Packing Edges in Random Regular Graphs

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Abstract. A k-separated matching in a graph is a set of edges at distance at least k from one another (hence, for instance, a 1-separated matching is just a matching in the classical sense). We consider the problem of approximating the solution to the maximum k-separated matching problem in random r-regular graphs for each fixed integer k and each fixed $r \geq 3$. We prove both constructive lower bounds and combinatorial upper bounds on the size of the optimal solutions.

1 Introduction

In this paper we consider graphs generated according to the $\mathcal{G}(n,r\text{-reg})$ model (see, for example, [9, Chapter 9]). Given n urns, each containing r balls (with rn even), a set of $rn/2$ distinct pairs of balls is chosen *uniformly at random* (u.a.r.). Then a (random) n-vertex graph, $G = (V, E)$, is obtained by identifying the n urns with the vertices of the graph and letting $\{i, j\} \in E$ if and only if there is at least one pair with one ball belonging to urn i and the other ball belonging to urn j. The maximum degree of a vertex in G (i.e. the maximum number of edges incident to a vertex) is at most r. Moreover, for every integer $r > 0$, there is a positive fixed probability that the random pairing contains neither pairs with two balls from the same urn nor couples of pairs with balls coming from just two urns (see, for example, [16, Section 2.2]). In this case the graph is r-regular (all vertices have degree r). Notation $G \in \mathcal{G}(n,r\text{-reg})$ will signify that G is selected according to the $\mathcal{G}(n,r\text{-reg})$ model. An event, $\mathcal{E}_n$, describing a property of a random graph depending on a parameter n, holds *asymptotically almost surely* (a.a.s.), if the probability that $\mathcal{E}_n$ holds tends to one as n tends to infinity.

The distance between two vertices in a graph is the number of edges in a shortest path between the two vertices. The distance between two edges $\{u_1, u_2\}$ and $\{v_1, v_2\}$ is the minimum distance between any two vertices u_i and v_j. For any positive integer k, a *k-separated matching* of a graph, is a set of edges, $\mathcal{M}$, such that the minimum distance between any two edges in $\mathcal{M}$ is at least k (the qualifier "separated" will normally be omitted in the rest of this paper). Let $\nu_k(G)$ be the size of a largest k-matching in G. The *maximum k-matching* (MkM) problem asks for a k-matching of size $\nu_k(G)$. For $k = 1$ this is the classical maximum matching problem [11]. Stockmeyer and Vazirani [14] introduced the generalisation for $k \geq 2$, motivating it (for $k = 2$) as the "risk-free marriage problem" (find the maximum number of married couples such that each person is compatible only with the person (s)he is married to). The M2M problem (also known as the *maximum induced matching* problem) stimulated much interest in other areas of theoretical computer science and discrete mathematics as finding a maximum 2-matching of a graph is a sub-task of finding a strong edge-colouring of a graph (a proper colouring of the edges such that no edge is incident with more than one edge of the same colour as each other, see (for example) [7, 8, 10, 13]). The separation constraint imposed on the matching edges when $k > 2$ is a distinctive feature of the MkM problem and the main motivation for our algorithmic investigation of such problems.

MkM is NP-hard [14] (polynomial time solvable [6]), for each $k \geq 2$ (for $k = 1$). Improved complexity results are known for M1M [12] on random instances. In particular it has been proven that simple greedy heuristics a.a.s. produce sets of $\frac{n}{2} - o(n)$ independent edges [1] in dense random graphs and random regular graphs. A number of results are known on the approximability of an optimal 2-matching [2, 3, 17]. Zito [18] presented some simple results on the approximability of an optimal 2-matching in dense random graphs.

In this paper, we consider several natural (and simple) strategies for approximating the solution to the MkM problem, for each positive integer k, and analyse their performance on graphs generated according to the $\mathcal{G}(n,r\text{-reg})$ model. We also prove combinatorial upper bounds on $\nu_k(G)$ which hold a.a.s. if $G \in \mathcal{G}(n,r\text{-reg})$. The algorithm we present for M2M was analysed deterministically in [3] where it was shown to return a 2-matching of size at least $r(n-2)/2(2r-1)(r-1)$ in a connected r-regular graph on n vertices, for each $r \geq 3$. Furthermore, it was shown that there exist infinitely many r-regular graphs on n-vertices for which the algorithm only achieves this bound. For the case $r = 3$, the cardinality of a largest 2-matching $\mathcal{M}$ of a random 3-regular graph a.a.s. satisfies $0.26645n \leq |\mathcal{M}| \leq 0.282069n$ [5] (unfortunately the optimistic $0.270413n$ lower bound claimed in the paper is not correct). We analyse the performances of such algorithm when the input is distributed according to $\mathcal{G}(n,r\text{-reg})$. Generalisations of this algorithm to the case $k > 2$ fails. Any greedy process based on a sequence of local choices/updates has no permanent record of the original neighbourhood structure of each vertex. Hence, an edge chosen by the algorithm to add to $\mathcal{M}$ may cause the matching not to be k-separated. For $k > 2$, different strategies, based on a more selective updating, must be used.

