COMP108
Algorithmic Foundations

Dynamic Programming

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Dynamic programming

an efficient way to implement some divide and conquer algorithms

Learning outcomes

- Understand the basic idea of dynamic programming
- Able to apply dynamic programming to compute Fibonacci numbers
- Able to apply dynamic programming to solve the assembly line scheduling problem

Fibonacci numbers ...

Problem with recursive method

Fibonacci number F(n)

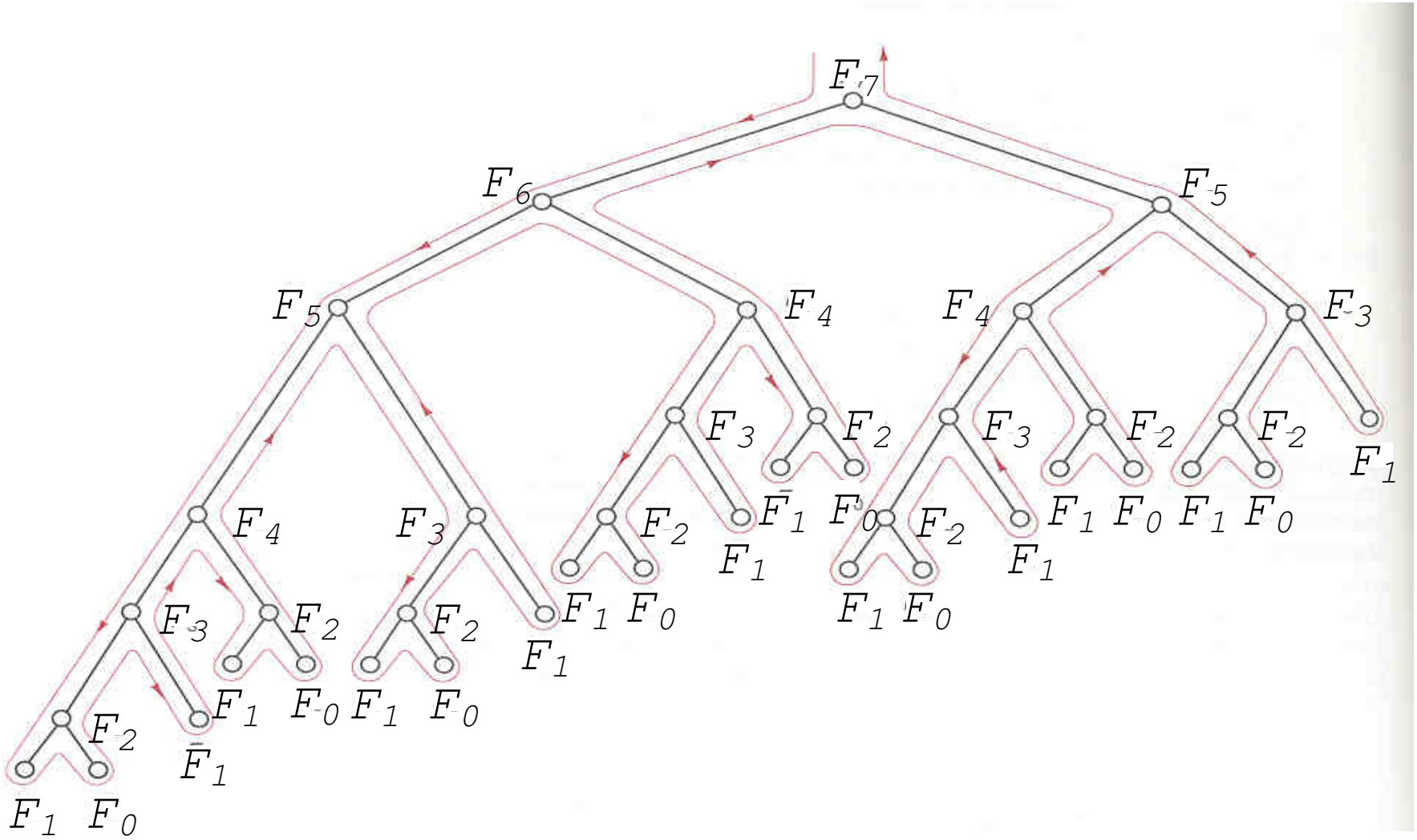
$$F(n) = \begin{cases} 1 & \text{if } n = 0 \text{ or } 1 \\ F(n-1) + F(n-2) & \text{if } n > 1 \end{cases}$$

n	0	1	2	3	4	5	6	7	8	9	10
F(n)	1	1	2	3	5	8	13	21	34	55	89

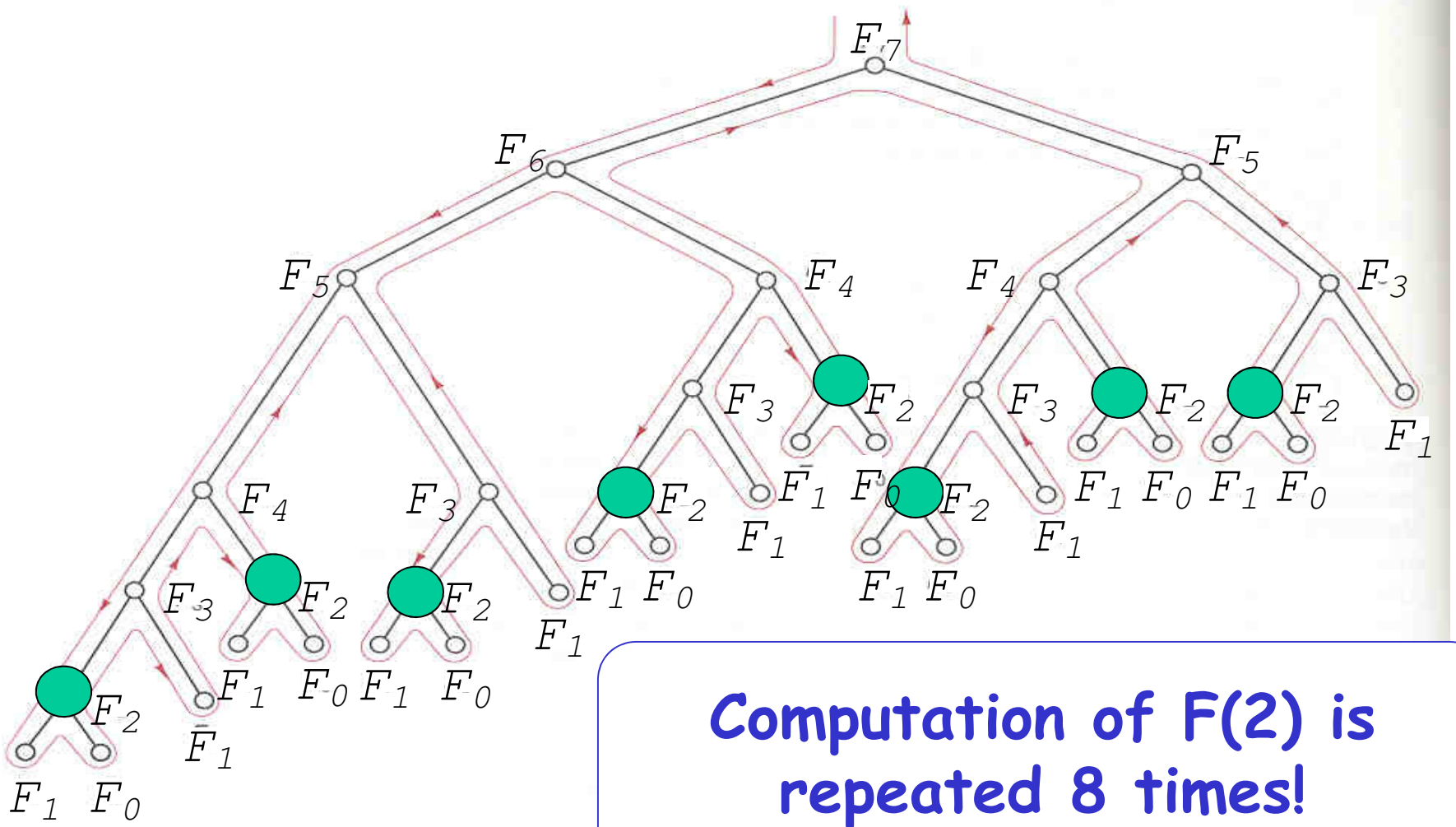
Pseudo code for the recursive algorithm:

```
Procedure F(n)
    if n==0 or n==1 then
        return 1
    else
        return F(n-1) + F(n-2)
```

The execution of F(7)

