Dominating Sets in Web Graphs

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Abstract. In this paper we study the size of generalised dominating sets in two graph processes which are widely used to model aspects of the world-wide web. On the one hand, we show that graphs generated this way have fairly large dominating sets (i.e. linear in the size of the graph). On the other hand, we present efficient strategies to construct small dominating sets.
The algorithmic results represent an application of a particular analysis technique which can be used to characterize the asymptotic behaviour of a number of dynamic processes related to the web.

1 Introduction

In recent years the world wide web has grown dramatically. Its current size is measured in billions of pages [20], and pages are added to it every day. As this graph (nodes correspond to web pages and edges to links between pages) continues to grow it becomes increasingly important to study mathematical models which capture its structural properties [6, 19]. Such models can be used to design efficient algorithms for web applications and may even uncover unforeseen properties of this huge evolving structure. Several mathematical models for analysing the web have been proposed (for instance [5, 9, 19]). The (evolution of the) web graph is usually modelled by a (random) process in which new vertices appear from time to time. Such vertices may be linked randomly to the existing structure through some form of *preferential attachment*: existing vertices with many neighbours are somewhat more likely to be linked to the newcomers.
The main focus of research so far has been on capturing empirically observed [3, 6] features of the web. No attempt has been made to characterise graph-theoretic sub-structures of such graphs. We initiate such investigation by looking at sets of vertices that, in a sense, cover all other vertices. More formally, a vertex in a graph *dominates* all vertices that are adjacent to it (parallel edges give multiple domination). In the spirit of Harary and Haynes [14], an h*-dominating set* for a graph $G = (V, E)$ is a set $\mathcal{S} \subseteq V$ such that each vertex in $V \setminus \mathcal{S}$ is dominated at least h times by vertices in $\mathcal{S}$. Let $\gamma_h = \gamma_h(G)$ denote the size of the smallest h-dominating sets in G. The *minimum* h*-dominating set* problem (MhDS) asks for an h-dominating set of size γ_h.
Dominating sets play an important role in many practical applications, e.g. in the context of distributed computing or mobile ad-hoc networks [2, 10, 22]. The reader is referred to [15, 16] for an in-depth view of the subject. The typical fundamental task in such applications is to select a subset of nodes in the network that will 'provide' a certain service to all other vertices. For this to be time-efficient, all other vertices must be directly connected to the selected nodes, and in order for it to be cost-effective, the number of selected nodes must be minimal. In relation to web graphs, a dominating set may be used to devise efficient web searches. A crawler storing the pages in a dominating set could use such pages to visit any other page. For $h > 1$ an h-dominating set can be considered as a more fault-tolerant structure. If up to $h - 1$ vertices or edges fail, the domination property is still maintained (i.e. it is still possible to provide the service).
The MhDS problem is NP-hard [13, 18] and, moreover, it is not likely that it may be approximated effectively [12, 18]. Polynomial time algorithms exist on special classes of graphs (e.g. [21]). The M1DS

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problem has been studied also in random graphs. In the *binomial model* $G(n,p)$ [24] the value of γ_1 can be pin-pointed quite precisely, provided p is not too small compared to n. In random regular graphs of degree r (see for example results in the *configuration model* [26] and references therein) upper and lower bounds are known on γ_1.

In this paper we look at simple and efficient algorithms for building small h-dominating sets in graphs. The performance guarantees of these algorithms are analysed under the assumption that the input is a random web graph. We also analyse the tightness of the performances of such algorithms, by proving combinatorial lower bounds on γ_h, for any fixed $h \geq 1$. Such bounds, often disappointingly weak, offer nevertheless a proof that, most of the times, the sets returned by the various algorithms are only a constant factor away from an optimal answer. Finally, we compare the quality of the solutions returned by (some of) our algorithms with empirical average values of γ_1.

The main outcome of this paper can be stated informally by saying that web graphs have fairly large dominating sets. Hence a crawler who wants to use a dominating set to explore the web will need to store a large proportion of the whole graph. Interestingly, the results in this paper also uncover a(nother) difference between models of the web based on classical random graph processes, and models based on preferential attachment. The tendency to choose neighbours of high degree in the preferential attachment model affects the size of the smallest dominating sets.

Most of our algorithms are *on-line* in the sense that the decision to add a particular vertex to the dominating set is taken without total information about the web graph under consideration, and *greedy* in the sense that decisions, once taken, are never changed. The algorithms are also quite efficient: only a constant amount of time is used per update of the dominating set. Such algorithms are of particular interest in the context of web graphs. As the web graph is evolving, one wants to decide whether a new vertex is to be added to the already existing dominating set without recomputing the existing dominating set and with minimal computational effort. On-line strategies for the dominating set problem have been considered in the past [11, 17] for general graphs. However the authors are not aware of any result on on-line algorithms for this problem in random graphs.