The following Theorem encompasses the results of this paper. Its proof appears in subsequent sections.

Theorem 1. *For each fixed positive integer k and fixed integer $r \geq 3$ there exist two positive real numbers $\lambda_k = \lambda_k(r)$ and $\mu_k = \mu_k(r)$ such that if $G \in \mathcal{G}(n, r\text{-reg})$, then $\lambda_k n \leq \nu_k(G) \leq \mu_k n$ a.a.s..*

The table below reports the values of λ_k and μ_k for the first few values of r and k.

r	M1M		M2M		M3M		M4M	
	λ_1	μ_1	λ_2	μ_2	λ_3	μ_3	λ_4	μ_4
3	0.5	0.5	0.26645	0.28207	0.11239	0.15605	0.03943	0.09455
4	0.5	0.5	0.22953	0.25	0.07314	0.10757	0.01672	0.05007
5	0.5	0.5	0.20465	0.22695	0.05071	0.07922	0.01856	0.02933
6	0.5	0.5	0.18615	0.2091	0.03801	0.0611	0.00478	0.01859
7	0.5	0.5	0.17157	0.19465		0.04876	0.00317	0.01251
8	0.5	0.5	0.15964	0.18259		0.03994	0.00201	0.00882
9	0.5	0.5	0.14962	0.17231		0.0334	0.00141	0.00646
10	0.5	0.5	0.14106	0.1634		0.02841	0.00104	0.00488

2 Lower Bounds

In this section we describe the simple greedy heuristics used to construct large k-matchings. The algorithms are quite general and may be applied to any graph. The analyses presented at the end of each sub-section give lower bounds on the size of the resulting k-matching if the input graph is selected according to the $\mathcal{G}(n, r\text{-reg})$ model.

2.1 Dense matchings

Next we describe the algorithm that will be used to find a large k-matching in a random r-regular graph, when[3] $k \leq 2$. Let $\Gamma(u) = \{v \in G : \{u, v\} \in E\}$ be the *neighbourhood of vertex u*.

Algorithm DegreeGreedy(G, k)
Input: a graph $G = (V, E)$ on n vertices.
$\mathcal{M} \leftarrow \emptyset$;
while $E \neq \emptyset$
let u be a vertex of minimum positive degree in $V(G)$;
if $|\Gamma(u)| > 0$
let v be a vertex of minimum positive degree in $\Gamma(u)$;
$\mathcal{M} \leftarrow \mathcal{M} \cup \{\{u, v\}\}$;
shrink$(G, \{u, v\}, k)$;

Ties in the selection of u and v may be broken by any reasonable rule, e.g. making selections at random or according to some predefined ordering on the graph vertices. For each iteration of the algorithm, procedure *shrink* updates G by removing u, v, all vertices within distance $k-1$ from $\{u, v\}$ and all edges incident with these vertices.

Let $V_i = \{v : |\Gamma(v)| = i\}$. In the following discussion G always denotes the subgraph of the input graph still to be dealt with at some point during the execution of DegreeGreedy. In each iteration of the main while loop a further portion of G is dealt with.

[3] we believe that the values reported in Section 1 justify the attribute "dense" in the title of this section.

The analysis below is based on the fact that if $G \in \mathcal{G}(n, r\text{-reg})$ the state of the dynamic process associated with the execution of DegreeGreedy can be described by the vector $(|V_1|, |V_2|, \ldots, |V_r|)$. Furthermore, the evolution of such a vector is (approximately) Markovian and quite peculiar. The algorithm essentially runs for $r-1$ *phases*: Phase 1, Phase 2, …, Phase $r-1$. In Phase j (for $j < r-1$) there are essentially no vertices of degree less than $r-j-1$, and few vertices of degree $r-j-1$ which are used up as soon as they are created. Therefore, the vertex u is selected predominantly from V_{r-j}, and occasionally (but rarely) from V_{r-j-1}. However, towards the end of the phase, the rate at which vertices of degree $r-j-1$ are generated becomes larger than the rate at which they are consumed. This determines the transition to Phase $j+1$. We refer to one iteration of the main while loop that selects u from V_j as a *selection of a vertex of degree* j. A *clutch* (or *step*) in a given Phase j is defined as a sequence of iterations of the main while loop. The first of these iterations selects u from V_{r-j}, and the remainder selects u from V_{r-j-1}. The last iteration in the clutch is the last one before the next u of degree $r-j$ is selected.

For $1 \le j \le r-1$, let $Y_i = Y_i^j(t)$ denote $|V_{r-i}|$ after step t of Phase j. Let $Y_0^1(0) = n$ and $Y_i^1(0) = 0$ for all $i > 0$. Furthermore, let $Y_i^{j+1}(0)$ be equal to the final value of Y_i in Phase j for all $i \ge 0$ and $1 \le j \le r-2$. Finally, let $X^j(t) = \sum_{i=0}^{r-1}(r-i)Y_i^j(t)$. From now on the dependency on t and j will be omitted unless ambiguity arises. The key ingredient in the analysis of the algorithm above is the use (in each Phase) of Theorem 6.1 from [16] which provides tight asymptotics for the most likely values of Y_i (for each $i \in \{0, \ldots, r-1\}$), as the algorithm progresses through successive Phases.