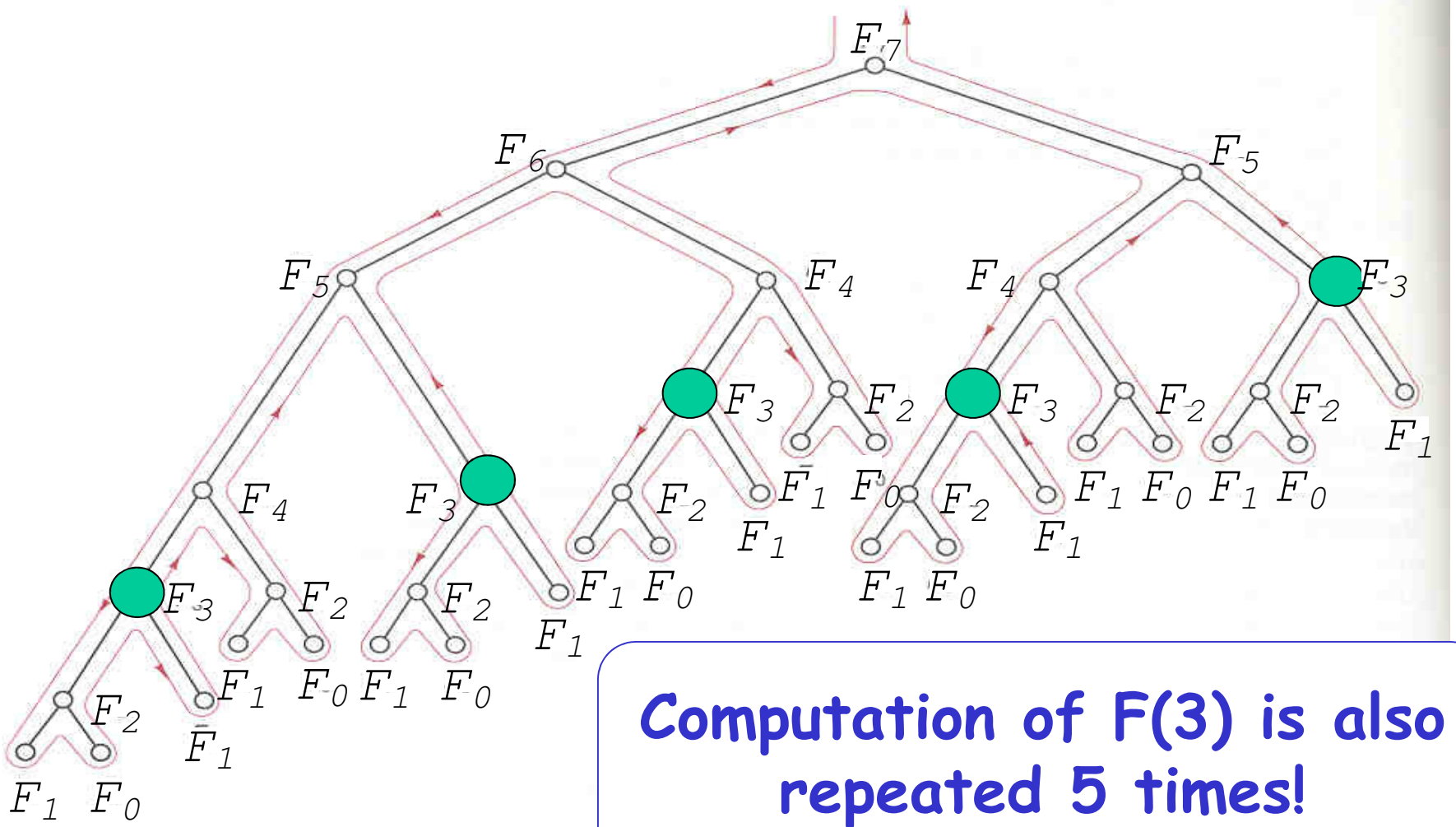


The execution of F(7)



Computation of F(2) is repeated 8 times!

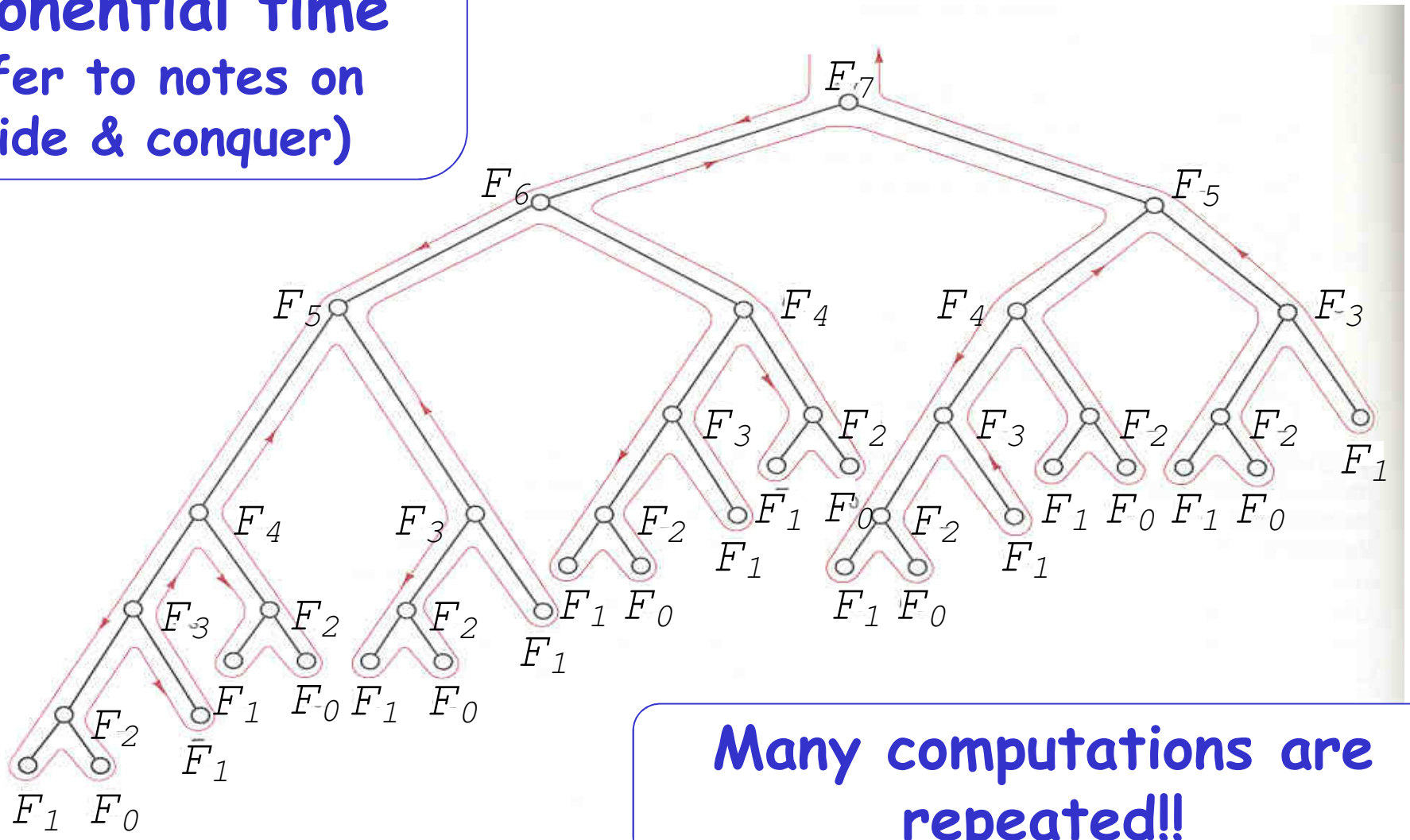
The execution of F(7)



Computation of F(3) is also repeated 5 times!

The execution of F(7)

How long it takes?
exponential time
(refer to notes on divide & conquer)



Many computations are repeated!!

Idea for improvement

Memorization:

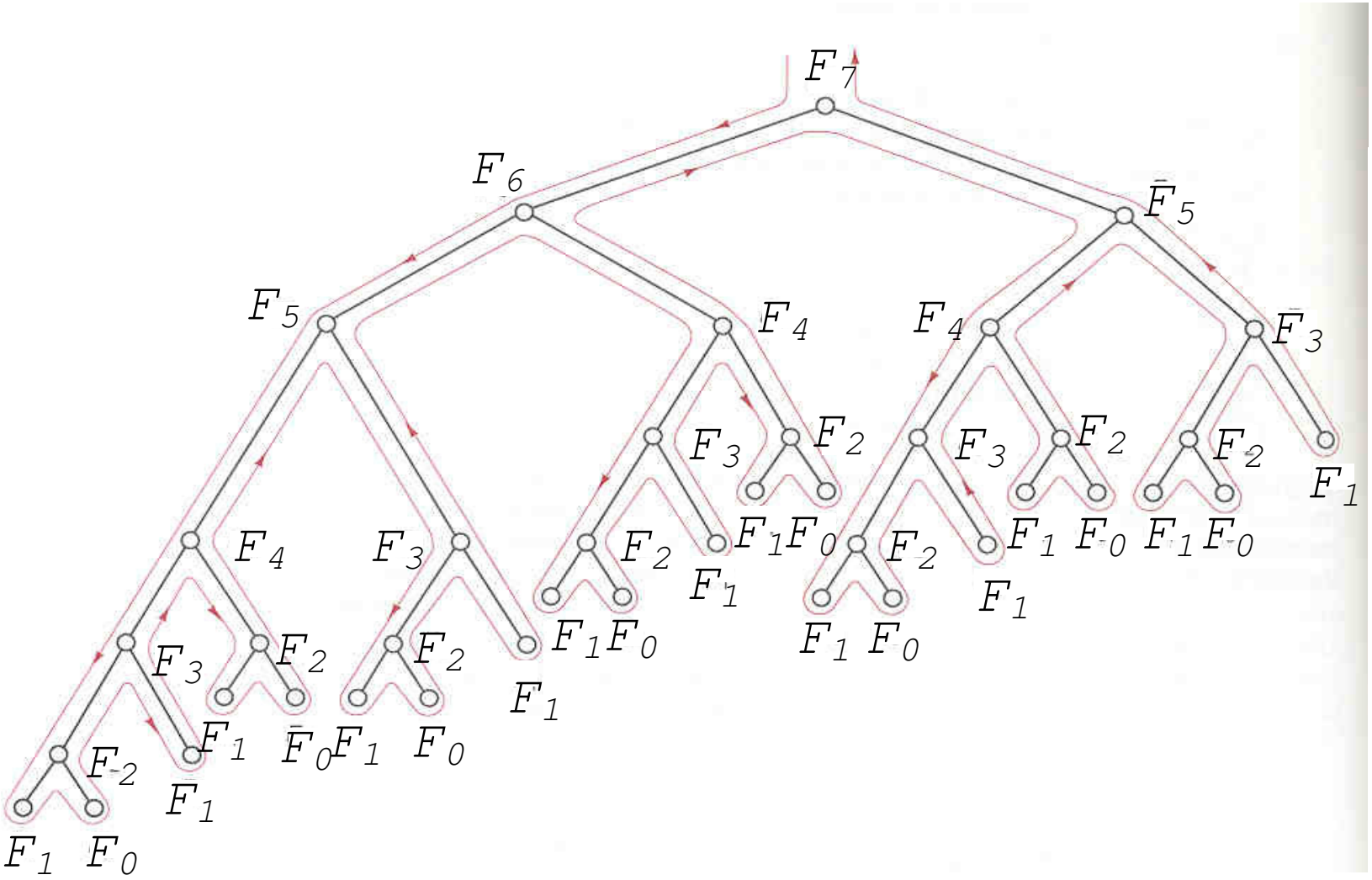
- Store F(i) somewhere after we have computed its value
- Afterward, we don't need to re-compute F(i); we can retrieve its value from our memory.

```
Procedure F(n)
  if (v[n] < 0) then
    v[n] =  F(n-1)+F(n-2)
  return v[n]
```

```
Main
  set v[0] = v[1] = 1
  for i = 2 to n do
    v[i] = -1
  output F(n)
```

