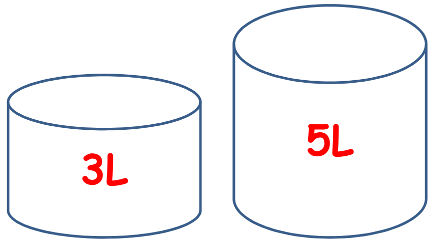
COMP108
Algorithmic Foundations

Graph Theory

Prudence Wong

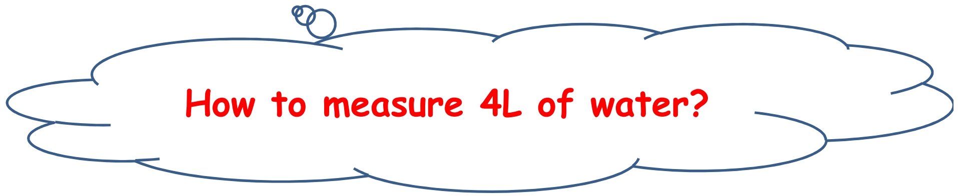
How to Measure 4L?



a 3L container &
a 5L container
(without mark)

infinite supply of water

You can pour water from one container to another



Learning outcomes

- Able to tell what an undirected graph is and what a directed graph is
  - Know how to represent a graph using matrix and list
- Understand what Euler circuit is and able to determine whether such circuit exists in an undirected graph
- Able to apply BFS and DFS to traverse a graph
- Able to tell what a tree is

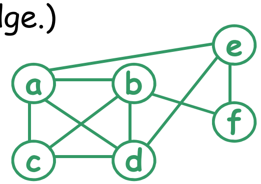
Graph ...

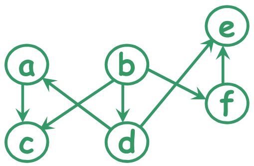
Graphs

introduced in the 18th century

Graph theory – an old subject with many modern applications.

An ***undirected*** graph G=(V,E) consists of a set of vertices V and a set of edges E. Each edge is an ***unordered*** pair of vertices. (E.g., {b,c} & {c,b} refer to the same edge.)





A ***directed*** graph G=(V,E) consists of … Each edge is an ***ordered*** pair of vertices. (E.g., (b,c) refer to an edge from b to c.)

Applications of graphs

In computer science, graphs are often used to model

- computer networks,
- precedence among processes,
- state space of playing chess (AI applications)
- resource conflicts, …

In other disciplines, graphs are also used to model the structure of objects. E.g.,

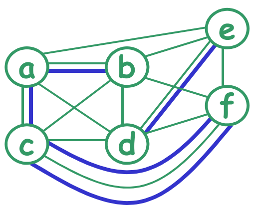
- biology - evolutionary relationship
- chemistry - structure of molecules

Undirected graphs

Undirected graphs:

- **simple graph:** at most one edge between two vertices, no self loop (i.e., an edge from a vertex to itself).
- **multigraph:** allows more than one edge between two vertices.

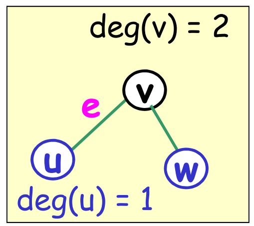
Reminder: An undirected graph G=(V,E) consists of a set of vertices V and a set of edges E. Each edge is an unordered pair of vertices.



Undirected graphs

In an undirected graph G, suppose that e = {u, v} is an edge of G

- u and v are said to be ***adjacent*** and called ***neighbors*** of each other.
- u and v are called ***endpoints*** of e.
- e is said to be ***incident*** with u and v.
- e is said to ***connect*** u and v.



- The ***degree*** of a vertex v, denoted by **deg(v)**, is the number of edges incident with it (a loop contributes twice to the degree)

Representation (of undirected graphs)

An undirected graph can be represented by **adjacency matrix**, **adjacency list**, **incidence matrix** or **incidence list**.

Adjacency matrix and adjacency list record the relationship between **vertex adjacency**, i.e., a vertex is adjacent to which other vertices

Incidence matrix and incidence list record the relationship between **edge incidence**, i.e., an edge is incident with which two vertices

Data Structure - Matrix

Rectangular / 2-dimensional array

- m-by-n matrix
  - m rows
  - n columns
- $a_{i,j}$
  - row i, column j

m-by-n matrix

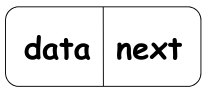
$a_{i,j}$ — **n columns**, **m rows**

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n} \end{pmatrix}$$

Data Structure - Linked List

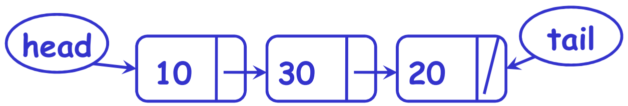
List of elements (nodes) connected together like a chain

Each node contains two fields:



- "data" field: stores whatever type of elements
- "next" field: pointer to link this node to the next node in the list

Head / Tail



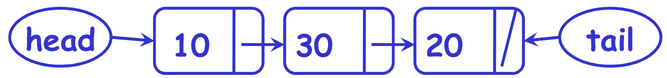
- pointer to the beginning & end of list