Note that it would be fairly simple to modify the graph generation process described in Section 1 to incorporate the decisions made by the algorithm DegreeGreedy: the input random graph and the output structure $\mathcal{M}$ would be generated at the same time. In this setting one would keep track of the degree sequence of the so called *evolving graph* H_t (see for instance [15]). H_0 would be empty, and a number of edges would be added to H_t to get H_{t+1} according to the behaviour of algorithm DegreeGreedy during step $t+1$. The random variables Y_i also denote the number of vertices of degree i in the evolving graph H_t. We prefer to give our description in terms of the original regular graph. In this way the algorithm is easier to understand, and all details about the analysis of its performances are kept away from the programmer.

Let $\mathrm{E}_j(\Delta Y_i)$ denote the expected change of Y_i during step $t+1$ in some given Phase j, conditioned to the history of the algorithm execution from the start of the phase until step t. This is asymptotically the sum of two terms: the expected change due to the updating following the selection of a vertex of degree $r-j$ at the beginning of a clutch and the one due to the updating in the remaining part of the clutch. We denote by $\boldsymbol{d} \equiv (d_1, d_2, \ldots, d_{|\Gamma(u)|})$ the degrees of u's neighbours in $G \setminus \{u\}$. In what follows we assume that a.a.s. $r-j-1 \le d_1 \le d_2 \le \ldots \le d_{|\Gamma(u)|} \le r-1$. If Births_{j+1} denotes the expected number of vertices of degree $r-(j+1)$ generated during a clutch when $j < r-1$ (and $\text{Births}_r = 0$) then the asymptotic expression for $\mathrm{E}_j(\Delta Y_i)$ is

$$\sum_{\boldsymbol{d}} \mathrm{E}_j(\Delta Y_i \mid \boldsymbol{d}) \Pr[\boldsymbol{d}] + \mathrm{Births}_{j+1} \sum_{\boldsymbol{d}'} \mathrm{E}_{j+1}(\Delta Y_i \mid \boldsymbol{d}) \Pr[\boldsymbol{d}'] \tag{1}$$

where $\mathrm{E}_j(\Delta Y_i \mid \boldsymbol{d})$ is the expected change of Y_i conditional to the degree(s) of u's neighbour(s) in $G \setminus \{u\}$ being described by $\boldsymbol{d}$. The expression $\Pr[\boldsymbol{d}]$ denotes (asymptotically) the probability that configuration $\boldsymbol{d}$ occurs conditioned to the history of the algorithm so far. The first (second) sum is over all possible configurations $\boldsymbol{d}$ (resp. $\boldsymbol{d}'$) with $|\boldsymbol{d}| = r - j$ (resp. $|\boldsymbol{d}'| = r - j - 1$). Also the expected change in the size of the structure output by the algorithm, $\mathrm{E}_j(\Delta|\mathcal{M}|)$, is asymptotically

$$1 + \mathrm{Births}_{j+1} \tag{2}$$

since an edge is added to $\mathcal{M}$ following each selection. Setting $x = t/n$, $y_i^j(x) = Y_i^j/n$ (again dependency on j will be usually omitted), and $\lambda = \lambda_k(x) = |\mathcal{M}|/n$, the following system of differential equations is associated with each Phase j,

$$\frac{\mathrm{d}y_i}{\mathrm{d}x} = \tilde{\mathrm{E}}_j(\Delta Y_i)\Big|_{t=xn, Y_i=y_i n} \qquad \frac{\mathrm{d}\lambda}{\mathrm{d}x} = \tilde{\mathrm{E}}_j(\Delta|\mathcal{M}|)\Big|_{t=xn, Y_i=y_i n} \tag{3}$$

where $\tilde{\mathrm{E}}_j(\Delta Y_i)$ and $\tilde{\mathrm{E}}_j(\Delta|\mathcal{M}|)$ denote the asymptotic expressions for the corresponding expectations obtained from (1) and (2) using the estimates on $\Pr[\boldsymbol{d}]$ and $\mathrm{E}_j(\Delta Y_i \mid \boldsymbol{d})$ given later on. Since x does not occur in $\tilde{\mathrm{E}}_j(\Delta Y_i)$ and $\tilde{\mathrm{E}}_j(\Delta|\mathcal{M}|)$, we actually solve, one after the other, the systems

$$\frac{\mathrm{d}y_i}{\mathrm{d}\lambda} = \frac{\tilde{\mathrm{E}}_j(\Delta Y_i)}{\tilde{\mathrm{E}}_j(\Delta|\mathcal{M}|)}\Big|_{Y_i=y_i n} \tag{4}$$

where differentiation is w.r.t. λ, setting $y_0^0(0) = 1$ and $y_i^0(0) = 0$ for all $i \geq 1$ and using the final conditions of Phase j, as initial conditions of Phase $j+1$ for each $j > 0$. The value of λ_k reported in the table in Section 1 is the sum of the final values of λ in each Phase.