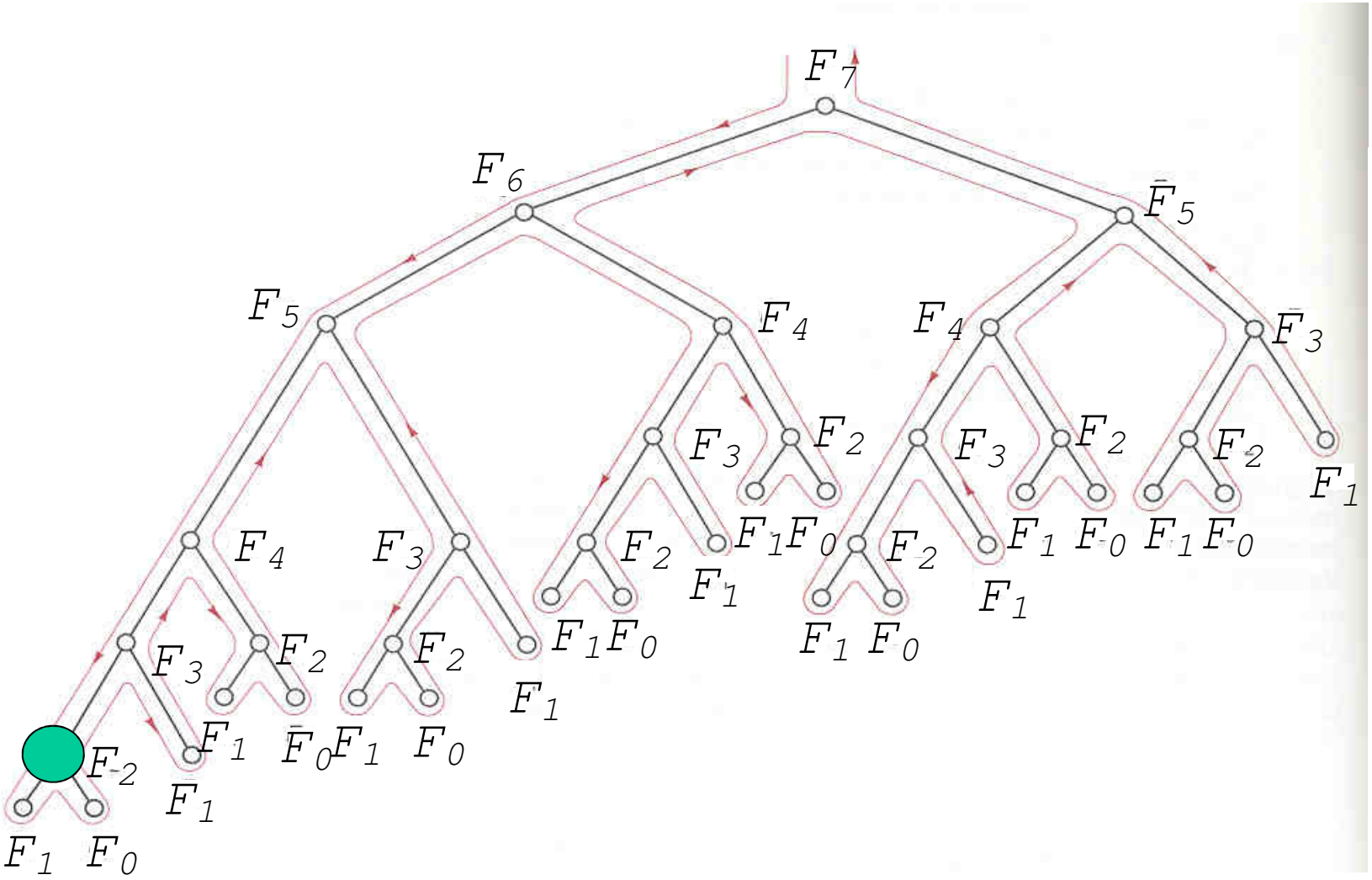
[ ] refers to array
( ) is parameter for calling a procedure

Look at the execution of F(7)



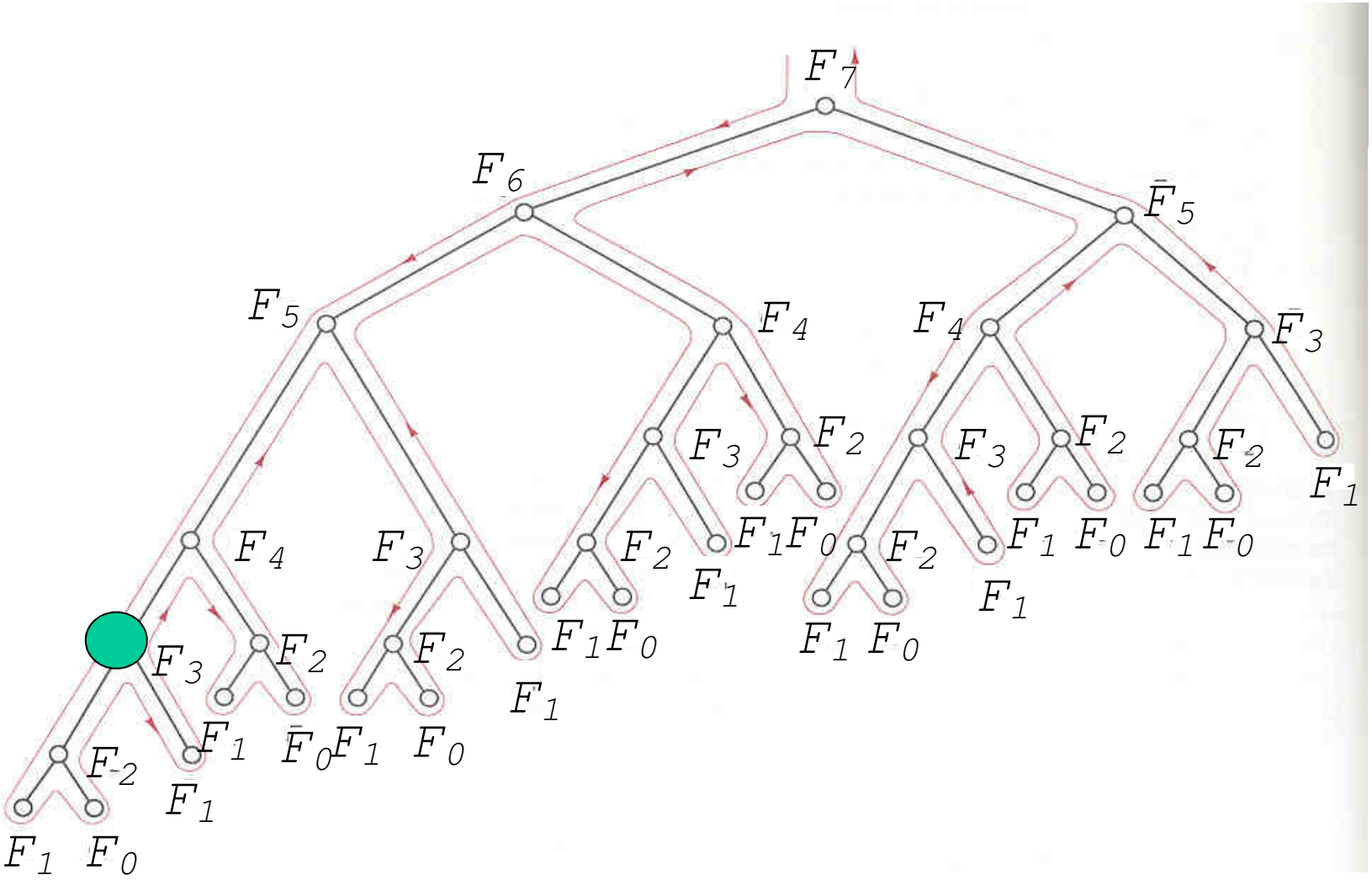
v[0]	1
v[1]	1
v[2]	-1
v[3]	-1
v[4]	-1
v[5]	-1
v[6]	-1
v[7]	-1

Look at the execution of F(7)