Data Structure - Linked List

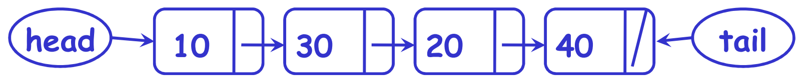
Queue (FIFO: first-in-first-out)

Insert element (enqueue) to tail

Remove element (dequeue) from head

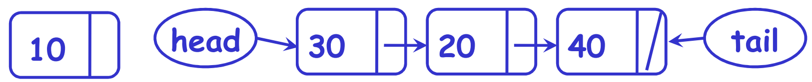


Insert 40



create newnode of 40; tail.next = newnode; tail = tail.next

Remove



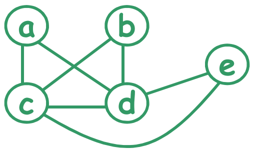
return whatever head points to; head = head.next

Adjacency matrix / list

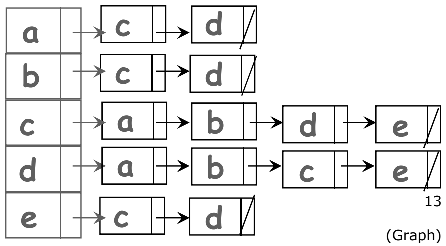
Adjacency matrix M for a simple **undirected** graph with n vertices is an **nxn** matrix

- M(i, j) = 1 if vertex i and vertex j are adjacent
- M(i, j) = 0 otherwise

Adjacency list: each vertex has a list of vertices to which it is adjacent



	a	b	c	d	e
a	0	0	1	1	0
b	0	0	1	1	0
c	1	1	0	1	1
d	1	1	1	0	1
e	0	0	1	1	0

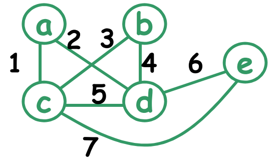


Incidence matrix / list

Incidence matrix M for a simple **undirected** graph with n vertices and m edges is an **mxn** matrix

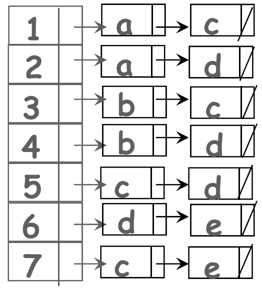
- M(i, j) = 1 if edge i and vertex j are incidence
- M(i, j) = 0 otherwise

Incidence list: each edge has a list of vertices to which it is incident with



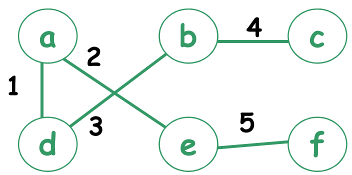
labels of edge are edge number

	a	b	c	d	e
1	1	0	1	0	0
2	1	0	0	1	0
3	0	1	1	0	0
4	0	1	0	1	0
5	0	0	1	1	0
6	0	0	0	1	1
7	0	0	1	0	1



Exercise

Give the adjacency matrix and incidence matrix of the following graph



labels of edge are edge number

	a	b	c	d	e	f
a						
b						
c						
d						
e						
f						

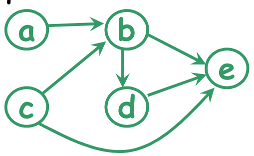
	a	b	c	d	e	f
1						
2						
3						
4						
5						

Directed graph ...

Directed graph

Given a directed graph G, a vertex ***a*** is said to be **connected to** a vertex ***b*** if there is a path from ***a*** to ***b***.

E.g., G represents the routes provided by a certain airline. That means, a vertex represents a city and an edge represents a flight from a city to another city. Then we may ask question like: Can we fly from one city to another?



E = { (a,b), (b,d), (b,e), (c,b), (c,e), (d,e) }

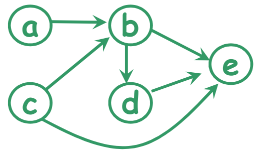
N.B. (a,b) is in E, but (b,a) is NOT

Reminder: A directed graph G=(V,E) consists of a set of vertices V and a set of edges E. Each edge is an ordered pair of vertices.

In/Out degree (in directed graphs)

The **in-degree** of a vertex ***v*** is the number of edges ***leading to*** the vertex ***v***.

The **out-degree** of a vertex ***v*** is the number of edges ***leading away*** from the vertex ***v***.



v	in-deg(v)	out-deg(v)
a	0	1
b	2	2
c	0	2
d	1	1
e	3	0
sum:	6	6

Always equal?

Representation (of directed graphs)

Similar to undirected graph, a directed graph can be represented by **adjacency matrix**, **adjacency list**, **incidence matrix** or **incidence list**.