Our results hold *asymptotically almost surely* (a.a.s.), i.e. with probability approaching one as the size of the web graph grows to infinity. The algorithmic results are based on the analysis of a number of (Markovian) random processes. In each case the properties of the process under consideration lead to the definition of a (deterministic) continuous function that is very close (in probability) to the values of the process, as the size of the graph grows. It should be pointed out at this stage that the proposed analysis methodology is quite general. We apply it to analyse heuristics for the MhDS problem only, but it would allow to prove results about other graphs parameters such as the independence or the chromatic number. The method is closely related to the so called *differential equation method* [25]. In fact a version of the main analytical tool proposed by Wormald can be adapted to work for the processes considered in this paper. However the machinery employed in [25] is not needed to analyse the processes considered in this paper. Our results are obtained by proving concentration of the various processes of interest around their mean and by devising a method for getting close estimates on the relevant expectations. In Section 2 we review the definitions of the models of web graphs that we will use. We also state our main result in the context of these models, and present more detailed comments on our general analysis method. In the following section we consider a very simple algorithm and apply the proposed method to obtain non-trivial upper bounds on γ_1. Refined algorithms are introduced and analysed in Section 4 and 5. In Section 6 we discuss generalisations to $h > 1$. Then we turn to lower bounds. In Section 7 and 8 we present our argument for the lower bounds stated in Section 2. Finally we briefly comment on some empirical work carried out on a sub-class of the graphs considered in this paper.

2 Models and Results

The models used in this paper are based on the work of Albert and Barabasi [3]. A *web graph* (see also [9]) can be defined as an ever growing structure in which, at each step, new vertices, new edges or a combination of these can be added. Decisions on what to add to the existing graph are made at random based on the values of a number of defining parameters. The existence of these parameters makes the model very general. For the purposes of this paper, to avoid cluttering the description of our results, we prefer to make a drastic

m	$\alpha_{\rm lo}^R$	$\alpha_{\rm up}^R$	$\alpha_{\rm lo}^C$	$\alpha_{\rm up}^C$
1	0.3678	0.5	0.21	0.3333
2	0.0585	0.3714	0.0286	0.2342
3	0.0422	0.3054	0.0148	0.1777
4	0.0352	0.2634	0.0097	0.1422
5	0.0261	0.2335	0.0066	0.1178
6	0.0247	0.2110	0.0049	0.1000
7	0.0208	0.1932	0.0038	0.0865

Table 1. Numerical values defined in Theorem 1 for the minimum dominating set problem.

simplification. We will consider graphs generated according to two rather extreme procedures derived from the general model. In each case the generation process is governed by a single integer parameter m.

Random graph. The initial graph $G_0^{R,m}$ is a single vertex v_0 with m loops attached to it. For $t \geq 1$, let $G_{t-1}^{R,m}$ be the graph generated in the first $t-1$ steps of this process, to define $G_t^{R,m}$ a new vertex v_t is generated, and it is connected to $G_{t-1}^{R,m}$ through m (undirected) edges. The neighbours of v_t are chosen uniformly at random (u.a.r.) with replacement from $\{v_0, \ldots, v_{t-1}\}$.

Pure copy. The initial graph $G_0^{C,m}$ is a single vertex v_0 with m loops attached to it. For $t \geq 1$, graph $G_t^{C,m}$ is defined from $G_{t-1}^{C,m}$ by generating a new vertex v_t, and connecting it to $G_{t-1}^{C,m}$ through m edges. A vertex $u \in \{v_0, \ldots, v_{t-1}\}$ is connected to v_t with probability $\frac{|\Gamma(u)|}{2mt}$ (where $\Gamma(u) = \{w : \{u, w\} \in E(G_{t-1}^{C,m})\}$).

Both models are dynamic, with new vertices and edges continuously increasing the size of the graph. However they represent two extreme cases. In the random graph model connections are completely random, whereas in the copy model they are chosen based on the current popularity of the various vertices. The results in this paper are summarised by the following Theorem.

Theorem 1 *Let $M \in \{R, C\}$, and m be a positive integer. Then for each $h \geq 1$ there exist positive real constants $\alpha_{\rm lo}^M$ and $\alpha_{\rm up}^M$ (dependent on M, m and h but independent of t) with $\alpha_{\rm lo}^M < \alpha_{\rm up}^M < 1$ such that $\alpha_{\rm lo}^M t \leq \gamma_h(G_t^{M,m}) \leq \alpha_{\rm up}^M t$ a.a.s.*

For $h = 1$, the values of the constants mentioned in the Theorem are reported in Table 1. Bounds for $h > 1$ are reported (and briefly commented) in Section 6. In particular the upper bounds are proved by analysing the size of the dominating set returned by a number of simple polynomial time algorithms.

