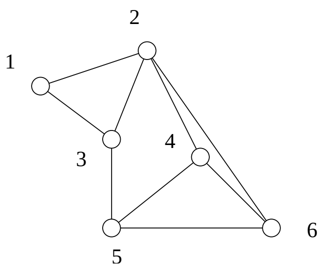
Hall's condition

König's equality will be needed to prove the correctness of our matching algorithm.

Hall's condition has even more algorithmic implications.

Notation. If X is a set of vertices in a graph G, let denote by $\Gamma(X)$ the set of vertices (not belonging to X itself) that are adjacent to at least one vertex in X. Sometimes we may write $\Gamma_G(X)$ to make clear what is the graph we are working on.

In the figure to the right $\Gamma(\{3\}) = \{1, 2, 5\}$, whereas $\Gamma(\{4, 5\}) = \{2, 3, 6\}$.



Scenario. A team of 8 cyclists has available 10 bicycles to run a race. The trainer has to assign a bicycle to each cyclist, assuming each cyclist is really confident with only few different bicycles. For instance the first cyclist likes bike one and five, the second one, bike one, two and ten, and so on.

Can the trainer assign a bicycle for each person in his team so that nobody has to drive a machine he does not like?

Hall's condition states that if any group of cyclists has at least as many options as the number of its elements, then the trainer can make it.

General statement

Let $G = (V_1, V_2, E)$ be a bipartite graph. Then G has a matching of V_1 into V_2 if and only if $|\Gamma(X)| \geq |X|$ for all $X \subseteq V_1$.

A set of edges M is "a matching of V_1 into V_2", if M is a collection of independent edges such that each vertex in V_1 belongs to some edge in M. (Note that V_2 may be larger than V_1 and therefore some vertices in V_2 may NOT belong to any edge of M.)

Does this make sense?

What does this result mean?

1. If we can find a set of, say, 3 cyclists who are only comfortable driving just 2 bicycles we know the trainer won't make it!
2. Conversely if any group of cyclists has "sufficiently many options" then the trainer will solve his problem.

Why is this true?

1. Few years ago I had a very intelligent Greek student in this course who after listening to this, shrugged and said:

   Isn't this obvious?

2. Well, the result is in fact true but don't we need a slightly more formal argument?

   Suppose you said:

   > If any set of x cyclists has at least x bicycles available then the trainer will manage to solve his problem.

   If I were not as confident as Aris what would I need to do to prove that this is in fact the case?

Let's consider one possibility first.

> If G has a matching of V_1 into V_2 then $|\Gamma(X)| \geq |X|$ for all $X \subseteq V_1$.

First notice that if G does have a matching M of V_1 into V_2 then $|\Gamma(X)| \geq |X|$ for all $X \subseteq V_1$. Take an $X \subseteq V_1$. For each of its elements x, we can find a distinct element in V_2, namely the vertex v such that $\{x, v\} \in M$, such that $v \in \Gamma(X)$. Hence $\Gamma(X)$ is at least as large as X.

How about the opposite implication?

> If $|\Gamma(X)| \geq |X|$ for all $X \subseteq V_1$ then there is a matching of V_1 into V_2.

We are back to the question of Aris the Greek!

Some may say it's obvious and leave the room.

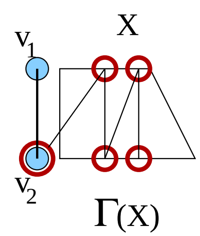
Others may think back to what is needed: we need to prove the *existence* of some structure in a graph. This is one of those occasions when Mathematics is actually borrowing from Computer Science.

If we devise an algorithm that, under the given condition, gives us the matching we will be fine!

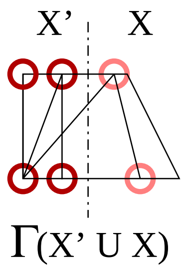
In Mathematics they call it induction (on the cardinality of V_1).