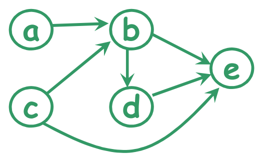
Adjacency matrix / list

Adjacency matrix M for a **directed** graph with n vertices is an **nxn** matrix

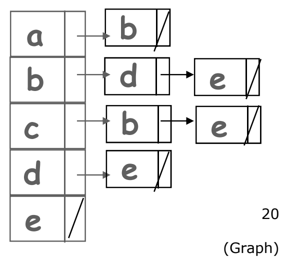
- M(i, j) = 1 if (i,j) is an edge
- M(i, j) = 0 otherwise

Adjacency list:

- each vertex ***u*** has a list of vertices pointed to by an edge leading away from ***u***



	a	b	c	d	e
a	0	1	0	0	0
b	0	0	0	1	1
c	0	1	0	0	1
d	0	0	0	0	1
e	0	0	0	0	0

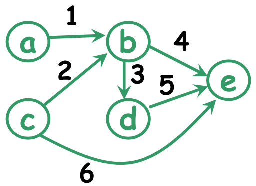


Incidence matrix / list

Incidence matrix M for a **directed** graph with n vertices and m edges is an **mxn** matrix

- M(i, j) = 1 if edge i is leading away from vertex j
- M(i, j) = -1 if edge i is leading to vertex j

Incidence list: each edge has a list of two vertices (leading away is 1st and leading to is 2nd)

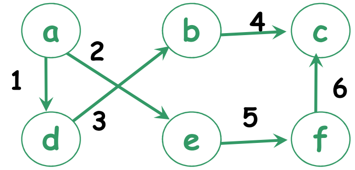


	a	b	c	d	e
1	1	-1	0	0	0
2	0	-1	1	0	0
3	0	1	0	-1	0
4	0	1	0	0	-1
5	0	0	0	1	-1
6	0	0	1	0	-1

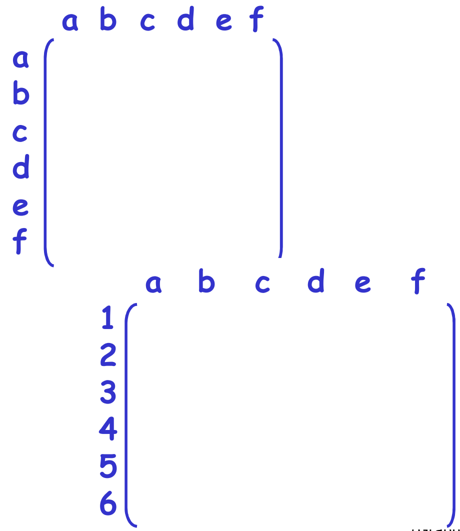
1	a	b
2	c	b
3	b	d
4	b	e
5	d	e
6	c	e

Exercise

Give the adjacency matrix and incidence matrix of the following graph



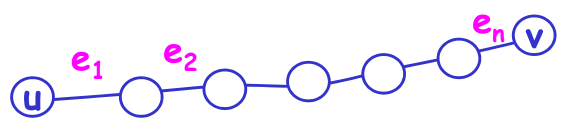
labels of edge are edge number



Euler circuit ...

Paths, circuits (in undirected graphs)

- In an undirected graph, a **path** from a vertex ***u*** to a vertex ***v*** is a sequence of edges $e_1 = \{u, x_1\}$, $e_2 = \{x_1, x_2\}$, …$e_n = \{x_{n-1}, v\}$, where $n \geq 1$.
- The **length** of this path is **n**.
- Note that a path from ***u*** to ***v*** implies a path from ***v*** to ***u***.
- If u = v, this path is called a **circuit** (cycle).

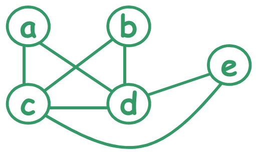


Euler circuit

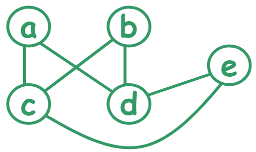
A simple circuit visits an edge at most once.

An Euler circuit in a graph G is a circuit visiting every edge of G exactly once.
(NB. A vertex can be repeated.)

Does every graph has an Euler circuit ?

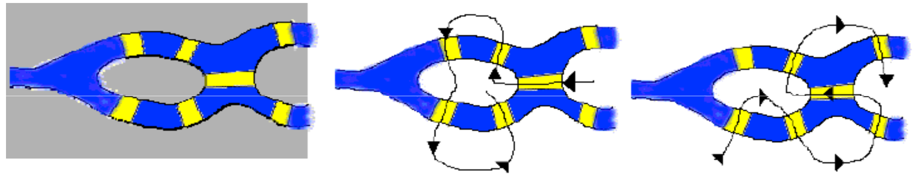


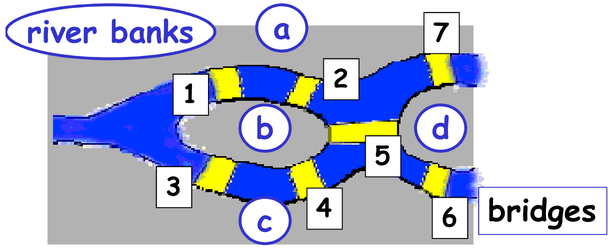
a c b d e c d a

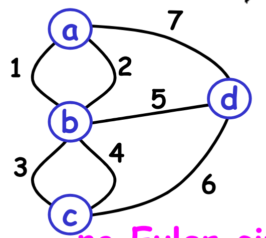


no Euler circuit

History: In Konigsberg, Germany, a river ran through the city and seven bridges were built. The people wondered whether or not one could go around the city in a way that would involve crossing each bridge exactly once.