We use Theorem 6.1 (see Appendix) from [16] to show that, during each phase, the values of $|\mathcal{M}|/n$ and Y_i/n a.a.s. remain within $o(1)$ of the corresponding deterministic solutions to the differential equations (4). To apply such result to the multi-dimensional random process $(t/n, |\mathcal{M}|/n, Y_0/n, \ldots, Y_{r-1}/n)$ we need to define a bounded connected open set in $\mathbb{R}^{r+2}$ containing the closure of the set

$$\{(0, z^{(1)}, \ldots, z^{(r+1)}) : \Pr[Y_i(t) = z^{(i+2)}n, 0 \leq i \leq r-1] \neq 0 \text{ for some } n\},$$

and then verify that the boundedness, trend, and Lipschitz hypotheses hold uniformly for each t smaller than the minimum t such that $(t/n, |\mathcal{M}|/n, Y_0/n, \ldots, Y_{r-1}/n) \notin \mathcal{D}$. For Phase j, where $j < r - 1$, and for an arbitrarily small $\epsilon > 0$, define $\mathcal{D}_{\epsilon,j}$ to be the set of all $(x, z^{(1)}, \ldots, z^{(r+1)})$ for which $x > -\epsilon$, $\sum_{i=2}^{r+1} z^{(i)} > \epsilon$, $\mathrm{Births}_{j+1} < (1-\epsilon)n$, $z^{(1)} > -\epsilon$ and $z^{(i)} < 1 + \epsilon$ where $1 \leq i \leq r+1$. It is fairly simple, if tedious, to verify that the functions under consideration satisfy the three main hypotheses of Wormald's theorem in the domain $\mathcal{D}_{\epsilon,j}$. A similar argument applies to Phase $r-1$ as well except that, here, a clutch is equivalent to processing just one vertex.

Some additional work is required to ensure that once the end of Phase j is reached, the process proceeds as described informally into the next phase. This involves ensuring that the expected number of births in a clutch (divided by n) rises strictly above 1. This may be achieved by computing partial derivatives of the equations representing the expected changes in the variables Y_i for processing a vertex (not a clutch) in Phase j. For reasons of brevity, we do not include these details here.

From the point in Phase $r-1$ after which Wormald's analysis tool does not apply until the completion of the algorithm, the change in each variable per step is bounded by a constant. Hence, the change in the random variables Y_i and $|\mathcal{M}|$ is $o(n)$. This completes the proof of the lower bounds in Theorem 1.

In the remainder of this section details are given on how to compute asymptotic expressions for $\Pr[\boldsymbol{d}]$, $\mathrm{E}_j(\Delta Y_i \mid \boldsymbol{d})$, for each $\boldsymbol{d}$, and Births_{j+1}, and some comments are made on how to solve the systems in (4).

Probability of a configuration. The formula for $\Pr[\boldsymbol{d}]$ is better understood if we think of the algorithm DegreeGreedy as embedded in the graph generation process. Each configuration $\boldsymbol{d} \equiv (d_1, d_2, \ldots, d_{|\Gamma(u)|})$ occurs at the neighbourhood of a given u if the $|\Gamma(u)|$ balls still available in the urn U associated with u are paired up with balls from random urns containing respectively $d_1+1, d_2+1, \ldots, d_{|\Gamma(u)|}+1$ free balls. The probability of pairing up one ball from U with a ball from an urn with $r-i$ free balls is $P_i = \frac{(r-i)Y_i}{X}$ for $i \in \{0, \ldots, r\}$ at the beginning of a step, and this only changes by a $o(1)$ factor during each step due to the multiple edge selections which are part of a step. Let $\Pr[\boldsymbol{d}]$ be $\binom{|\Gamma(u)|}{m_1, \ldots, m_{|\Gamma(u)|}} P_{r-(d_1+1)} \cdot P_{r-(d_2+1)} \cdot \ldots \cdot P_{r-(d_{|\Gamma(u)|}+1)}$ where m_z are the multiplicities of the possibly $|\Gamma(u)|$ distinct values occurring in $\boldsymbol{d}$.

Conditional expectations. Note that the sequence $\boldsymbol{d}$ contains all the information needed to compute asymptotic expressions for all the conditional expected changes of the variables Y_i. We define

$$\mathrm{E}_j(\Delta Y_i \mid \boldsymbol{d}) = -\mathrm{rm}(r, j, i, \boldsymbol{d}) - \partial(r, j, i, \boldsymbol{d})$$

where $\mathrm{rm}(r, j, i, \boldsymbol{d})$, gives, for each $\boldsymbol{d}$ in a given Phase j, the number of vertices of degree $r-i$ removed from the subgraph of G induced by $\{u\} \cup \Gamma(u)$ and ∂ is a function accounting for the degree changes occurring outside such a subgraph. Let $Q_i(0) = P_i - P_{i-1}$ for $i \in \{0, \ldots, r\}$ (with $P_{-1} = 0$) and $Q_i(1) = \sum_{z=0}^{r-1} P_z(\delta_{i,z} + (r-z-1)Q_i(0))$, for $i \in \{0, 1, \ldots, r\}$ (where $\delta_{x,y} = 1$ if $x = y$ and zero otherwise). The expression $-Q_i(0)$ describes asymptotically the contribution to $\mathrm{E}_j(\Delta Y_i \mid \boldsymbol{d})$ given by the removal from G of one edge incident to a vertex whose degree is decreased by one during one execution of *shrink*. Expression $-Q_i(1)$ is (asymptotically) the contribution to $\mathrm{E}_j(\Delta Y_i \mid \boldsymbol{d})$ given by the removal from G of the vertices in $\Gamma(v)$ (and all their incident edges). We have