If $|V_1| \leq 1$ the result is trivial. Otherwise, we need to distinguish two cases:

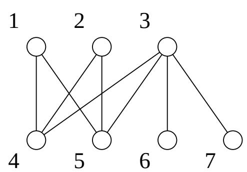
(Case 1.) $|\Gamma(X)| > |X|$ for all X such that $\emptyset \subset X \subset V_1$. Let v_1 and v_2 be adjacent with $v_i \in V_i$ for $i = 1, 2$. Let $G' = G - v_1 - v_2$ (this is the white trapezium in the figure on the right) and X be any subset of $V_1 - v_1$ (for instance the two circled vertices in the figure). Since $|\Gamma(X)| > |X|$ by our assumption, counting w.r.t. G', $|\Gamma_{G'}(X)| \geq |\Gamma(X)| - 1 \geq |X|$. Thus we can apply the induction hypothesis (or invoke a recursive call), find a matching of $V_1 - v_1$ into $V_2 - v_2$, and add $\{v_1, v_2\}$ to it, end of story.



(Case 2.) There is at least one subset of V_1, X' such that $|\Gamma(X')| = |X'|$ (X is called *critical set*).
Let G_1 be the subgraph of G induced by X' and $\Gamma(X')$ and G_2 the graph induced by $V \setminus X' \setminus \Gamma(X')$. If we can prove that Hall's condition holds for G_1 and G_2 we can apply induction on each of these graphs (in Computer Science terms, we define the matching for G as the union of the results of two recursive calls, one on G_1, the other one on G_2).
Suppose $X \subseteq X'$, then $\Gamma(X)$ must be a subset of $\Gamma(X')$. Therefore, as for every subset of V_1, $|\Gamma_{G_1}(X)| \geq |X|$.
Secondly if $X \subseteq V_1 \setminus X'$, then we look at $\Gamma(X \cup X')$. This is equal to $\Gamma_{G_2}(X) \cup \Gamma(X')$ (and the two sets are disjoint). Therefore $|\Gamma_{G_2}(X)| = |\Gamma(X \cup X')| - |\Gamma(X')| \geq |X \cup X'| - |\Gamma(X')| = |X \cup X'| - |X'| = |X|$.



Mathematical results hiding algorithms



Does this graph have a matching of $V_1 = \{1, 2, 3\}$ into V_2? We find out by walking through the proof of Hall's condition.

First $|V_1| > 1$. We look for a critical set.

X	$\{1\}$	$\{2\}$	$\{3\}$	$\{\mathbf{1}, \mathbf{2}\}$	$\{1, 3\}$	$\{2, 3\}$
$\Gamma(X)$	$\{4, 5\}$	$\{4, 5\}$	$\{4, 5, 6, 7\}$	$\{\mathbf{4}, \mathbf{5}\}$	$\{4, 5, 6, 7\}$	$\{4, 5, 6, 7\}$

Found it: $X = \{1, 2\}$. We can split the analysis in two parts (cross-edges simply deleted).

Mathematical results hiding algorithms

We reapply the same analysis to each of these graphs individually.

The graph on the left will need one more iteration: the set $X = \{1\}$ has $\Gamma(X) = \{4\}$ and therefore $|X| = |\Gamma(X)|$. So the graph is split in two again. Eventually we get the matching $M_1 = \{\{1, 4\}, \{2, 5\}\}$.

The graph on the right has $|V_1| = 1$ hence we just return a matching formed by a single edge, say, $M_2 = \{\{3, 6\}\}$.

The matching of the original graph will then be

$$M = M_1 \cup M_2 = \{\{1, 4\}, \{2, 5\}, \{3, 6\}\}.$$

Final remarks

1. Can you design a recursive algorithm based on the proof of Hall's theorem given above?
2. What is the complexity of the algorithm under point 1.?
3. A *perfect matching* (or *1-factor*) is a matching which covers all points of G. The following is an obvious corollary of Hall's theorem.

   (Frobenius' Marriage theorem) A bipartite graph G has a perfect matching if and only if Hall's condition holds with respect to V_1 and $|V_1| = |V_2|$.