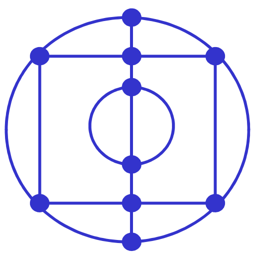


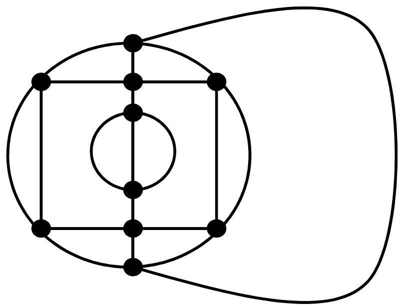
no Euler circuit

Necessary and sufficient condition

Let G be a connected graph.

Lemma: G contains an Euler circuit if and only if degree of every vertex is even.





Hamiltonian circuit

Let G be an undirected graph.

A **Hamiltonian circuit** is a circuit containing **every vertex** of G **exactly once**.

Note that a Hamiltonian circuit may NOT visit all edges.

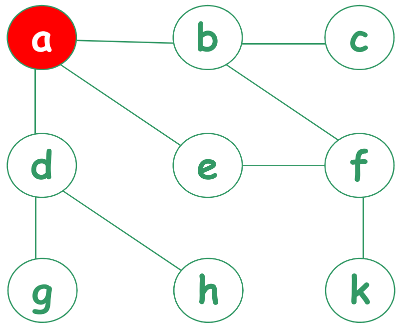
Unlike the case of Euler circuits, determining whether a graph contains a Hamiltonian circuit is a very ***difficult*** problem. (NP-hard)

Breadth First Search BFS ...

Breadth First Search (BFS)

All vertices at distance k from s are explored before any vertices at distance k+1.

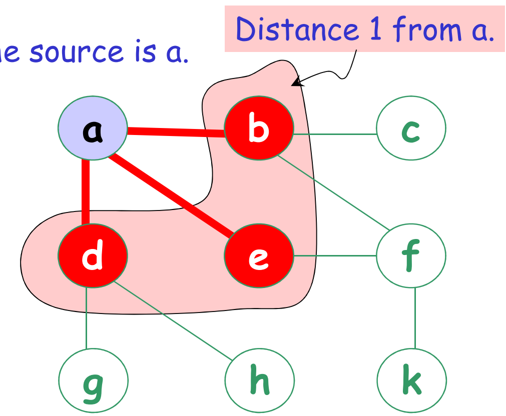
The source is a.



Order of exploration
a,

Breadth First Search (BFS)

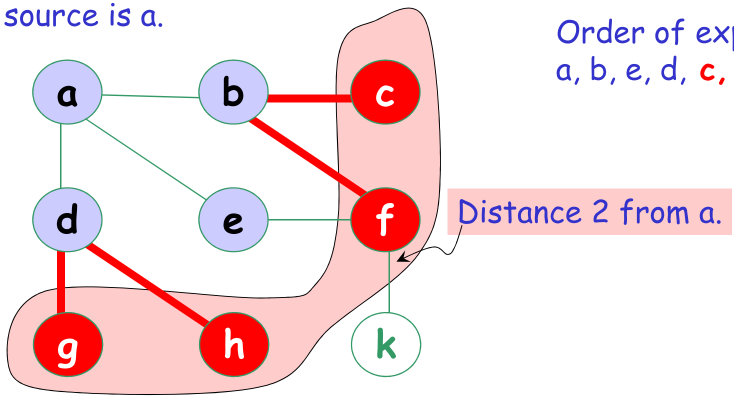
The source is a.



Order of exploration
a, **b, e, d**

Breadth First Search (BFS)

The source is a.

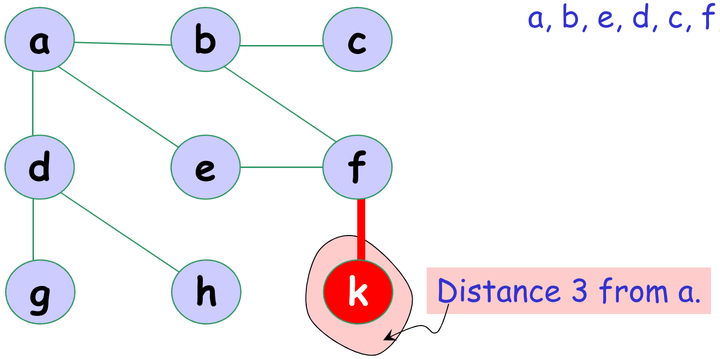


Order of exploration
a, b, e, d, **c, f, h, g**

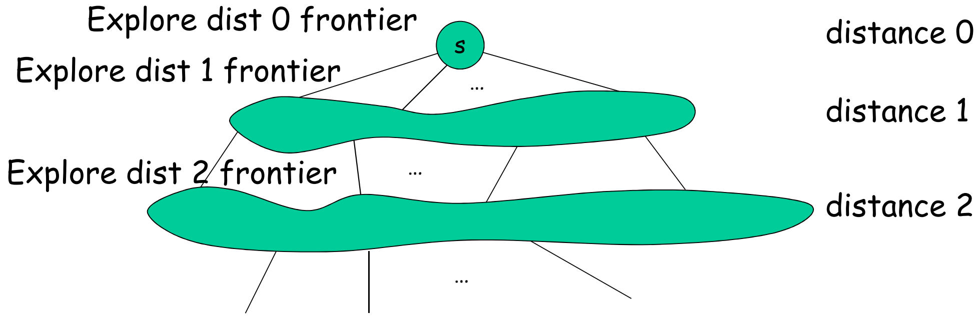
Breadth First Search (BFS)

The source is a.