$$\begin{aligned}
\mathrm{rm}(r, j, i, \boldsymbol{d}) &= \delta_{i,j} + \delta_{i,r-(d_1+1)} + \delta_{k,2} \textstyle\sum_{h=2}^{|\Gamma(u)|} \delta_{i,r-(d_h+1)}, \\
\partial(r, j, i, \boldsymbol{d}) &= d_1 Q_i(k-1) + \textstyle\sum_{h=2}^{|\Gamma(u)|} (\delta_{k,1}(\delta_{i,r-(d_h+1)} - \delta_{i,r-d_h}) + \delta_{k,2} d_h Q_i(0)).
\end{aligned}$$

Vertices of degree $r-j-1$ in Phase j. We remind the reader that Births_{j+1} denotes the expected number of vertices of degree $r-j-1$ generated during a clutch. This quantity can be computed by modelling the generation of these vertices as a discrete branching process. The first selection in a clutch will, asymptotically, generates $\mathrm{E}_j(\Delta Y_{j+1})$ vertices of degree $r-j-1$: this can be considered as the first generation of the branching process. In general, the bth generation will contain approximately $\mathrm{E}_j(\Delta Y_{j+1})(\mathrm{E}_{j+1}(\Delta Y_{j+1}))^{b-1}$ vertices, and therefore, provided $\mathrm{E}_{j+1}(\Delta Y_{j+1})$ is strictly smaller than one, the asymptotic expression for Births_{j+1} is

$$\frac{\mathrm{E}_j(\Delta Y_{j+1})}{1-\mathrm{E}_{j+1}(\Delta Y_{j+1})}$$

(and after all these vertices have been removed from G, there is no vertex of degree $r-j-1$ left).

> *Computational aspects.* (STILL TO BE CHANGED) The systems of differential equations have been solved using high precision methods (Cash-Karp) with error correction. In all cases the equations are quite stable and the numbers provided in the table in Section 1 are accurate to the fifth decimal digit.

2.2 Sparse matchings

Any obvious generalisation of the algorithm DegreeGreedy to the case $k>2$ fails. The DegreeGreedy process, which repeatedly picks sparsely connected edges $\{u,v\}$ and removes their neighbourhood within distance $k-1$, has no permanent record of the original neighbourhood structure of each vertex. Hence, an edge chosen to add to $\mathcal{M}$ may cause the matching not to be k-separated. For $k>2$ we therefore resort to a different heuristic to approximate ν_k. Such an algorithm is based on a more selective process that repeatedly picks vertices of degree r from the given graph, explores their neighbourhood at distance at most $\lfloor k/2 \rfloor$ and performs different updatings depending on the degree structure around the chosen vertex. We treat the k even and k odd cases separately, as some extra care is needed to define a large k-matching when k is odd.

Let $t_0(r)$ be the trivial tree formed by a single vertex. Let $t_d(r)$ be the (rooted) tree obtained by taking r copies of $t_{d-1}(r)$ and joining their roots to a new vertex u. For any integer $k \geq 2$, the tree $T_k(r)$ is a rooted tree whose root u has a child v which is the root of a copy of $t_{\lfloor k/2 \rfloor -1}(r-1)$ and, when $k \geq 4$, $r-1$ other children $v_2, \ldots, v_r$ which are roots of copies of $t_{\lfloor k/2 \rfloor -2}(r-1)$. It can be easily verified that $T_k(r)$ contains $\mathrm{rem}(r,k) = \frac{2(r-1)^{\lfloor k/2 \rfloor}-2}{r-2}$ vertices.

The k-matching algorithm can be described as follows :

Algorithm Sparse(G,k)
Input: an r-regular graph $G=(V,E)$ on n vertices.
    $\mathcal{M} \leftarrow \emptyset$;
    while $V_r \neq \emptyset$
        let u be a vertex of degree r in $V(G)$;

if (u is the root of a copy of $T_k(r)$ such that the
vertex v is chosen at u.a.r. in $\Gamma(u)$ and
$\mathcal{M} \cup \{\{u, v\}\}$ is a k-matching)
$\mathcal{M} \leftarrow \mathcal{M} \cup \{\{u, v\}\}$;
remove $T_k(r)$ from G;
else
remove u from G;

It would be quite simple to modify the description above so that the algorithm could handle any graph on n vertices. For clarity of exposition we preferred to state the process for r-regular graphs only.