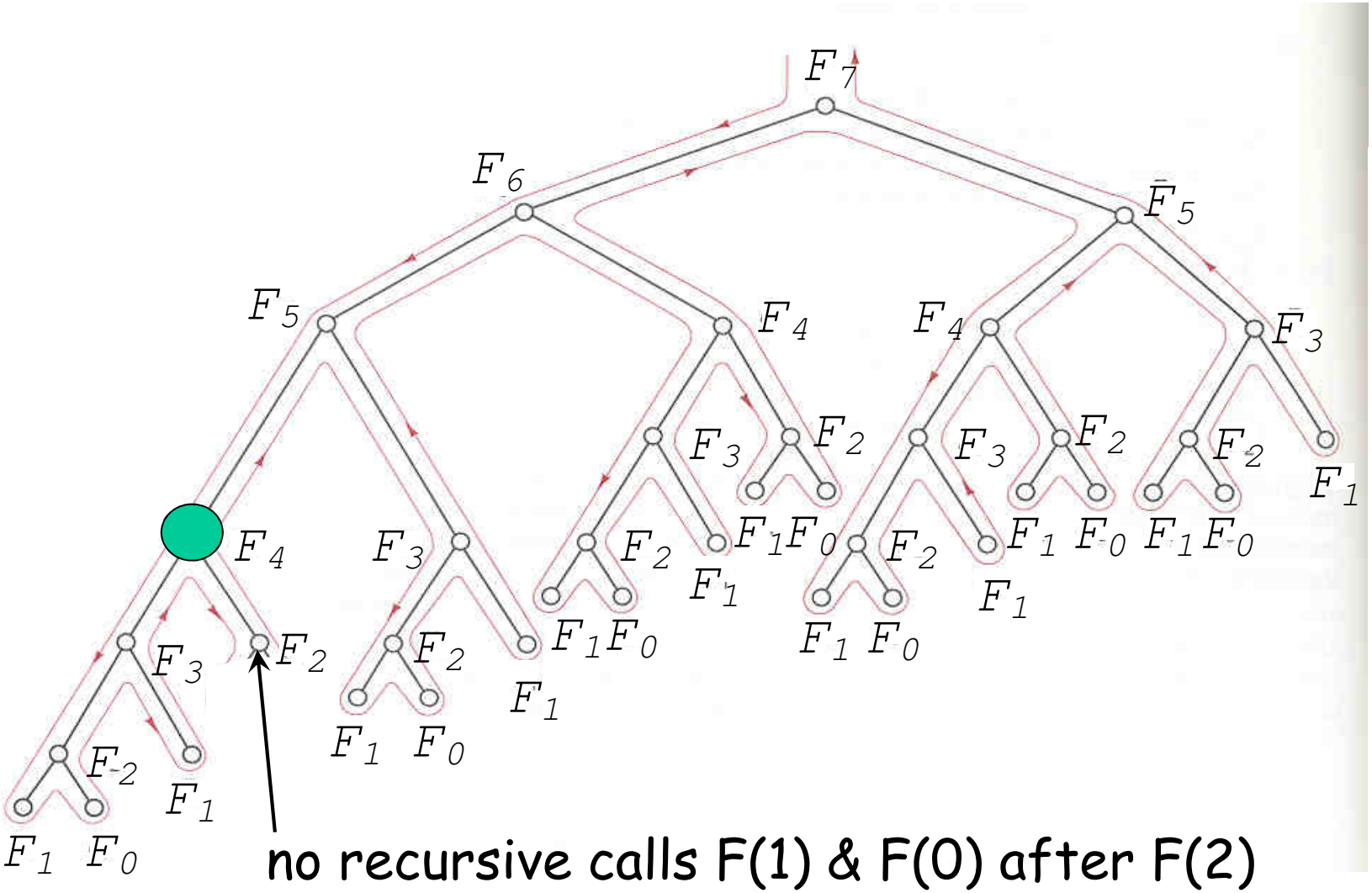
v[0]	1
v[1]	1
v[2]	2
v[3]	-1
v[4]	-1
v[5]	-1
v[6]	-1
v[7]	-1

Look at the execution of F(7)



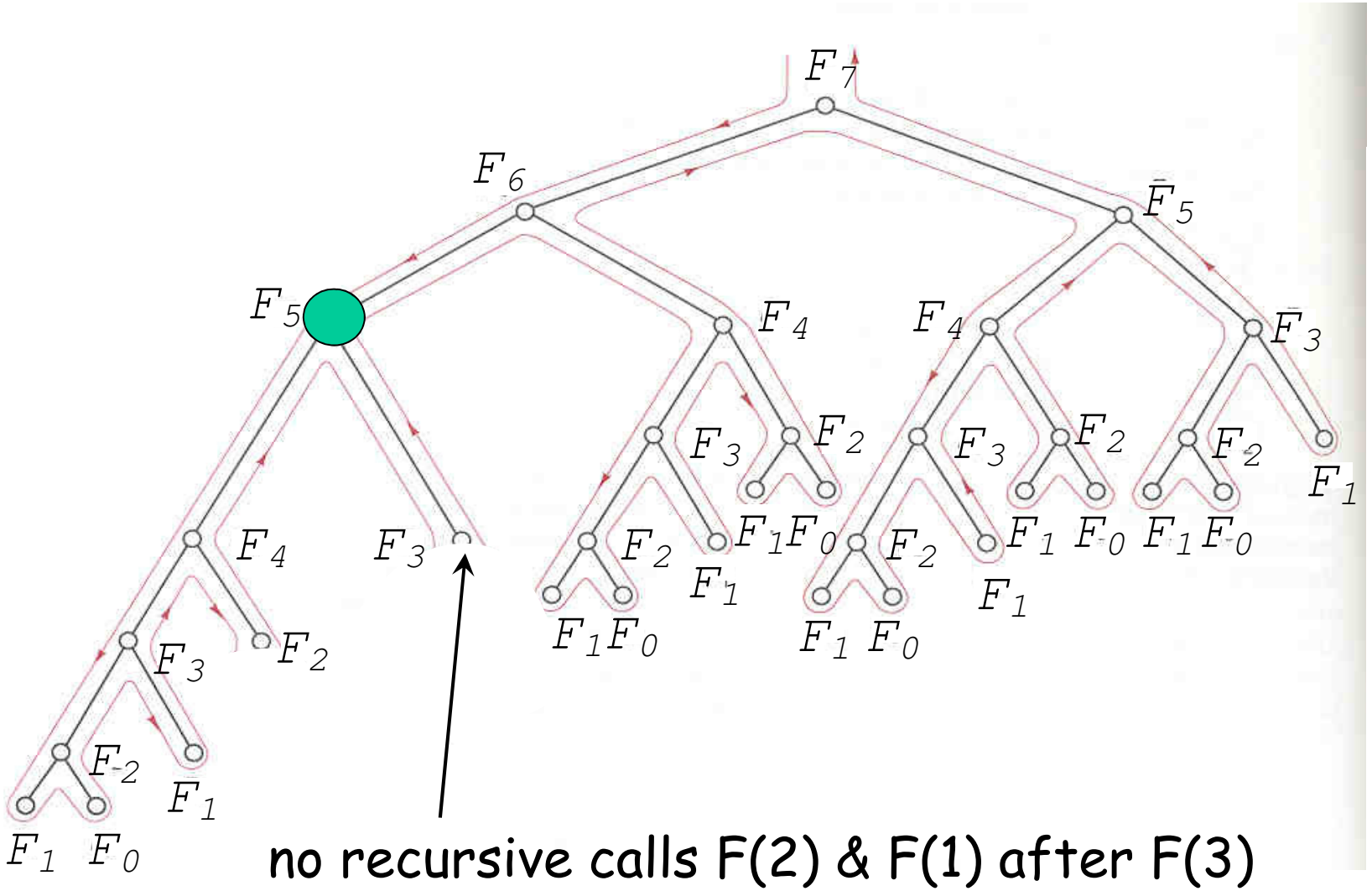
v[0]	1
v[1]	1
v[2]	2
v[3]	3
v[4]	-1
v[5]	-1
v[6]	-1
v[7]	-1

Look at the execution of F(7)



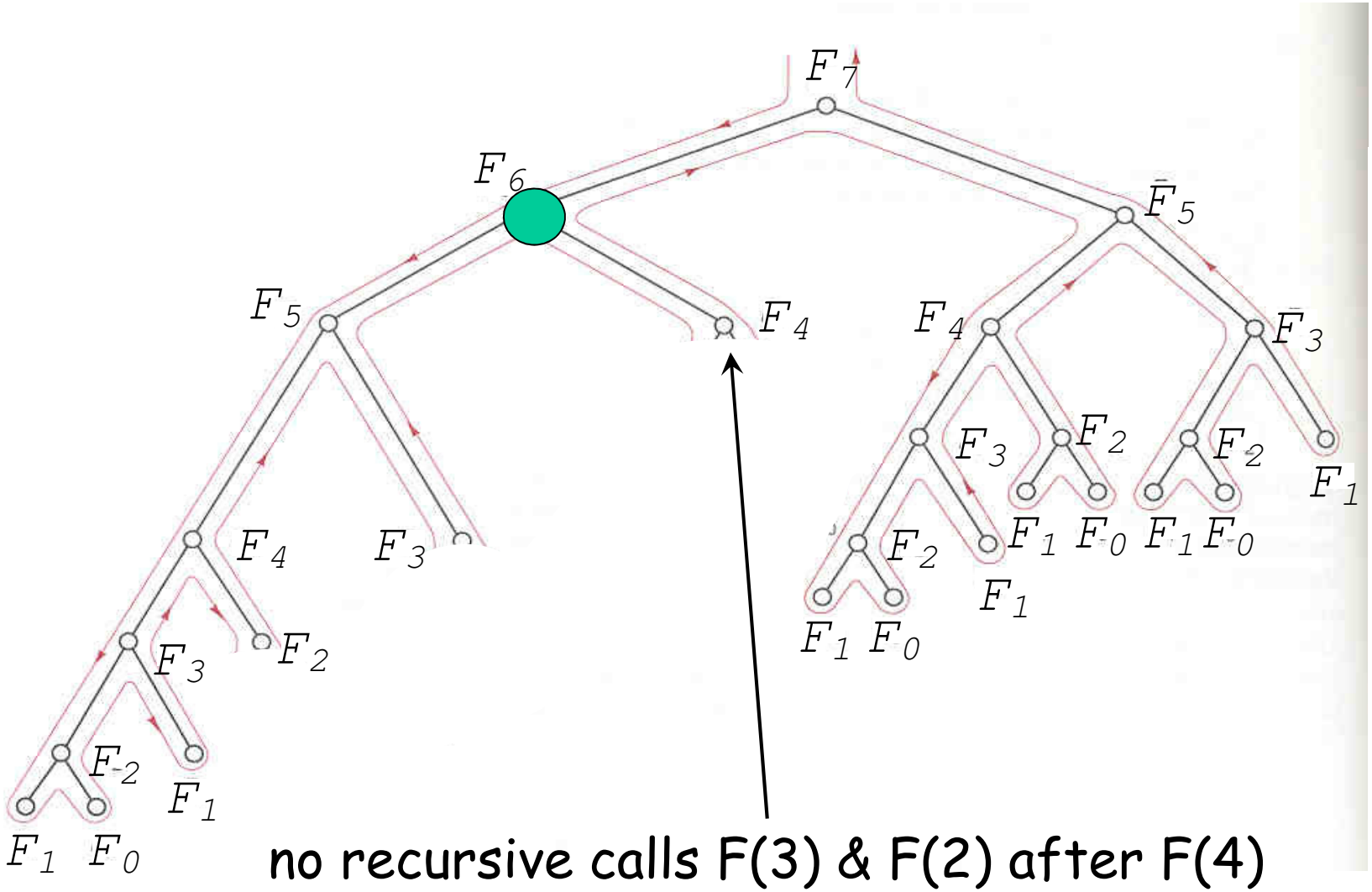
v[0]	1
v[1]	1
v[2]	2
v[3]	3
v[4]	5
v[5]	-1
v[6]	-1
v[7]	-1