The proof of Theorem 1 is based on the fact that natural edge-exposure martingales can be defined on the graph processes under consideration [9]. More precisely, if $f(G)$ is any graph theoretic function (e.g. the size of the dominating set returned by a particular algorithm), the random process defined by setting $Z_0 = \mathrm{E}(f(G_t^{M,m}))$, and Z_i (for $i \in \{1, \ldots, mt\}$) to be the expectation of $f(G_t^{M,m})$ conditioned on the "exposure" of the first i edges in the graph process is a martingale. Notice that the space of all graphs which can be generated according to the given model $G_t^{M,m}$ is partitioned into classes (or *i-blocks*) containing all those graphs which coincide w.r.t. the first i edge exposures.

In the forthcoming sections we will repeatedly use the following concentration result (for a proof see, for instance, [1]).

Theorem 2 *Let $c = Z_0, \ldots, Z_n$ be a martingale with $|Z_{i+1} - Z_i| \leq 1$ for all $i \in \{0, \ldots, n-1\}$. Then* $\Pr[|Z_n - c| > \lambda\sqrt{n}] \leq 2\mathrm{e}^{-\lambda^2/2}$.

In all our applications $c = \mathrm{E}(f(G_t^{M,m}))$, $n = mt$ and $\lambda = O(\log t)$. In order to apply Theorem 2 one needs to prove that $|Z_{i+1} - Z_i| \leq 1$. Such inequality follows from the smoothness of f (i.e. $|f(G) - f(H)| \leq 1$ if G and H differ w.r.t. the presence of a single edge) and the ability to demonstrate the existence of a measure preserving bijection between $i+1$-blocks in a same i-block. This is obvious in the random graph process as edges are inserted independently. In the case of the copy model it is convenient to identify each

graph with the projection of a particular type of *configuration*, i.e. an ordered collection of mt labelled pairs of points (see [4]). Let C_1 and C_2 be two such configurations that are identical up to the ith pair. Suppose the $i+1$-st pair is $\{a,b\}$ in C_1 and $\{a,c\}$ in C_2. If C_1 never uses point b again then the image of C_1 under the measure preserving bijection will be a configuration C' identical to C_1 except that pair $\{a,b\}$ is replaced by pair $\{a,c\}$. If b is used in a following pair (say $\{d,b\}$) of C_1 then C' will have $\{a,c\}$ instead of $\{a,b\}$ and $\{d,c\}$ instead of $\{d,b\}$, and so on. Similar construction is presented in [9].
Finally we will need the following

Lemma 1 *If Z_n is a martingale and there exist constants $c_1, c_2 > 0$ such that $\mu_n = \mathrm{E}(Z_n) \in [c_1 n, c_2 n]$, then for each fixed integer $j > 0$, there exists positive constant K and ϵ such that $|(Z_n)^j - (\mu_n)^j| \leq K n^{j-\epsilon}$ a.a.s..*

Proof. If Z_n is a martingale, then by Theorem 2, $\mu_n - \lambda\sqrt{n} \leq Z_n \leq \mu_n + \lambda\sqrt{n}$ where w.l.o.g. we assume $\lambda = o(\sqrt{n})$. From this we also have, for each fixed integer $j > 0$,

$$(\mu_n)^j(1 - \tfrac{\lambda\sqrt{n}}{\mu_n})^j \leq (Z_n)^j \leq (\mu_n)^j(1 + \tfrac{\lambda\sqrt{n}}{\mu_n})^j.$$

The result now follows (provided K is chosen big enough) since, the assumptions on λ and μ_n entail that $(1 + \frac{\lambda\sqrt{n}}{\mu_n})^j$ is at most $1 + \frac{j^2\lambda\sqrt{n}}{\mu_n}$, whereas $(1 - \frac{\lambda\sqrt{n}}{\mu_n})^j$ is at least $1 - \frac{2j\lambda\sqrt{n}}{\mu_n}$ □

3 Simplest algorithm

The algorithm presented in this section is a very simple "first attempt" solution for the problem at hand. Although in many cases it does not lead to a very small dominating set, it represents a natural benchmark for any more refined heuristic.

Algorithm 1. Before the first step of the algorithm the graph consists of a single vertex v_0 and $\mathcal{S} = \{v_0\}$. At step t if the newly generated vertex v_t does not have any neighbours in $\mathcal{S}$ (i.e. $\Gamma(v_t) \cap \mathcal{S} = \emptyset$) then v_t is added to $\mathcal{S}$.