Analysis for even k. The dynamics of this algorithm can again be described by looking at the evolution of the (random) vector $(|V_1|, \ldots, |V_r|)$. Furthermore the analysis in such case is much simpler than in the case $k \leq 2$. During each iteration of the main while loop one of two things can happen. If a copy of $T_k(r)$ is found "around" u, then the matching is updated and $\frac{2(r-1)^{\frac{k}{2}}-2}{r-2} = r^{O(k)}$ vertices are removed from G. Otherwise we simply remove u (along with its incident edges) from G. In other words the algorithm is often mimicking a simple process which builds an independent dominating set of vertices in a regular graph by repeatedly stripping off vertices of degree r. Such process has been analysed in [15]. Therefore it is fairly easy to verify that $\mathrm{E}(\Delta Y_i)$ is asymptotically

$$-\delta_{i,0} - rQ_i(0) + ((r+1-\mathrm{rem}(r,k))\delta_{i,0} + r\delta_{i,1} - 2(r-1)^{\frac{k}{2}} Q_i(0)) \times \mathrm{E}(\Delta \mathcal{M})$$

where $\mathrm{E}(\Delta \mathcal{M})$ can be approximated (asymptotically) by $\left(\frac{rY_0}{X}\right)^{\mathrm{rem}(r,k)-1}$. Solving numerically the associated differential equations leads, for $k = 4$ to the values reported in the table in Section 1.

Analysis for odd k. When k is odd the analysis needs some additional book-keeping to model the process that ensures that the matching is actually k-separated. Let V_i^+ (V_i^-), for each $i \in \{1, \ldots, r-1\}$, be the collection of vertices of degree i in G which are at distance at least two (resp. one) from a matching edge, and define $Y_{r-i}^{\mathrm{sgn}} = |V_i^{\mathrm{sgn}}|$ for each sgn $\in \{``+", ``-"\}$ (with $Y_0 = |V_r|$ as usual). Let $\boldsymbol{D}$ denote an arbitrary configuration around u. For arbitrary k, configurations describe, essentially, the sequence of degrees of the vertices at distance at most $\lceil k/2 \rceil$ from u. *Good* configurations are those leading to an increas in $|\mathcal{M}|$. The expected change in Y_i^{sgn} can be expressed in the "usual" way in terms of a linear combination of conditional expectations. Function $\mathrm{E}(\Delta Y_i^{\mathrm{sgn}} \mid \boldsymbol{D})$ takes different expressions depending on whether $\boldsymbol{D}$ is a good configuration or not. If $\boldsymbol{D}$ is not good then the conditional expectation is a.a.s.

$$-\delta_{i,0} - \sum(\delta_{i,r-(|d_h|+1)} - \delta_{i,r-|d_h|})$$

(the sum being over all d_h referring to vertices in $\Gamma(u)$ with sign sgn). If $\boldsymbol{D}$ is good then $\mathrm{E}(\Delta Y_i^{\mathrm{sgn}} \mid \boldsymbol{D})$ has a contribution $-\delta_{i,0}$ for each vertex of $T_k(r)$, another one $-\sum \delta_{i,r-(|d_h|+1)}$ for each vertex not in $T_k(r)$ adjacent to a leaf of such tree and a third one, $\sum \delta_{i,r-|d_h|}$, present only if sgn $\equiv$ "$-$". In other words in the case of a good configuration each of the Y_i^+ is decreased, whereas the Y_i^- can be increased due to constributions from either Y_{i-1}^+ or Y_{i-1}^-.

3 The Upper Bounds

Consider a random n-vertex r-regular graph G generated using the pairing model given in Section 1. Let Q_k be the event "G contains at least one k-matching of size y". We will show that there exists a positive real number μ_k such that when $\mu > \mu_k$ and $y = \mu n$, then $\Pr[Q_k] = o(1)$, thus proving the upper bound in Theorem 1. Let Q'_k be the corresponding event defined on pairings that correspond to n-vertex r-regular graphs. It is well known [16, Section 2.2] that there is a constant c_r such that $\Pr[Q_k]$ is, asymptotically $c_r \Pr[Q'_k]$, hence we can estimate $\Pr[Q_k]$ by performing all our calculations using the pairing model.

Let $N(i) = \frac{(2i)!}{i!2^i}$. Let $X_k = X_k(G, r, y)$ be the number of k-matchings of size y in a random pairing. We calculate an asymptotic expression for $\mathrm{E}(X_k)$ and show that when $y = \mu_k n$, then $\mathrm{E}(X_k) = o(1)$, thus proving the upper bound in Theorem 1 by Markov's inequality. Such number can be computed using linearity of expectation. If $X_{\mathcal{M}}$ is a random indicator equal to one if $\mathcal{M}$ is a k-matching of size y in a random pairing, then $\mathrm{E}(X_k) = \sum_{\mathcal{M}} \mathrm{E}(X_{\mathcal{M}})$. Notice that $\mathrm{E}(X_{\mathcal{M}})$ is simply the probability that $\mathcal{M}$ actually occurs in the random pairing, viz. $N(rn/2)^{-1}$ times the number of ways in which a pairing can be built with $\mathcal{M}$ embedded in it. Such pairings have a very special structure. If $\mathcal{M} = \{e_1, \ldots, e_y\}$ is a k-matching in the given pairing G, then it is possible to define y unlabelled vertex-disjoint induced pairings, $H_1, \ldots, H_y$ such that:

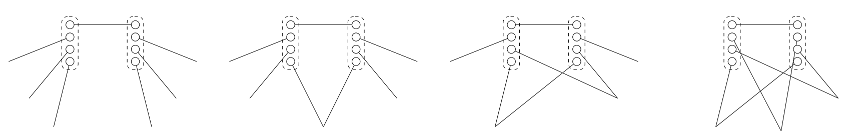

Fig. 1. Different graphs in $\mathcal{H}_{4,3}$. The vertices u_1 and u_2 are represented as a collection of points (in the pairing model).