Look at the execution of F(7)



v[0]	1
v[1]	1
v[2]	2
v[3]	3
v[4]	5
v[5]	8
v[6]	-1
v[7]	-1

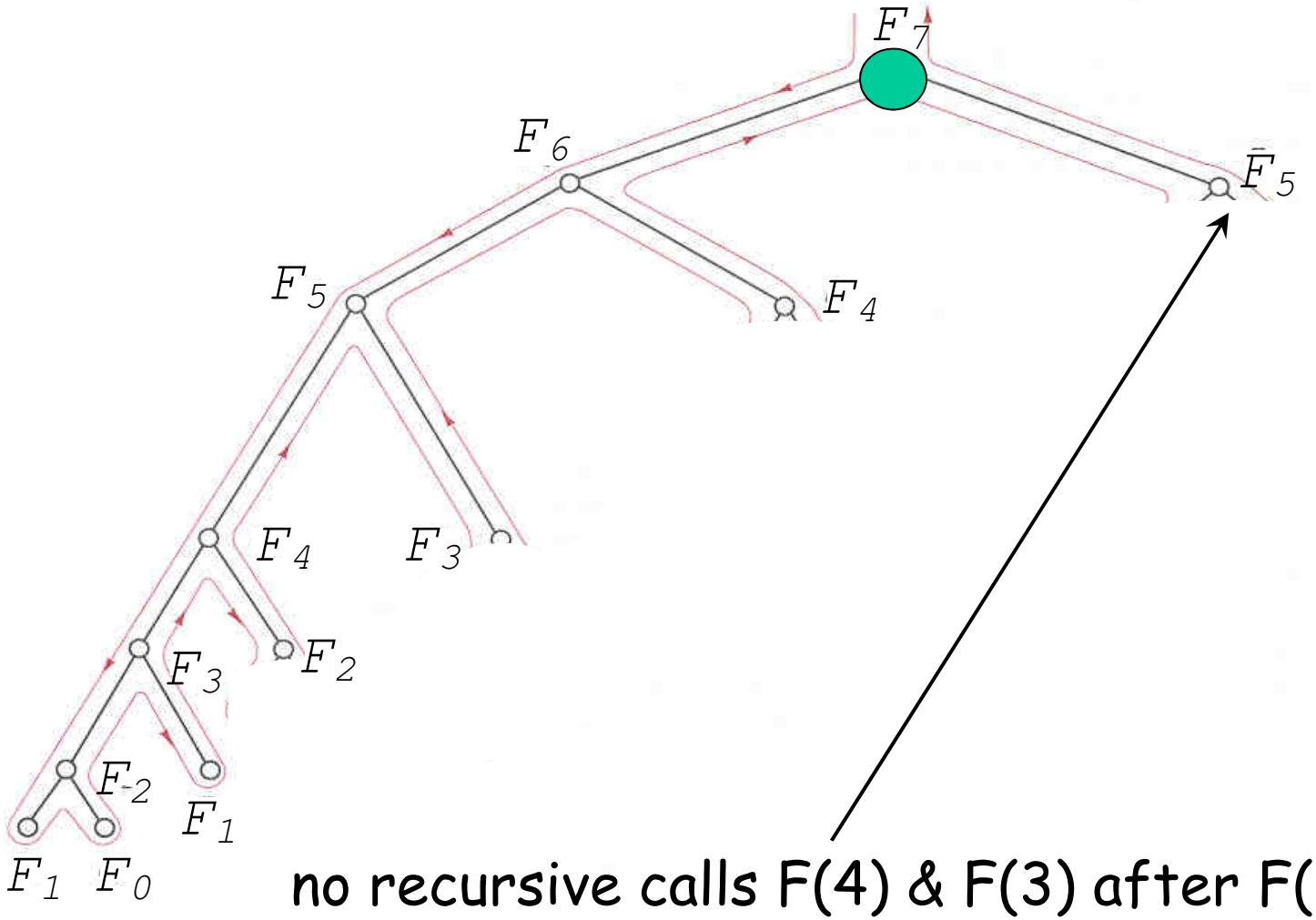
Look at the execution of F(7)



no recursive calls F(3) & F(2) after F(4)

v[0]	1
v[1]	1
v[2]	2
v[3]	3
v[4]	5
v[5]	8
v[6]	13
v[7]	-1

Look at the execution of F(7)



no recursive calls F(4) & F(3) after F(5)

v[0]	1
v[1]	1
v[2]	2
v[3]	3
v[4]	5
v[5]	8
v[6]	13
v[7]	21

Can we do even better?

Observation

- The 2nd version still makes many function calls, and each wastes time in parameters passing, dynamic linking, ...
- In general, to compute F(i), we need F(i-1) & F(i-2) only

Idea to further improve

- Compute the values in bottom-up fashion.
- That is, compute F(2) (we already know F(0)=F(1)=1), then F(3), then F(4)...


```
Procedure F(n)
   Set A[0] = A[1] = 1
   for i = 2 to n do
      A[i] = A[i-1] + A[i-2]
   return A[n]
```

Recursive vs DP approach

Recursive version:

```
Procedure F(n)
    if n==0 or n==1 then
        return 1
    else
        return F(n-1) + F(n-2)
```

Too Slow!
exponential

Dynamic Programming version:

```
Procedure F(n)
    Set A[0] = A[1] = 1
    for i = 2 to n do
        A[i] = A[i-1] + A[i-2]
    return A[n]
```

Efficient!
Time complexity is O(n)

Summary of the methodology

- Write down a formula that relates a solution of a problem with those of sub-problems.
  E.g. F(n) = F(n-1) + F(n-2).
- Index the sub-problems so that they can be **stored** and **retrieved** easily in a table (i.e., array)
- Fill the table in some **bottom-up** manner; start filling the solution of the smallest problem.
  - This ensures that when we solve a particular sub-problem, the solutions of all the smaller sub-problems that it depends are available.

For historical reasons, we call such methodology **Dynamic Programming**.

In the late 40's (when computers were rare), programming refers to the "tabular method".

Exercise

Consider the following function

$$G(n) = \begin{cases} 1 & \text{if } 0 \le n \le 2 \\ G(n-1) + G(n-2) + G(n-3) & \text{if } n > 2 \end{cases}$$

1. Write a recursive procedure to compute G(n)
2. Draw the execution tree of computing **G(6)** recursively
3. Using dynamic programming, write a pseudo code to compute G(n) efficiently
4. What is the time complexity of your algorithm?