Order of exploration
a, b, e, d, c, f, h, g, **k**



In general (BFS)



Breadth First Search (BFS)

A simple algorithm for searching a graph.

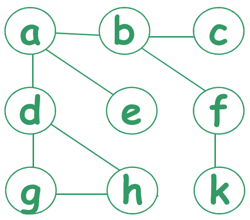
Given G=(V, E), and a distinguished source vertex ***s***, BFS systematically explores the edges of G such that

- all vertices at ***distance k*** from s are explored ***before*** any vertices at ***distance k+1***.

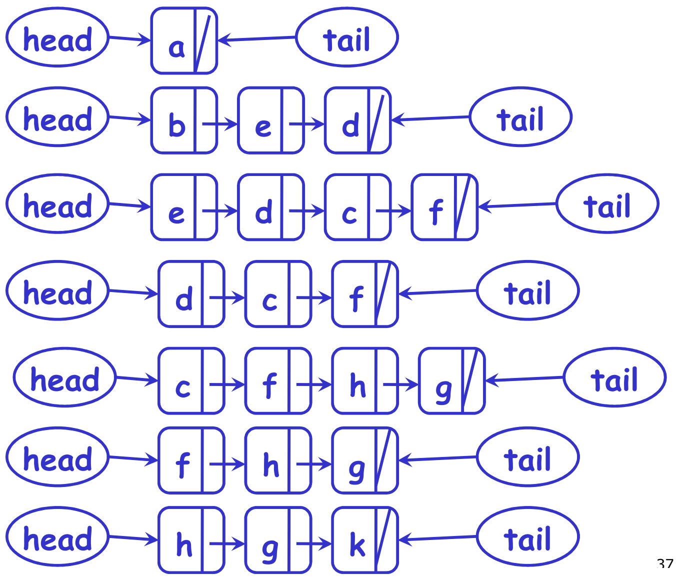
BFS – Pseudo code

```
unmark all vertices
choose some starting vertex s
mark s and insert s into tail of list L
while L is nonempty do
begin
  remove a vertex v from front of L
  visit v
  for each unmarked neighbor w of v do
      mark w and insert w into tail of list L
end
```

BFS using linked list



a, b, e, d, c, f, h, g, k



& so on …

Depth First Search DFS …

Depth First Search (DFS)

Edges are explored from the most recently discovered vertex, backtracks when finished

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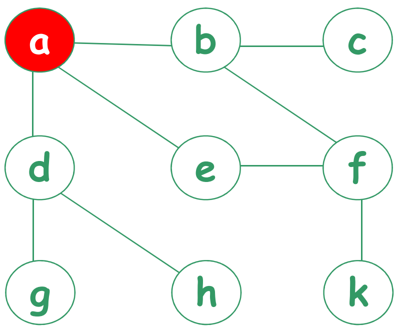
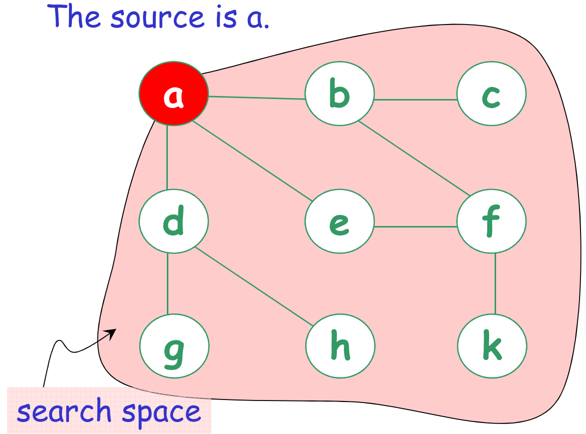


Order of exploration
a,

DFS searches "**deeper**" in the graph whenever possible

Depth First Search (DFS)

The source is a.



Order of exploration
a,

DFS searches "**deeper**" in the graph whenever possible

Depth First Search (DFS)

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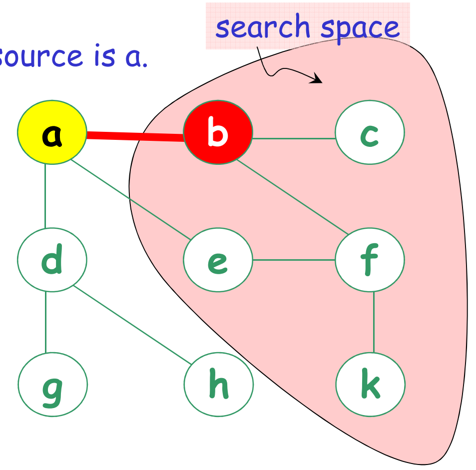