Let X_t denote the size of the dominating set before v_t is added to the current graph and let $\mu_t = \mathrm{E}(X_t)$. For graphs generated according to the $G_t^{R,m}$ model, the probability that v_t misses the dominating set is $(1 - \frac{X_t}{t})^m$. Hence we can write

$$\mu_{t+1} = \mu_t + \mathrm{E}[(1 - \tfrac{X_t}{t})^m].$$

Let $x = x(m)$ be the unique solution of the equation $x = (1-x)^m$ in $(0,1)$. Table 2 gives the values of x for the first few values of m.

Lemma 2 *For any $\frac{1}{2} < \rho < 1$ constant, there exists an absolute positive constant C such that for all $t > 0$, $|\mu_t - xt| \leq Ct^\rho$ a.a.s.*

Proof. We claim that the difference $|\mu_t - xt|$ satisfies a recurrence of the form

$$|\mu_{t+1} - x(t+1)| \leq |\mu_t - xt| + O\left(\sqrt{\tfrac{\log t}{t}}\right).$$

m	x
1	0.5
2	0.382
3	0.3177
4	0.2755
5	0.2451
6	0.2219
7	0.2035

Table 2. Numerical values defined in Lemma 2.

This can be proved by induction on t. By definition $X_1 = 1$, hence $|\mu_1 - x| = 1 - x$. We also have $|\mu_{t+1} - x(t+1)| = |\mu_t - xt + \mathrm{E}[(1 - \frac{X_t}{t})^m] - (1-x)^m|$. The difference $\mathrm{E}[(1 - \frac{X_t}{t})^m] - (1-x)^m$ can be rewritten as $-\frac{m}{t}(\mu_t - xt) + \{\mathrm{E}[(1 - \frac{X_t}{t})^m] - 1 + m\frac{\mu_t}{t} - (1-x)^m + 1 - mx\}$. Hence

$$|\mu_{t+1} - x(t+1)| = |(1 - \tfrac{m}{t})(\mu_t - xt) + \{\mathrm{E}[(1 - \tfrac{X_t}{t})^m] - 1 + m\tfrac{\mu_t}{t} - (1-x)^m + 1 - mx\}|.$$

To complete the proof notice that, by Lemma 1,

$$\mathrm{E}[(1 - \tfrac{X_t}{t})^m] - 1 + m\tfrac{\mu_t}{t} = (1 - \tfrac{\mu_t}{t})^m - 1 + m\tfrac{\mu_t}{t} + O(\sqrt{\tfrac{\log t}{t}}).$$

and the function $f(z) = (1-z)^m - 1 + mz$ satisfies $|f(z_1) - f(z_2)| \le m|z_1 - z_2|$, for $z_1, z_2 \in [0,1]$. □
The following Theorem is a direct consequence of Lemma 2 and the concentration result mentioned in the previous section.

Theorem 3 $|X_t - xt| = o(t)$ *a.a.s.*

4 Improved approximations in the random graph process

Although Algorithm 1 is quite simple, it seems difficult to beat, as a glance at α^R_{up} in Tables 1 and 2 shows. This is especially true for $m = 1$ where no improvement could be obtained. For larger values of m, a better way of finding small dominating sets is obtained by occasionally allowing vertices to be dropped from $\mathcal{S}$. It is convenient to classify the vertices in the dominating set as *permanent* (set $\mathcal{P}$) and *replaceable* (set $\mathcal{R}$). Thus $\mathcal{S} = \mathcal{P} \cup \mathcal{R}$. Let P_t and R_t denote the sizes of such sets at time t (set $P_1 = 0$ and $R_1 = 1$).

Algorithm 2. Before the first step of the algorithm the graph consists of a single vertex v_0 and $\mathcal{R} = \{v_0\}$. After v_t is created and connected to m neighbours, if $\Gamma(v_t) \cap \mathcal{P} \neq \emptyset$ then nothing happens. Otherwise v_t is added to $\mathcal{R}$ if $\Gamma(v_t) \cap \mathcal{R} = \emptyset$, otherwise v_t is added to $\mathcal{P}$ and all vertices in $\Gamma(v_t) \cap \mathcal{R}$ are moved to $V \setminus \mathcal{S}$.