Exercise

$$G(n) = \begin{cases} 1 & \text{if } 0 \le n \le 2 \\ G(n-1) + G(n-2) + G(n-3) & \text{if } n > 2 \end{cases}$$

Recursive version:

```
Procedure G(n)
```

Dynamic Programming version:

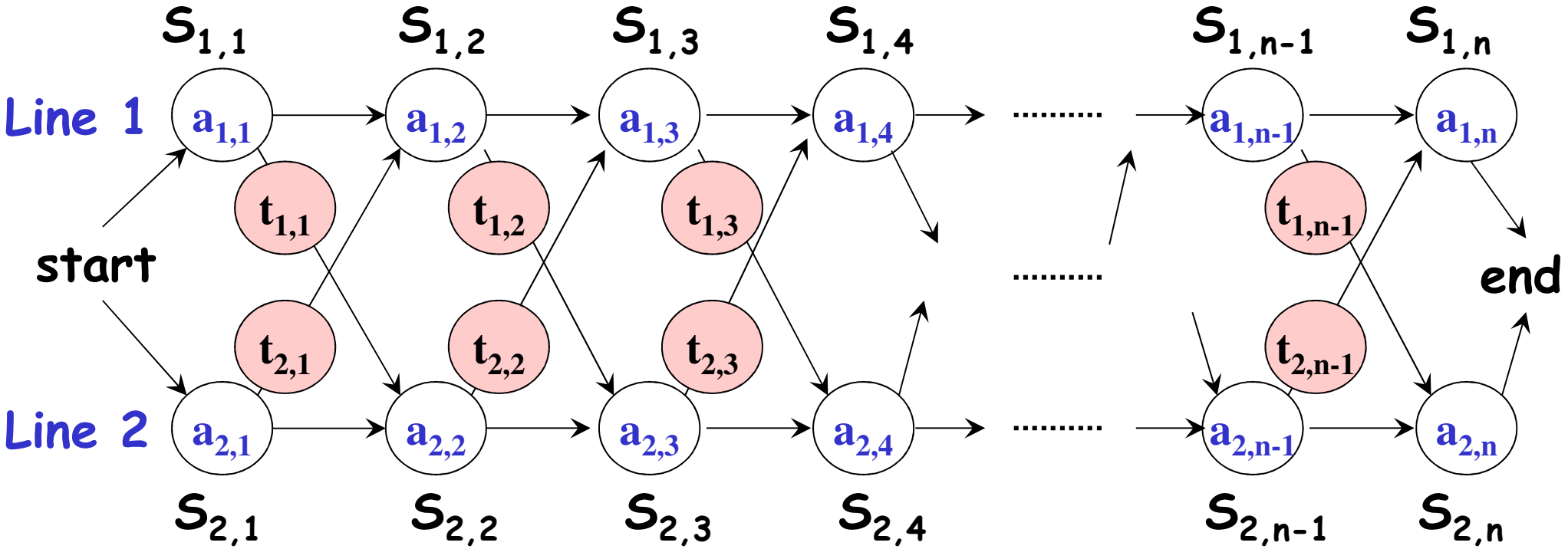
```
Procedure G(n)
```

O(?)

Assembly line scheduling ...

Assembly line scheduling

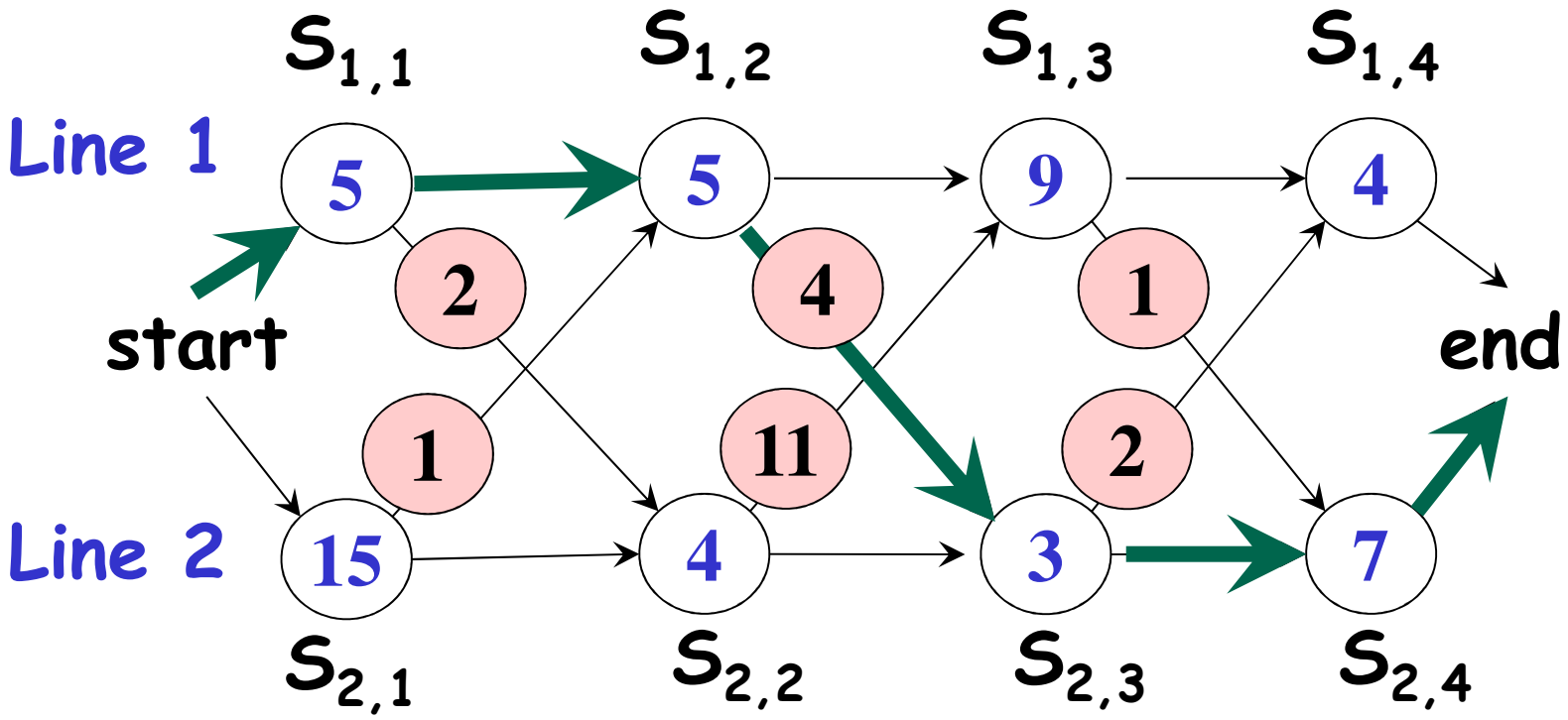
2 assembly lines, each with n stations ($S_{i,j}$: line i station j)
$S_{1,j}$ and $S_{2,j}$ perform same task but time taken is different



$a_{i,j}$: assembly time at $S_{i,j}$
$t_{i,j}$: transfer time after $S_{i,j}$

Problem: To determine which stations to go in order to **minimize** the total time through the n stations

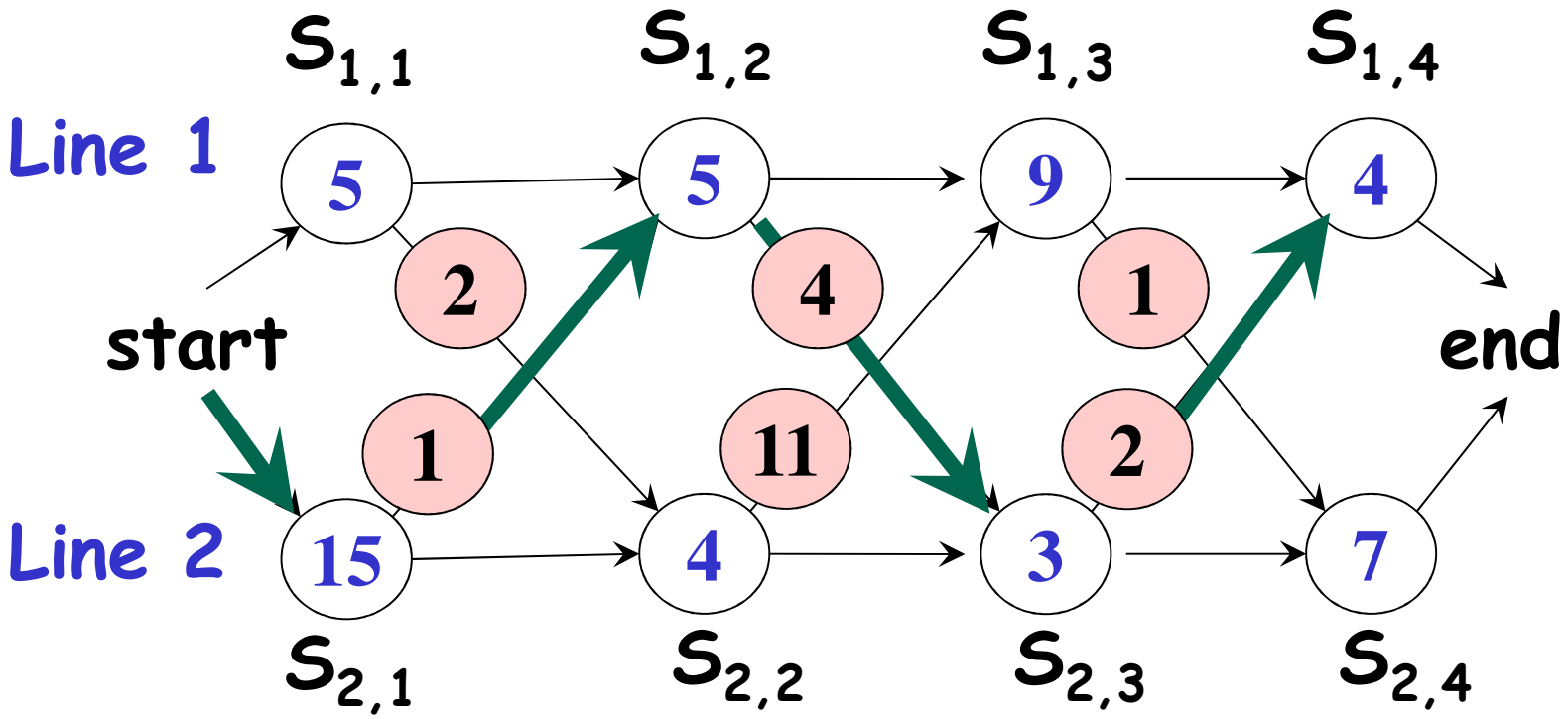
Example (1)



stations chosen: $S_{1,1}$ $S_{1,2}$ $S_{2,3}$ $S_{2,4}$

time required: 5 5 4 3 7 = 24

Example (2)