1. e_j belongs to (a labelled version of) H_j for each $j \in \{1, \ldots, y\}$;
2. each such copy of H_j is connected[4] to the rest of G through a set of *socket* urns;
3. all non-socket urns of H_j have degree r in H_j.

[4] Of course the term *connected* is quite ambiguous in this context. A pairing G is a collection of pairs on the given set of points, therefore no part of it can be connected in the classical graph-theoretic sense. Clearly we are referring here to a property of the regular graph corresponding to G.

Let ψ be a mapping that associates a tuple $(H_{i_1}, \ldots, H_{i_y})$ to each k-matching of size y, where for each $j \in \{1, \ldots, y\}$, (a labelled copy of) H_{i_j} is the largest sub-pairing of G satisfying the three properties described above. It is possible to count the number of k-matchings of a given size by finding the size of the range of ψ and multiplying such number by the number of ways in which H_{i_j} can be linked to the rest of G.

Let $\mathcal{H}_{r,k}$ be the collection of all distinct unlabelled pairings H whose labelled versions may occur in $\psi(\mathcal{M})$ for some k-matching $\mathcal{M}$. It should be remarked that $\mathcal{H}_{r,k} = \mathcal{H}_{r,k+1}$, for every odd k and consequently each sequence $(H_{i_1}, \ldots, H_{i_y})$ will characterise a number of k-matchings and $k+1$-matchings. In the case of a k-matching edges are allowed to connect pairs of sockets in distinct subgraphs, whereas for $k+1$-matchings this is not allowed. For instance, $\mathcal{H}_{r,3}$ is formed by the r distinct graphs H_i, for $i \in \{0, \ldots, r-1\}$, having two vertices of degree r, u_1 and u_2, joined by an edge (this is the edge added to the 3-matching), i vertices of degree two each joined to both u_1 and u_2, and $2(r-1)-i$ vertices of degree one, each joined to either u_1 or u_2 (see Fig. 1). The subgraph H_i has $2r-i$ vertices and it consists of a pairing of $2r + 2(r-1) - i + 2i = 4r + i - 2$ points.

A final additional approximation is needed to make the counting "viable". It is well known [9] that w.h.p. random regular graphs contain very few short cycles. This implies that, w.h.p. most of the k-matchings are described by tuples $(H_0, \ldots, H_0)$.

These remarks in particular imply that

$$\mathrm{E}(X_3) \sim \binom{n}{2y, 2y(r-1)} N(y) r^{2yr} (2y(r-1))! \frac{N((rn/2 - y(2r-1))}{N(rn/2)}$$
$$\mathrm{E}(X_4) \sim \binom{n}{2y, 2y(r-1)} N(y) r^{2yr} (2y(r-1))! \frac{rn - 2yr^2}{rn - 2yr^2 - 2y(r-1)^2} \frac{N((rn/2 - y(2r-1) - 2y(r-1)^2)}{N(rn/2)}$$

and a formula valid for arbitrary k can be easily deduced by generalising the reasoning above. Setting $y = \mu n$, and using standard Stirling's approximations to the various factorials, the expressions above have both the form $f_r(\mu)^n$, with f_r being a continuous unimodal function of μ. Therefore, for each r, one can find a value $\mu_k(r)$ such that $f_r(\mu) < 1$ for $\mu > \mu_k(r)$.

Maximal vs. non-maximal k-matchings. It should be remarked that slightly smaller values of $\mu_k(r)$ than those reported in the table in Section 1 can be numerically computed counting *maximal* k-matchings (a stronger 0.28206915 value for $\mu_2(3)$ is reported in [5]). However we preferred to keep the simpler exposition presented above as the magnitude of the improvements (less than 10^{-5} for each $r \geq 3$ and $k = 2$ and even smaller for larger k) makes the more convoluted analysis required to perform the counting correctly rather uninteresting.