Order of exploration
a, b

DFS searches "deeper" in the graph whenever possible

Depth First Search (DFS)

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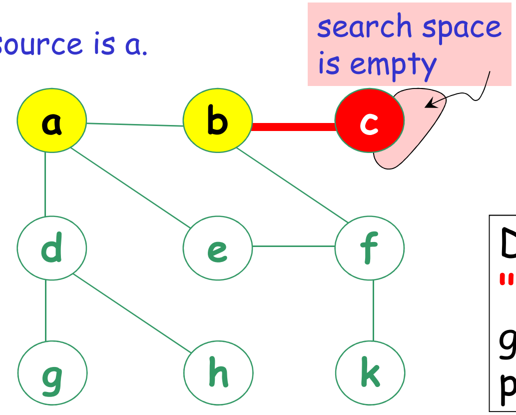

Order of exploration
a, b, c

DFS searches "deeper" in the graph whenever possible

Depth First Search (DFS)

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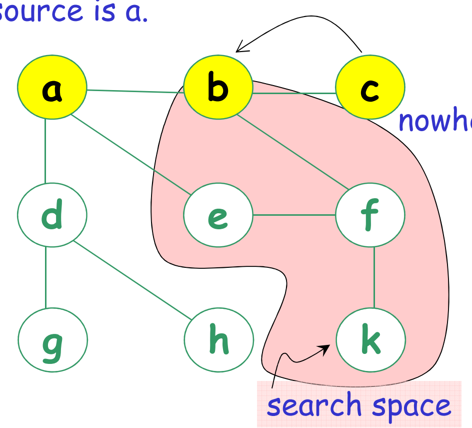

nowhere to go, backtrack

Order of exploration
a, b, c

DFS searches "deeper" in the graph whenever possible

Depth First Search (DFS)

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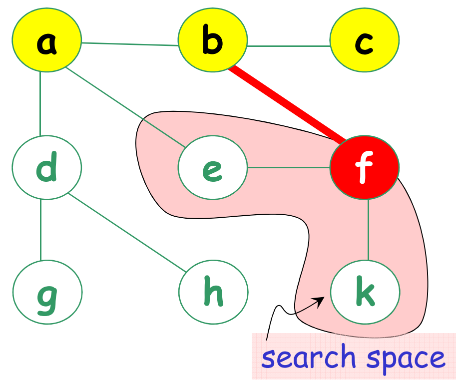
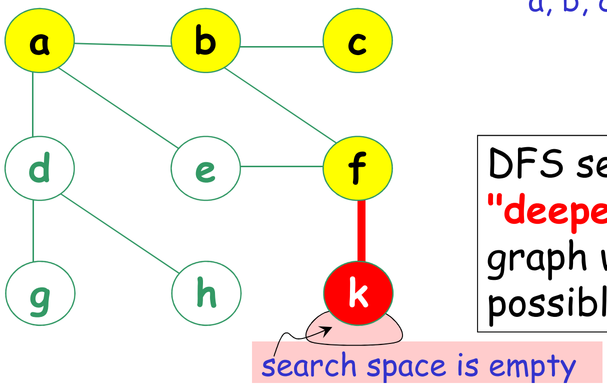


Order of exploration
a, b, c, f

DFS searches "deeper" in the graph whenever possible

Depth First Search (DFS)

The source is a.

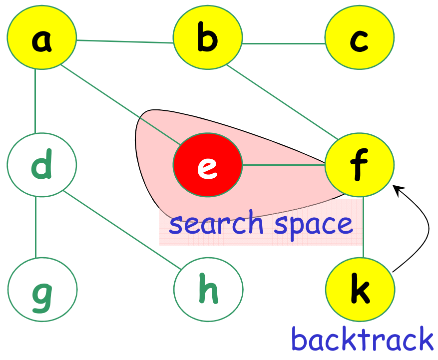


Order of exploration
a, b, c, f, **k**

DFS searches **"deeper"** in the graph whenever possible

Depth First Search (DFS)

The source is a.



Order of exploration
a, b, c, f, k, **e**

DFS searches **"deeper"** in the graph whenever possible

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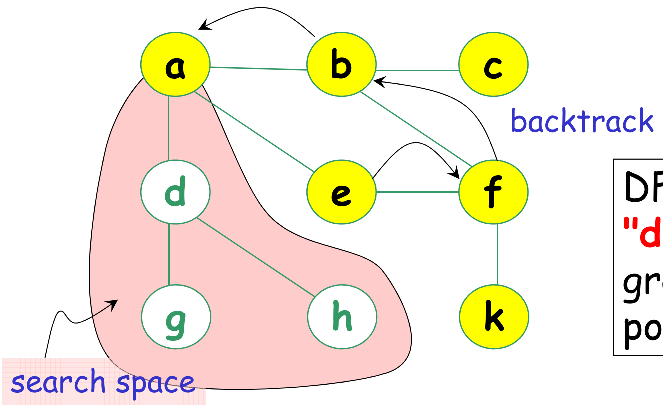

