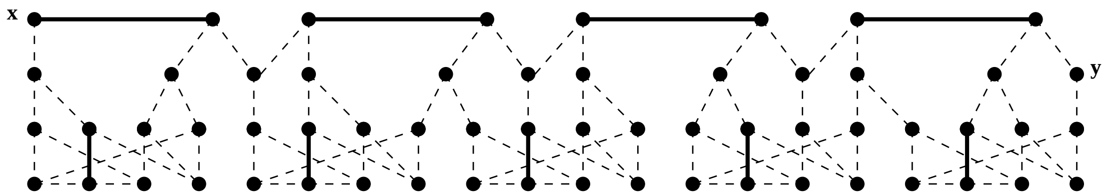

Figure 7: Repeating cubic component.

pair to the vertex labelled x in the second of the pair (these vertices are shown in Figure 7). Finally we obtain a d-regular graph $G_{d,k}$ by connecting the subgraph for the first step of the algorithm to the vertices labelled x and y at either end of the chain of repeating components.

In the worst case Algorithm MinGreedy would initially choose the edge given in the first step which exposes the chain of repeating components. It would then go on to process each repeating component in turn, adding $d(2d-3)$ edges to the induced matching from each repeating component. Hence, in the worst case, the matching M satisfies $|M| = kd(2d-3)+1$. It may be verified that $G_{d,k}$ has $kd\alpha(2d-3)+\alpha+d-1$ edges, where $\alpha = 2d^2 - 3d + 1$. The initial operation deletes $\alpha + d$ edges, and each operation performed thereafter deletes at most α edges. Hence the lower bound of Theorem 3.1 gives $|M| \geq kd(2d-3) + 1 - \frac{1}{\alpha}$, i.e. $|M| \geq kd(2d-3) + 1$ since $d \geq 3$. Thus in the worst case the size of the induced matching returned by Algorithm MinGreedy on $G_{d,k}$ is equal to the lower bound given by Theorem 3.1.

References

[1] P. Alimonti and V. Kann. Hardness of approximating problems on cubic graphs. In *Proceedings of the 3rd Italian Conference on Algorithms and Complexity*, volume 1203 of *Lecture Notes in Computer Science*, pages 288–298, 1997.

[2] B.S. Baker. Approximation algorithms for NP-complete problems on planar graphs. *Journal of the ACM*, 41(1):153–180, 1994.

[3] M. Behzad, G. Chartrand, and L. Lesniak-Foster. *Graphs and Digraphs.* Prinder, Weber and Schmidt International Series, 1979.

[4] P. Berman and M. Karpinski. On some tighter inapproximability results. In *Proceedings of ICALP '99: the 26th International Colloquium on Automata, Languages, and Programming*, volume 1644 of *Lecture Notes in Computer Science*, pages 200–209. Springer-Verlag, 1999.

[5] K. Cameron. Induced matchings. *Discrete Applied Mathematics*, 24:97–102, 1989.

[6] P. Crescenzi. A short guide to approximation preserving reductions. In *Proceedings of CCC '97: the 12th Annual IEEE Conference on Computational Complexity*, pages 262–273, Ulm, 1997. IEEE Computer Society.

[7] W. Duckworth, N.C. Wormald, and M. Zito. A polynomial-time approximation scheme for approximating a sparse 2-spanner of a 4-connected planar triangulation. Submitted for publication in *Journal of Discrete Algorithms*, 2000.

[8] P. Erdős. Problems and results in combinatorial analysis and graph theory. *Discrete Mathematics*, 72:81–92, 1988.

[9] R.J. Faudree, A. Gyárfas, R.H. Schelp, and Z. Tuza. Induced matchings in bipartite graphs. *Discrete Mathematics*, 78:83–87, 1989.

[10] G. Fricke and R. Laskar. Strong matchings in trees. *Congressus Numerantium*, 89:239–244, 1992.

[11] M.R. Garey and D.S. Johnson. *Computers and Intractability.* Freeman, San Francisco, CA., 1979.

[12] M.C. Golumbic and R.C. Laskar. Irredundancy in circular arc graphs. *Discrete Applied Mathematics*, 44:79–89, 1993.

[13] M.C. Golumbic and M. Lewenstein. New results on induced matchings. *Discrete Applied Mathematics*, 101:157–165, 2000.

[14] T. Haynes, S.T. Hedetniemi, and P.J. Slater. *Fundamentals of Domination in Graphs.* Marcel Dekker, 1998.

[15] P. Horák, H. Qing, and W.T. Trotter. Induced matchings in cubic graphs. *Journal of Graph Theory*, 17(2):151–160, 1993.

[16] T. Kikuno, N. Yoshida, and Y. Kakuda. The NP-completeness of the dominating set problem in cubic planar graphs. *Transactions of the Institute of Electronics and Communication Engineers of Japan*, E63(6):443–444, 1980.

[17] C.W. Ko and F.B. Shepherd. Adding an identity to a totally unimodular matrix. Working paper LSEOR.94.14, London School of Economics, Operational Research Group, 1994. Submitted for publication in *SIAM Journal of Discrete Mathematics.*

[18] C.H. Papadimitriou and M. Yannakakis. Optimization, approximation and complexity classes. *Journal of Computer and System Sciences*, 43(3):425–440, 1991.

[19] A. Steger and M. Yu. On induced matchings. *Discrete Mathematics*, 120:291–295, 1993.

[20] L.J. Stockmeyer and V.V. Vazirani. NP-completeness of some generalizations of the maximum matching problem. *Information Processing Letters*, 15(1):14–19, 1982.

[21] M. Zito. Maximum induced matchings in regular graphs and trees. In *Proceedings of WG '99: the 25th International Workshop on Graph-Theoretic Concepts in Computer Science*, volume 1665 of *Lecture Notes in Computer Science*, pages 89–100. Springer-Verlag, 1999.