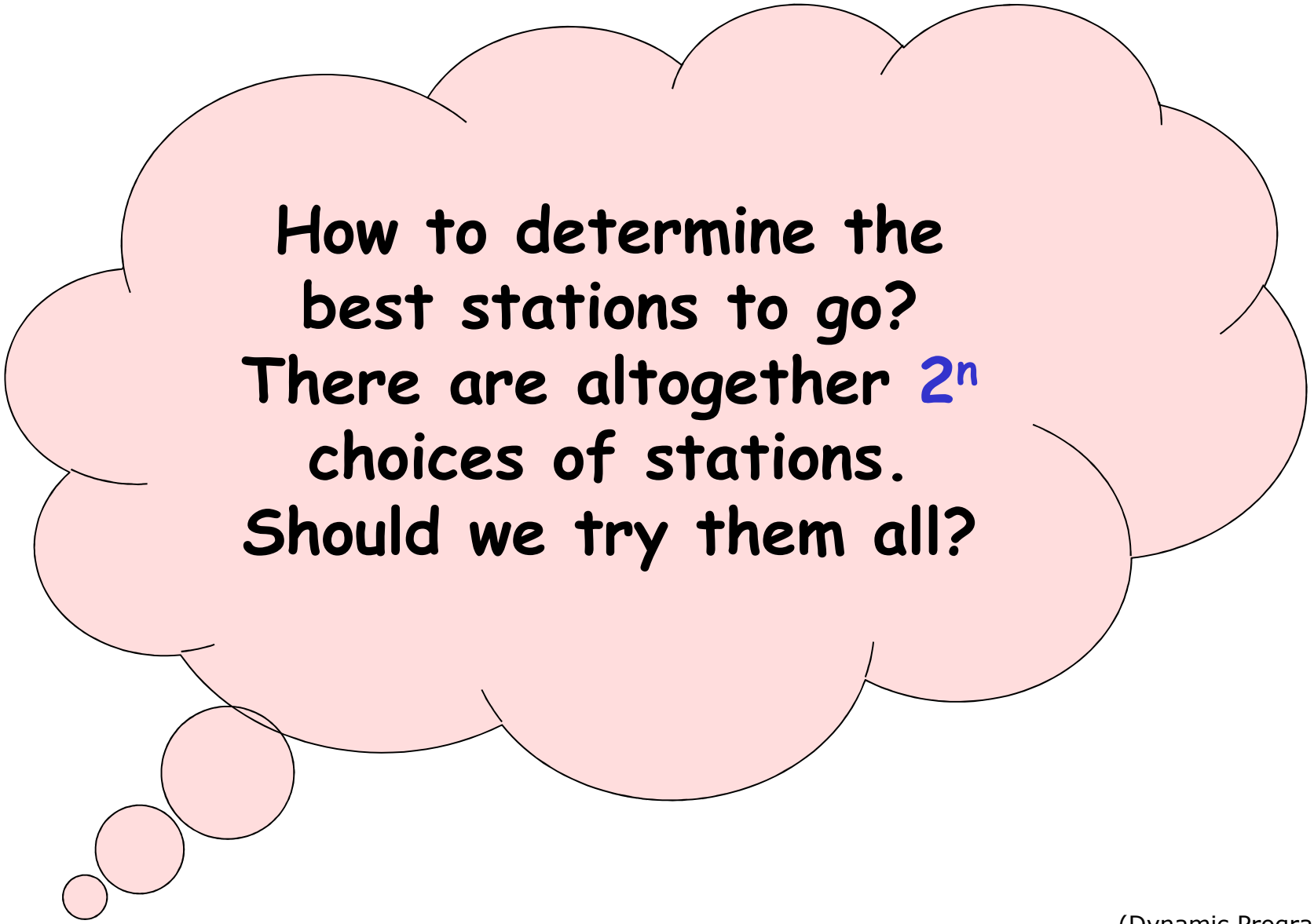


stations chosen: $S_{1,1}$ $S_{1,2}$ $S_{2,3}$ $S_{2,4}$

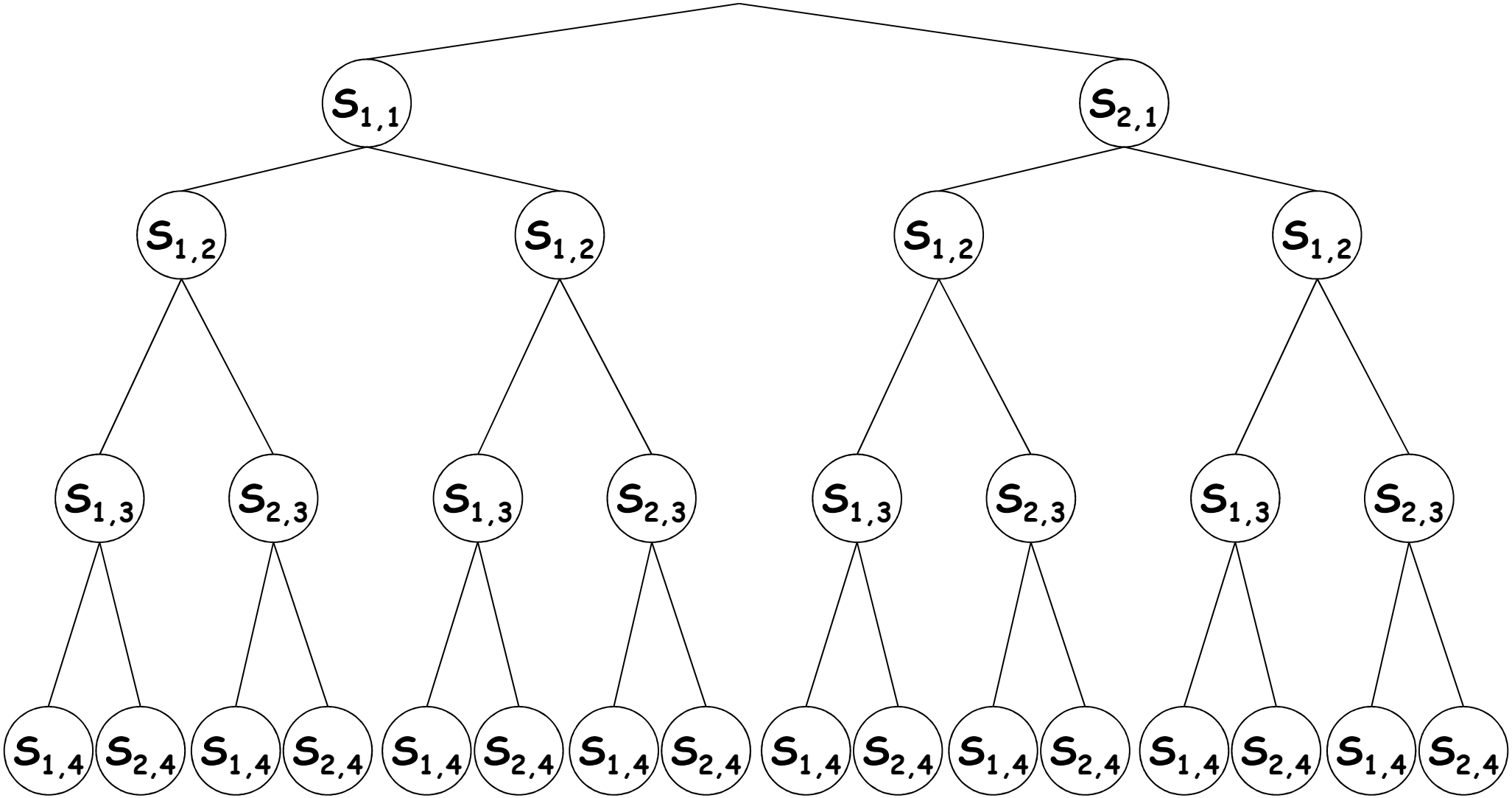
time required: 5 5 4 3 7 = 24

stations chosen: $S_{2,1}$ $S_{1,2}$ $S_{2,3}$ $S_{1,4}$

time required: 15 1 5 4 3 2 4 = 34

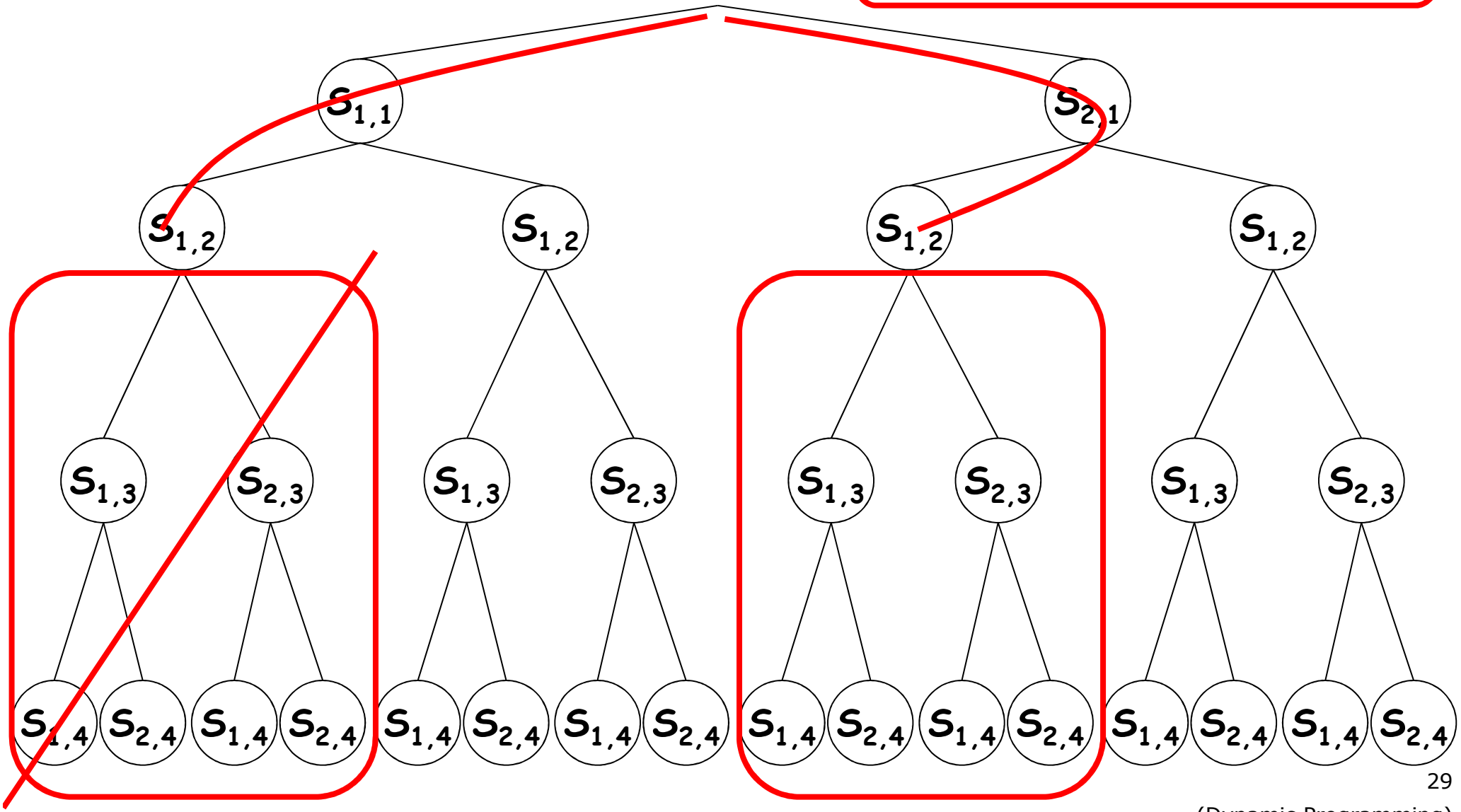