The expectations $\pi_t = \mathrm{E}(P_t)$ and $\rho_t = \mathrm{E}(R_t)$ satisfy:

$$\begin{aligned}\pi_{t+1} &= \pi_t + \mathrm{E}[(1 - \tfrac{P_t}{t})^m] - \mathrm{E}[(1 - \tfrac{P_t}{t} - \tfrac{R_t}{t})^m],\\ \rho_{t+1} &= \rho_t + \mathrm{E}[(1 - \tfrac{R_t}{t} - \tfrac{P_t}{t})^m] - m\mathrm{E}[\tfrac{R_t}{t}(1 - \tfrac{P_t}{t})^{m-1}].\end{aligned}$$

Define α^R_{up} as $p + r$, where $p = p(m)$ and $r = r(m)$ satisfy

$$\begin{aligned} r &= \tfrac{(1-p)^m - p}{1 + m(1-p)^{m-1}},\\ p &= (1-p)^m - (1-p-r)^m.\end{aligned}$$

Lemma 3 *For any $\frac{1}{2} < \rho < 1$ constant, there exist absolute positive constants C_1 and C_2 such that for all $t > 0$, $|\pi_t - pt| \le C_1 t^\rho$ and $|\rho_t - rt| \le C_2 t^\rho$ a.a.s.*

Proof. The proof is, essentially, a generalisation of that of Lemma 2. We present the argument in some details for π_t. At the inductive step:

$$|\pi_{t+1} - p(t+1)| = |\pi_t - pt + \mathrm{E}[(1 - \tfrac{P_t}{t})^m] - (1-p)^m - \{\mathrm{E}[(1 - \tfrac{P_t}{t} + \tfrac{R_t}{t})^m] - (1-p-r)^m\}|.$$

The proof is completed by decomposing the differences $\mathrm{E}[(1 - \frac{P_t}{t})^m] - (1-p)^m$ and $\mathrm{E}[(1 - \frac{P_t}{t} - \frac{R_t}{t})^m] - (1-p-r)^m$ in parts that are proportional to either $P_t - \pi_t$ or $\pi_t - pt$.
The result about ρ_t is proved similarly after noticing that r satisfies:

$$r = (1-p-r)^m - mr(1-p)^{m-1}.$$

□

Theorem 2 implies that the sum $p + r$ is in fact very close to $\frac{|\mathcal{S}|}{t} = \frac{P_t}{t} + \frac{R_t}{t}$.

5 Improved approximations in the pure copy process

Algorithm 1, described in Section 3, can be analysed in the copy model as well. The expected change in the variable X_t can be computed by keeping track of the total degree of the dominating set, D_t. In particular the following relationships hold

$$\begin{aligned}\mathrm{E}(X_{t+1}) &= \mathrm{E}(X_t) + \mathrm{E}[(1-\tfrac{D_t}{2mt})^m],\\ \mathrm{E}(D_{t+1}) &= (1+\tfrac{1}{2t})\mathrm{E}(D_t) + m\mathrm{E}[(1-\tfrac{D_t}{2mt})^m].\end{aligned}$$

Not surprisingly an analysis similar to the one described in the previous sections inplies that such algorithm returns dominating sets of size xt in $G_t^{C,m}$. However, in the copy model we can improve on this by pushing high degree vertices in the dominating set. The natural way to accomplish this would be, for any newly generated uncovered vertex, to select a neighbour of maximum degree and add it to $\mathcal{S}$. Unfortunately such algorithm is not easy to analyse because in the graph processes that we consider there may be few vertices of rather large degree. However a draconian version of this heuristic can be analysed. The following algorithm takes as input an additional integer parameter $k > 0$.

Algorithm 3. Before the first step of the algorithm the graph consists of a single isolated vertex v_0 and $\mathcal{S} = \{v_0\}$. After v_t is created and connected to m neighbours, the set Z of all newly updated vertices in $V \setminus \mathcal{S}$ of degree larger than km is added to $\mathcal{S}$. If v_t is still not dominated by some element of $\mathcal{S}$ then a vertex of maximum degree in $\Gamma(v_t) \setminus Z$ is added to the dominating set.