Order of exploration
a, b, c, f, k, e

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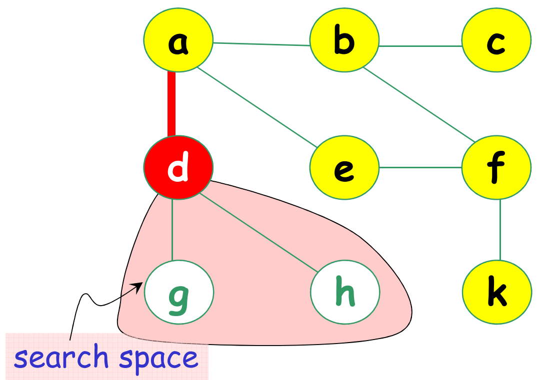


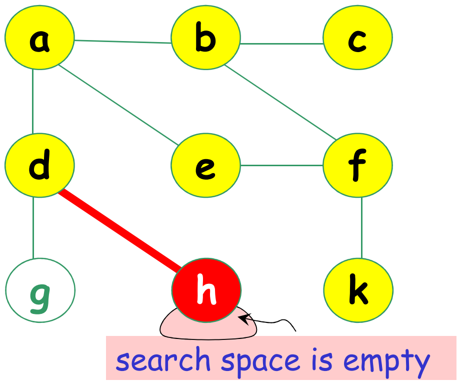
Order of exploration
a, b, c, f, k, e, **d**

DFS searches **"deeper"** in the graph whenever possible

Depth First Search (DFS)

The source is a.

Order of exploration
a, b, c, f, k, e, d, **h**

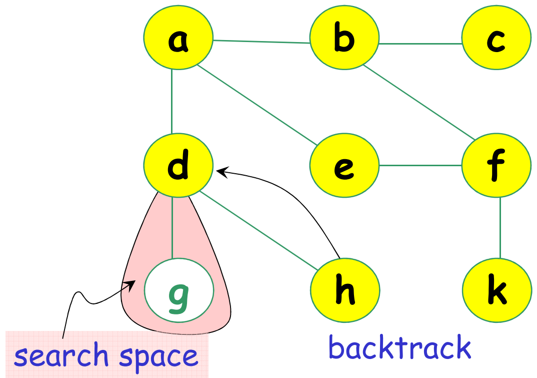


DFS searches **"deeper"** in the graph whenever possible

Depth First Search (DFS)

The source is a.

Order of exploration
a, b, c, f, k, e, d, h

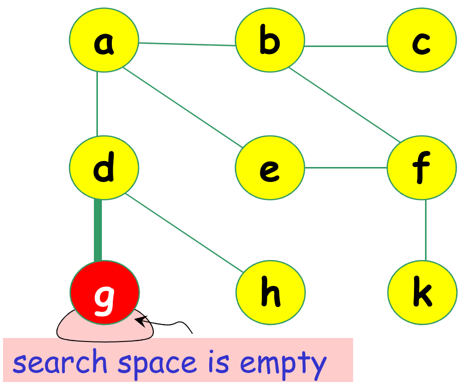


DFS searches **"deeper"** in the graph whenever possible

Depth First Search (DFS)

The source is a.

Order of exploration
a, b, c, f, k, e, d, h, **g**

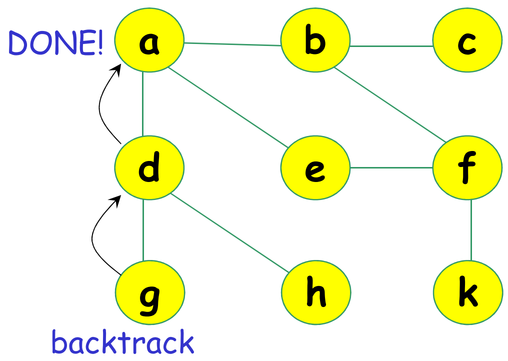


DFS searches **"deeper"** in the graph whenever possible

Depth First Search (DFS)

The source is a.

Order of exploration
a, b, c, f, k, e, d, h, g



DFS searches **"deeper"** in the graph whenever possible

Depth First Search (DFS)

Depth-first search is another strategy for exploring a graph; it search **"deeper"** in the graph whenever possible.

- Edges are explored from the most recently discovered vertex ***v*** that still has unexplored edges leaving it.
- When all edges of ***v*** have been explored, the search ***"backtracks"*** to explore edges leaving the vertex from which ***v*** was discovered.

DFS - Pseudo code (recursive)

```
Algorithm DFS(vertex v)
  visit v
  for each unvisited neighbor w of v do
  begin
      DFS(w)
  end
```


Tree ...