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Appendix

A *(discrete time) random process* is a probability space Ω defined over sequences of values $(q_0, q_1, \ldots)$ of some set S. We call $H_t = (q_0, q_1, \ldots, q_t)$ the *history* of the random process up to time t. If y is a function defined on histories, the random variable $y(H_t)$ will be denoted simply by $Y(t)$. The following theorem deals with sequences of random processes Ω_n for $n = 1, 2, 3, \ldots$. Hence, for instance, $q_i = q_i(n)$, and $S = S_n$, although the dependency on n will be usually dropped for simplicity. A function $f(x_1, \ldots, x_j)$ satisfies a *Lipschitz condition* on $\mathcal{D} \subseteq \mathbb{R}^j$ if there is a positive constant L such that

$$|f(x_1, \ldots, x_j) - f(y_1, \ldots, y_j)| \leq L \max_{1 \leq i \leq j} |x_i - y_i|$$

for all $(x_1, \ldots, x_j), (y_1, \ldots, y_j) \in \mathcal{D}$. In such a case f is said to be Lipschitz on $\mathcal{D}$. A useful sufficient condition for f to be Lipschitz on a given domain $\mathcal{D}$ is that all partial derivatives of f are continuous and bounded in $\mathcal{D}$.

Theorem 2. [16] *Let a be fixed and let $\mathcal{D}^* \subseteq \mathbb{R}^{a+1}$. For $1 \leq i \leq a$, let $y^{(i)} : \bigcup_n S_n^+ \to \mathbb{R}$ and $f_i : \mathbb{R}^{a+1} \to \mathbb{R}$, such that for some constant C_0 and for all i, $|Y_i(t)| < C_0 n$ for all $H_t \in S_n^+$ for all n. Define the* stopping time $T_{\mathcal{D}} = T_{\mathcal{D}}(Y_1, \ldots, Y_a)$ *to be the minimum t such that $(t/n, Y_1(t)/n, \ldots, Y_a(t)/n) \notin \mathcal{D}$ and let $T_{\mathcal{D}^*}$ be defined analogously w.r.t. $\mathcal{D}^*$. Assume the following three conditions hold, where in (ii) and (iii) $\mathcal{D}$ is some bounded connected open set containing the closure of $\{(0, z^{(1)}, \ldots, z^{(a)}) : \Pr[Y_i(t) = z^{(i)} n, 1 \leq i \leq a] \neq 0$ for some $n\}$.*

- *(i) (Boundedness hypothesis.) For some functions $\beta = \beta(n) \geq 1$ and $\gamma = \gamma(n)$, the probability that $\max_{1 \leq i \leq a} |Y_i(t+1) - Y_i(t)| < \beta$ conditional upon H_t, is at least $1 - \gamma$ for $t < \min\{T_{\mathcal{D}^*}, T_{\mathcal{D}}\}$.*
- *(ii) (Trend hypothesis.) For some function $\lambda_1 = \lambda_1(n) = o(1)$, for all $i \leq a$*

$$|\mathrm{E}(Y_i(t+1) - Y_i(t) \mid H_t)| - f_i(t/n, Y_1(t)/n, \ldots, Y_a(t)/n)| \leq \lambda_1$$

  uniformly for $t < \min\{T_{\mathcal{D}^*}, T_{\mathcal{D}}\}$.
- *(iii) (Lipschitz hypothesis.) Each function f_i is continuous and Lipschitz on the set $\mathcal{D} \cap \{(\xi, z^{(1)}, \ldots, z^{(a)}) : \xi \geq 0\}$ with the same Lipschitz constant for each i.*

Then the following are true

1. *For $(0, \hat{z}^{(1)}, \ldots, \hat{z}^{(a)}) \in \mathcal{D}$ the system of differential equations*

$$\frac{dy_i}{ds} = f_i(s, y_1, \ldots, y_a), \qquad i = 1, \ldots, a, \tag{5}$$

   has a unique solution in $\mathcal{D}$ for $y_i : \mathbb{R} \to \mathbb{R}$ passing through

$$y_i(0) = \hat{z}^{(i)}, \qquad 1 \leq i \leq a,$$

   and which extends to points arbitrarily close to the boundary of $\mathcal{D}$.
2. *Let $\lambda > \lambda_1 + C_0 n\gamma$ with $\lambda = o(1)$. For a sufficiently large constant C, with probability $1 - O(n\gamma + \frac{\beta}{\lambda} \exp(-\frac{n\lambda^3}{\beta^3}))$,*

$$Y_i(t) = ny_i(t/n) + O(\lambda n) \tag{6}$$

uniformly for $0 \le t \le \min\{\sigma n, T_{\mathcal{D}^*}$ *and for each* i*, where* $yz_i(x)$ *is the solution in (5) with* $\hat{z}^{(i)} = Y_i(0)/n$*, and* $\sigma = \sigma(n)$ *is the supremum of those* x *to which the solution can be extended before reaching within* l^∞*-distance* $C\lambda$ *the boundary of* $\mathcal{D}$*.*

To prove the results for $k \le 2$, Theorem 2 is applied to the variables $|\mathcal{M}|$ and $Y_0, \ldots, Y_{r-1}$, defined in Section 2.1, independently in each phase. The functions f_1, f_2, $\ldots$ are $\tilde{\mathrm{E}}(\Delta|\mathcal{M}|)$, $\tilde{\mathrm{E}}(\Delta Y_0), \ldots$ respectively. They clearly satisfy the trend hypothesis and simple calculations of their partial derivatives imply that all of them satisfy the Lipschitz hypothesis in $\mathcal{D}_{\epsilon,j}$.

Simpler versions of the same results [16] are sufficient to prove the results in Section 2.2.