The analysis of the evolution of $|\mathcal{S}|$ is based again on the definition of a random process that describes the algorithm dynamics and on the proof that such process behaves in a predictable way for large t.
Let $n = (k-1)m + 1$. For each $i \in \{0, \ldots, n-1\}$ and $t > 0$, define $Y_t^i = |V_{m+i} \setminus \mathcal{S}|$ in $G_{t-1}^{C,m}$ before v_t is added to the graph. Let Y_t^n denote the total degree inside $\mathcal{S}$ (i.e. $Y_t^n = \sum_{v\in\mathcal{S}} |\Gamma(v)|$) and X_t the size of the dominating set before v_t is added to the graph. The state of the system, at each step t, is modelled by the (random) vector $(Y_t^0, \ldots, Y_t^n, X_t)$. Notice that, for each $t > 0$, the variation in each of the variables is at most m. Also, $Y_t^n + \sum_{i=0}^{(k-1)m}(m+i)Y_t^i = 2mt$, and, at each step $t > 1$, when v_t is created the probability that it hits a vertex of degree $m+i$, for $i \in \{0, \ldots, (k-1)m\}$ (resp. the dominating set) in any one of the m trials available to it is approximately equal to $P_i = \frac{((m+i-1)(1-\delta_{i,n})+1)Y_t^i}{2mt}$ (where $\delta_{i,n} = 1$ if $i = n$ and zero otherwise).
For $i \in \{0, \ldots, n-1\}$ (resp. $i = n$), let E_i denote the event "v_t missed $\mathcal{S}$ and the maximum degree in $\Gamma(v_t)$ is $m+i$" (resp. "v_t did not miss $\mathcal{S}$"). The expected change to Y_t^i, conditioned to the process history up to time t can be computed by further conditioning family of events $(E_i)_{i\in\{0,\ldots,n\}}$. We can write such quantity as

$$\textstyle\sum_{d=0}^{n} \mathrm{E}(Y_{t+1}^i - Y_t^i \mid E_d)\,\Pr[E_d].$$

The probability in the expression above is approximately

$$\chi_d = (S_0^d)^m - (S_0^{d-1})^m$$

(notation S_a^b stands for $\frac{(m+a)Y_a}{2mt} + \ldots + \frac{(m+b)Y_b}{2mt}$, with $S_a^b \equiv 0$ if $a > b$). Furthermore, we can approximate $\mathrm{E}(Y_{t+1}^i - Y_t^i \mid E_d)$ by the expression:

$$\textstyle\sum C(h_0, \ldots, h_d, 0, \ldots, 0)\frac{m!}{h_0!\,h_1!\,\ldots,h_d!}\prod_{i=0}^{d} P_i^{h_i}\frac{1}{\chi_d}$$

where the sum is over all possible ordered tuples of values $h_0, \ldots, h_d$ such that $\sum_{i=0}^{d} h_i = m$ and $h_d > 0$, and $C(h_0, \ldots, h_d, 0, \ldots, 0)$ contains:

- a term for the addition of v_t to $G_{t-1}^{C,m}$;
- a term $\phi_{i,d}$ for the change to Y_t^i due to the handling of the chosen vertex of maximum degree in $\Gamma(v_t)$, and
- a term $\psi_{i,s}$ for the change to Y_y^i due to the handling of any other vertex in $\Gamma(v_t)$.

The first of these is simply $\delta_{i,0}$. We also have

$$\phi_{i,d} = \delta_{d,n} \times \delta_{i,d} + (1-\delta_{d,n}) \times \{(m+d+1)\delta_{i,n} - \delta_{i,d}\}$$

(if $d = n$ (i.e. if v_t hits the dominating set) then one is added to Y_t^n, otherwise Y_t^d is decreased and $m + d + 1$ units are added to Y_t^n), and

$$\psi_{i,s} = \delta_{s,n} \times \delta_{i,s} + (1 - \delta_{s,n}) \times \{((m + s)\delta_{s,n-1} + 1)\delta_{i,s+1} - \delta_{i,s}\}$$

(if $s = n$ then Y_t^n is increased by one, if $s = n - 1$ the newly created vertex of degree n must be added to $\mathcal{S}$, and finally, in any other case the vertex that has been hit is moved from V_{m+s} to V_{m+s+1}). Therefore

$$C(h_0, \dots, h_d, 0, \dots, 0) = \delta_{i,0} + \phi_{i,d} + (h_d - 1)\psi_{i,d} + \textstyle\sum_{s=0}^{d-1} h_s \psi_{i,s}.$$

Doing all the sums, for each $i \in \{0, \dots, n\}$, the expected change to Y_t^i is approximately equal to

$$\delta_{i,0} + \textstyle\sum_{d=0}^{n} \left\{ \chi_d \phi_{i,d} + \psi_{i,d} \left[m P_d (S_0^d)^{m-1} - \chi_d \right] + \frac{m[\chi_d - P_d (S_0^d)^{m-1}]}{S_0^{d-1}} \sum_{s=0}^{d-1} \psi_{i,s} P_s \right\}.$$

Also $\mathrm{E}(X_{t+1} - X_t)$ is approximately equal to

$$\tfrac{km\mathrm{E}(Y_t^{n-1})}{2t} + \mathrm{E}\left[\left(1 - \tfrac{Y_t^n}{2mt} - \tfrac{kY_t^{n-1}}{2t} \right)^m \right].$$

The value α_{up}^C reported in Table 1 is defined by $\alpha = kmy^{n-1} + (1 - \frac{y^n}{2m} - \frac{ky^{n-1}}{2})^m$ where y^i, for $i \in \{0, \dots, n\}$ satisfy $y^i = \left[\mathrm{E}(Y_{t+1}^i) - \mathrm{E}(Y_t^i) \right] \big|_{Y^i = y^i t, i \in \{0,\dots,n\}}$.