Trees

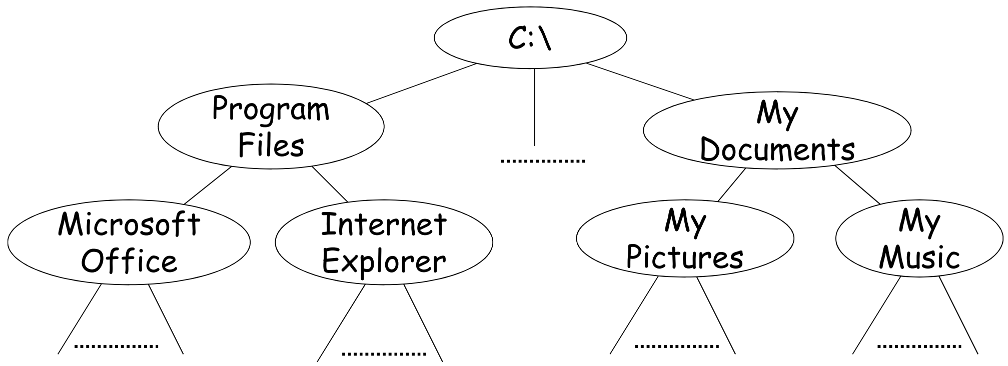
An undirected graph $G=(V,E)$ is a tree if G is connected and acyclic (i.e., contains no cycles)

Other equivalent statements:

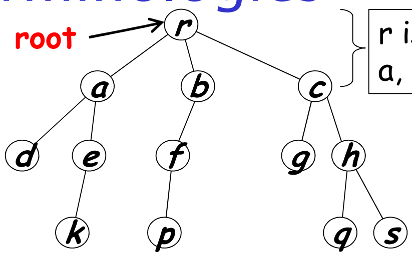
1. There is exactly one path between any two vertices in G
2. G is connected and removal of one edge disconnects G
3. G is acyclic and adding one edge creates a cycle
4. G is connected and $m=n-1$ (where $|V|=n$, $|E|=m$)

Rooted trees

Tree with hierarchical structure, e.g., directory structure of file system



Terminologies



r is **parent** of a, b & c;
a, b, & c are **children** of r

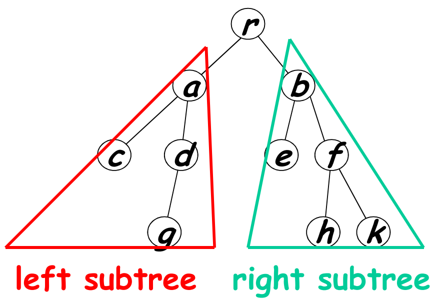
deg-0: d, k, p, g, q, s (**leaves**)
deg-1: b, e, f
deg-2: a, c, h
deg-3: r

What is the degree of this tree?

- Topmost vertex is called the **root**.
- A vertex ***u*** may have some **children** directly below it, ***u*** is called the **parent** of its children.
- **Degree** of a ***vertex*** is the no. of children it has. (N.B. it is different from the degree in an unrooted tree.)
- Degree of a ***tree*** is the max. degree of all vertices.
- A vertex with no child (degree-0) is called a **leaf**. All others are called **internal vertices**.

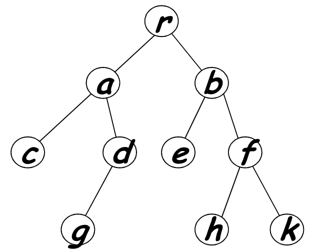
Binary tree

- a tree of degree at most TWO
- the two subtrees are called left subtree and right subtree (may be empty)



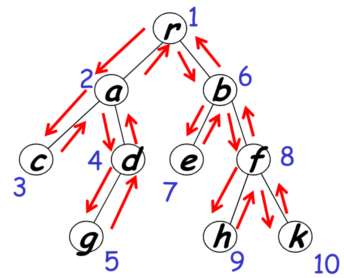
There are *three* common ways to traverse a binary tree:
- **preorder** traversal - vertex, left subtree, right subtree
- **inorder** traversal - left subtree, vertex, right subtree
- **postorder** traversal - left subtree, right subtree, vertex

Traversing a binary tree

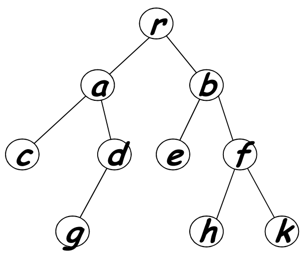


preorder traversal
- vertex, left subtree, right subtree

r -> a -> c -> d -> g -> b -> e -> f -> h -> k



Traversing a binary tree

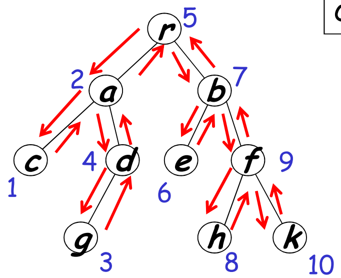


preorder traversal
- vertex, left subtree, right subtree

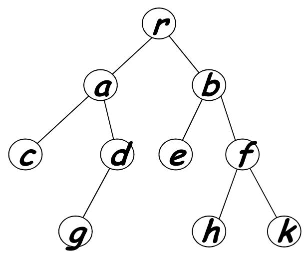
r -> a -> c -> d -> g -> b -> e -> f -> h -> k

inorder traversal
- left subtree, vertex, right subtree

c -> a -> g -> d -> r -> e -> b -> h -> f -> k



Traversing a binary tree



preorder traversal
- vertex, left subtree, right subtree

r -> a -> c -> d -> g -> b -> e -> f -> h -> k

inorder traversal
- left subtree, vertex, right subtree

c -> a -> g -> d -> r -> e -> b -> h -> f -> k

postorder traversal
- left subtree, right subtree, vertex

c -> g -> d -> a -> e -> h -> k -> f -> b -> r