6 Generalisations

The algorithms presented in the previous sections generalise naturally to larger values of h. We present here the generalisation of Algorithm 2 for finding an h-dominating set in the random graph process and of Algorithm 3 for the pure copy process. We then briefly sketch the analysis of the first heuristic, and comments on the numerical results presented in Table 3 below.

Algorithm 4. A vertex in the h-dominating set can be either *permanent* (set $\mathcal{P}$) as it will never be removed from the set or *i-removable* (set $\mathcal{R}_i$) meaning that it is dominated $i \in \{0, \dots, h-1\}$ times by other permanent vertices.
Before the first step of the algorithm the graph consists of a single vertex v_0 and $\mathcal{R}_0 = \{v_0\}$. After v_t is created and connected to m neighbours, v_t is added to $\mathcal{R}_0$ if $\Gamma(v_t) \cap (\mathcal{P} \cup \bigcup_i \mathcal{R}_i) = \emptyset$, otherwise if $\Gamma(v_t) \cap \mathcal{P} = \emptyset$, v_t is added to $\mathcal{P}$, any vertex in $\Gamma(v_t) \cap \mathcal{R}_i$, for $i < h - 1$ is moved to $\mathcal{R}_{i+1}$ and any vertex in $\Gamma(v_t) \cap \mathcal{R}_{h-1}$ is moved to $V \setminus \mathcal{S}$.

The process in Algorithm 4 can be modelled by h sequences of random variables: R_t^i for $i \in \{0, \dots, h-1\}$ with $R_t^i = |\mathcal{R}_i|$ in the graph $G_{t-1}^{R,m}$ just before v_t is added to it, and $P_t = |\mathcal{P}|$. The expected changes in these variables satisfy

$$\begin{aligned}
\mathrm{E}(P_{t+1}) &= \mathrm{E}(P_t) + \mathrm{E}[(1 - \tfrac{P_t}{t})^m] - \mathrm{E}[(1 - \tfrac{P_t}{t} - \textstyle\sum_i \tfrac{R_t^i}{t})^m] \\
\mathrm{E}(R_{t+1}^i) &= \mathrm{E}(R_t^i) + \mathrm{E}[(1 - \tfrac{P_t}{t} - \textstyle\sum_i \tfrac{R_t^i}{t})^m]\delta_{i,0} + m\mathrm{E}[(\tfrac{R_t^{i-1}}{t}(1 - \delta_{i,0}) - \tfrac{R_t^i}{t})(1 - \tfrac{P_t}{t})^{m-1}],
\end{aligned}$$

m	$h=2$	$h=3$	$h=4$	$h=5$
2	0.4484			
3	0.368	0.3836		
4	0.3162	0.3307	0.332	
5	0.2795	0.2929	0.2944	0.2931
6	0.2517	0.2641	0.2658	0.2647
7	0.2298	0.2413	0.2431	0.2422

Algorithm 4.

m	$h=2$	$h=3$	$h=4$	$h=5$
2	0.5			
3	0.4075	0.6		
4	0.3359	0.4852	0.6663	
5	0.282	0.4073	0.5423	0.7036
6	0.2428	0.3523	0.4613	0.5862
7	0.2132	0.3066	0.4035	0.5056

Algorithm 5.

Table 3. Upper bounds on γ_h/t, for $h > 1$.

therefore, we have $P_t \sim pt$ where p satisfies:

$$p = (1-p)^m - \{1 - [(1-p)^m - p][1 - (\tfrac{m(1-p)^{m-1}}{1+m(1-p)^{m-1}})^h] - p\}^m$$

and $R_t^i \sim r^i t$ where $r^0 = \frac{(1-p)^m - p}{1+m(1-p)^{m-1}}$, and $r^i = \frac{m(1-p)^{m-1}}{1+m(1-p)^{m-1}} r^{i-1}$. The values reported in Table 3 below are given by $p + \sum_{i=0}^{h-1} r^i$.

Algorithm 5. Before the first step of the algorithm the graph consists of a single vertex v_0 and $\mathcal{S} = \{v_0\}$. After v_t is created and connected to m neighbours, the set Z of all newly generated vertices of degree more than km are added to $\mathcal{S}$. If v_t is dominated $h - x$ times by elements of $\mathcal{S}$ then the x vertices of highest degree in $\Gamma(v_t) \setminus Z$ are added to the dominating set.

7 Tightness of the algorithmic results

An interesting and simple argument can be used to complement the algorithmic results presented in the previous sections. The argument is based on the following result (V_i is the set of vertices of degree i).

Lemma 4 *Let $\mathcal{S}$ be an h-dominating set in a graph $G = (V, E)$. If the total degree of the vertices in $\mathcal{S}$ is at least d, then $|\mathcal{S}| \geq \sum_{i>i_0} |V_i|$, where i_0 is the largest index i for which $\sum_{j>i} j|V_j| \geq d$.*

Proof. Let i_0 be defined as in the statement of the result. If $\Delta = \max_{v \in V} |\Gamma(v)|$, then the set $V_{i_0} \cup \ldots \cup V_\Delta$ is the smallest set of vertices in G with total degree at least d (any other vertex in G would have smaller degree and therefore it would contribute less to the total degree). □

The total degree of an h-dominating set in $G_t^{M,m}$ must be at least $h(t - |\mathcal{S}|) \geq ht(1 - \alpha_{\rm up}^M)$. Hence a lower bound on the size of any dominating set in a web-graph is obtained by using information on the proportional degree sequence.
In the pure copy web graphs Bollobas *et al.* [5] (see also Cooper [8]) proved that $|V_i| = tn_i + O(\sqrt{t \log t})$ a.a.s. for any $i \geq m$, where

$$n_i = \tfrac{2m(m+1)}{i(i+1)(i+2)}.$$

In the same paper (p. 288) it is possible to find a result about $G_t^{R,m}$. In the random graph process, for any $i \geq m$,

$$n_i = \tfrac{1}{m+1} \left(\tfrac{m}{m+1}\right)^{i-m}.$$

Again it is possible to prove that $|V_i|$ is concentrated around $n_i t$.
For $m > 1$, the bounds reported in Table 1 in Section 2 are obtained using Lemma 4 and the approximations above for $|V_i|$ in each case. Bounds for $h > 1$ are in Table 4.

m	$h=2$	$h=3$	$h=4$	$h=5$
2	0.1317			
3	0.1001	0.178		
4	0.0687	0.1342	0.1678	
5	0.0649	0.0935	0.1346	0.1938
6	0.0535	0.0849	0.1155	0.1573
7	0.0406	0.0692	0.1033	0.1349

Random graph.

m	$h=2$	$h=3$	$h=4$	$h=5$
2	0.0545			
3	0.0316	0.0351		
4	0.023	0.0333	0.0246	
5	0.0183	0.0284	0.0302	0.0183
6	0.0141	0.0233	0.0299	0.0269
7	0.0113	0.0203	0.0271	0.0283

Pure copy.

Table 4. Lower bounds on γ_h/t, for $h > 1$.

t	$\gamma_1(G^R_{1,t})/t$	$\gamma_1(G^C_{1,t})/t$
10000	3745.053	2943.157
20000	7489.3	5887.301
30000	11233.68	8829.288
40000	14980.384	11772.175
50000	18725.448	14714.073
...		
100000	37451.20312	29424.216

Table 5. Average values obtained over 1000 experiments for each value of t.

8 Trees

For $m = 1$ the graph processes under consideration generate a connected graph without cycles. Such structural property can be exploited to obtain improved lower bounds on γ_1. Without loss of generality, any vertex in such graphs that has at least one neighbour $u \in V_1$ must be part of a minimum size dominating set. The number of such vertices is precisely $t - |V_1| - |I|$ where $|I|$ is the number of vertices that have no neighbour in V_1. The cardinality of I can be estimated in both models via either a martingale argument similar to those used in previous sections or through the technique exploited in [8] to estimate $|V_i|$. The lower bounds in Table 1 for $m = 1$ come from this argument. Details of such analysis are left for the final version of this paper.

We end this Section reporting on some simple empirical results which help putting the mathematical analysis performed so far into context. It is well known [7] that minimum size dominating sets can be found efficiently in trees. We implemented Cockayne et al's algorithm and tested its performance. For different values of t, we repeatedly ran the two graph processes up to time t, and then applied the optimal algorithm mentioned above. Table 5 reports the average values we obtained. The least square approximation lines over the full set of data we collected are (coefficients rounded to the sixth decimal place) $y = 0.374509x - 0.214185$ for the random graph case, and $y = 0.294294x + 0.32284$ for the pure copy case. These results indicate that our algorithms are able to get better results for graphs generated according to the pure copy process ($\alpha^C_{\text{up}} = 0.3333$) than for graphs generated by the other process ($\alpha^R_{\text{up}} = 0.5$). We leave the finding of improved algorithms in the random graph process as an interesting open problem of this work.

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