How to determine the best stations to go?
There are altogether 2^n choices of stations.
Should we try them all?

All possible choices



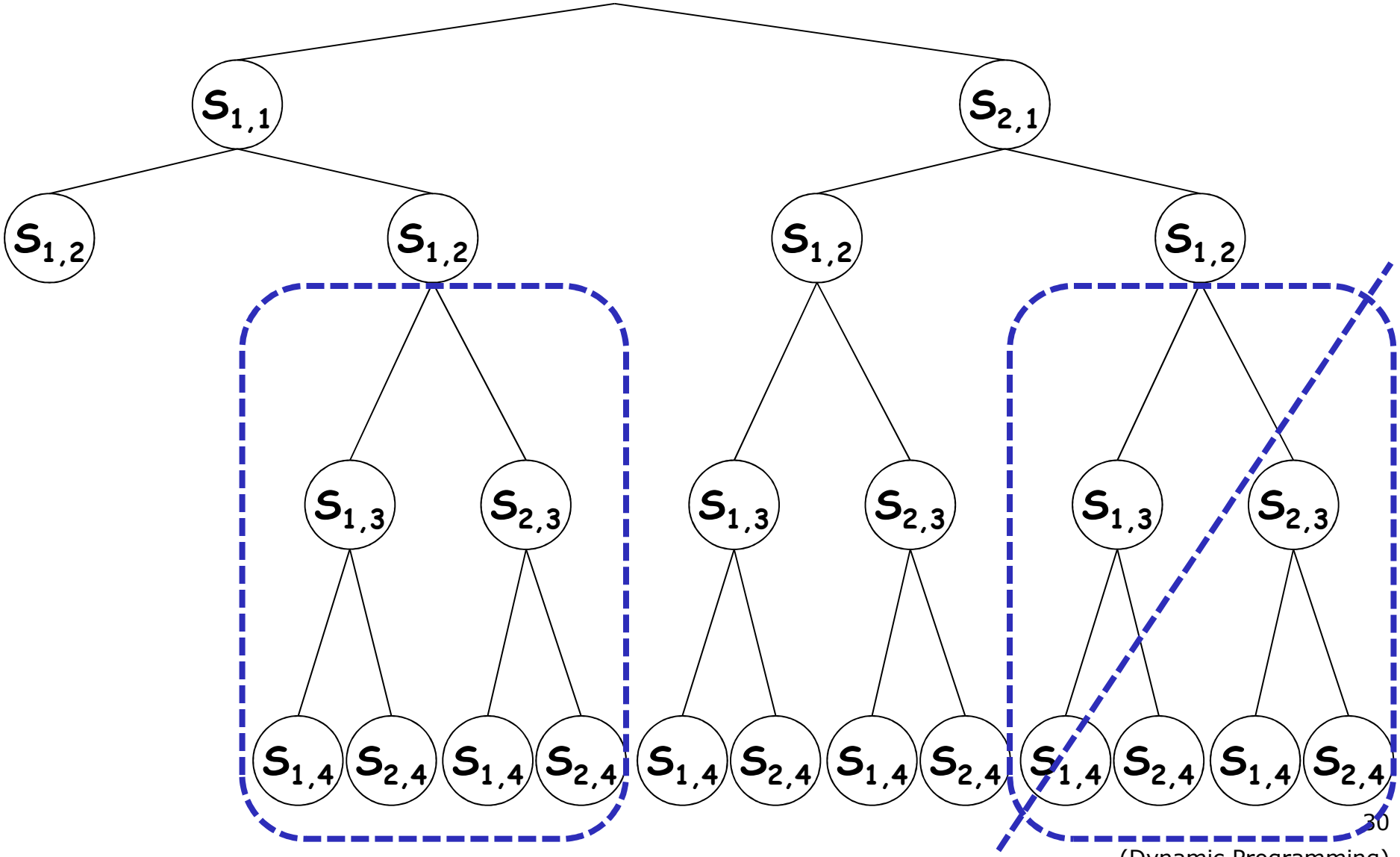
All possible choices

The two subtrees cost the same, only one path is needed.

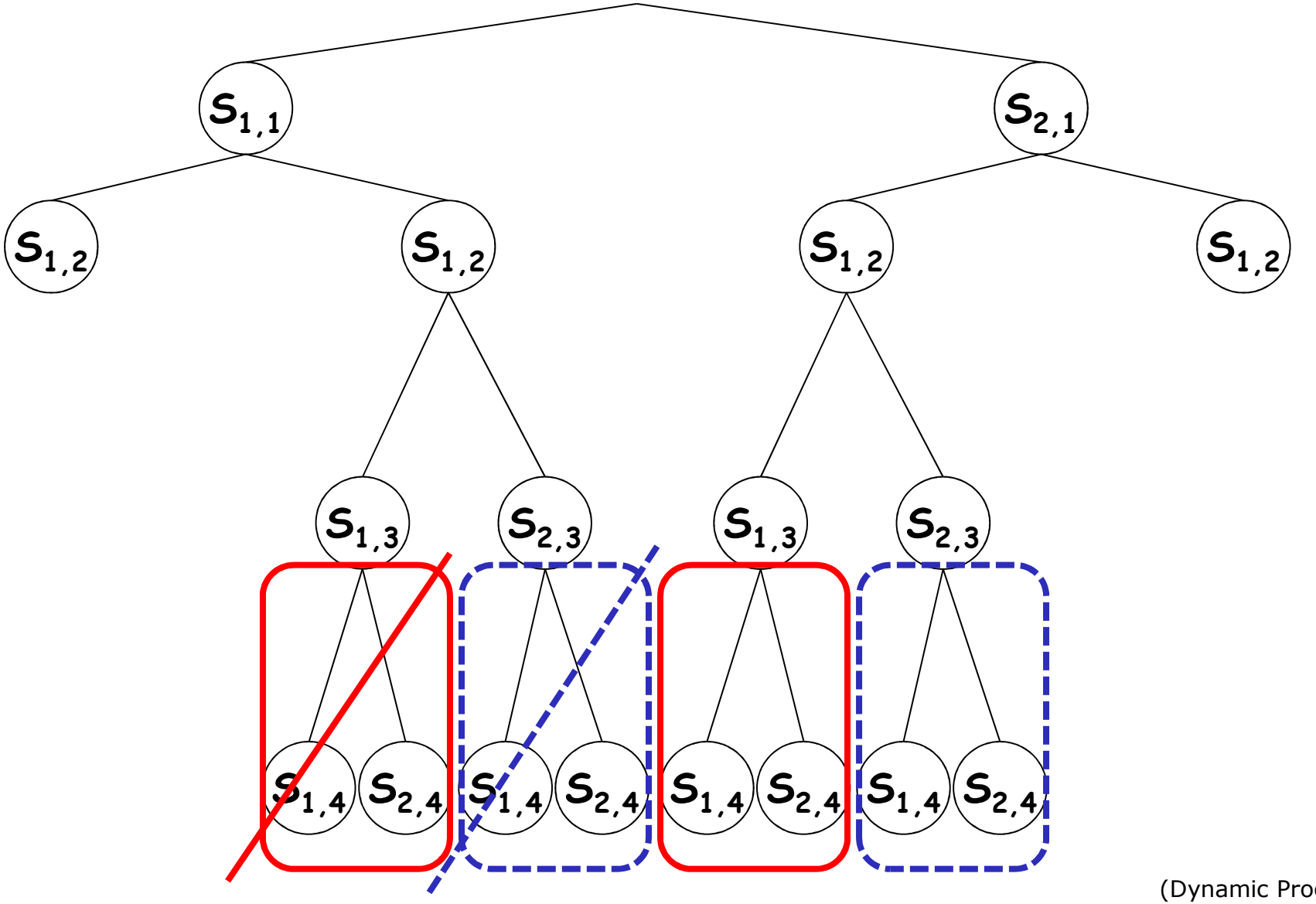


All possible choices

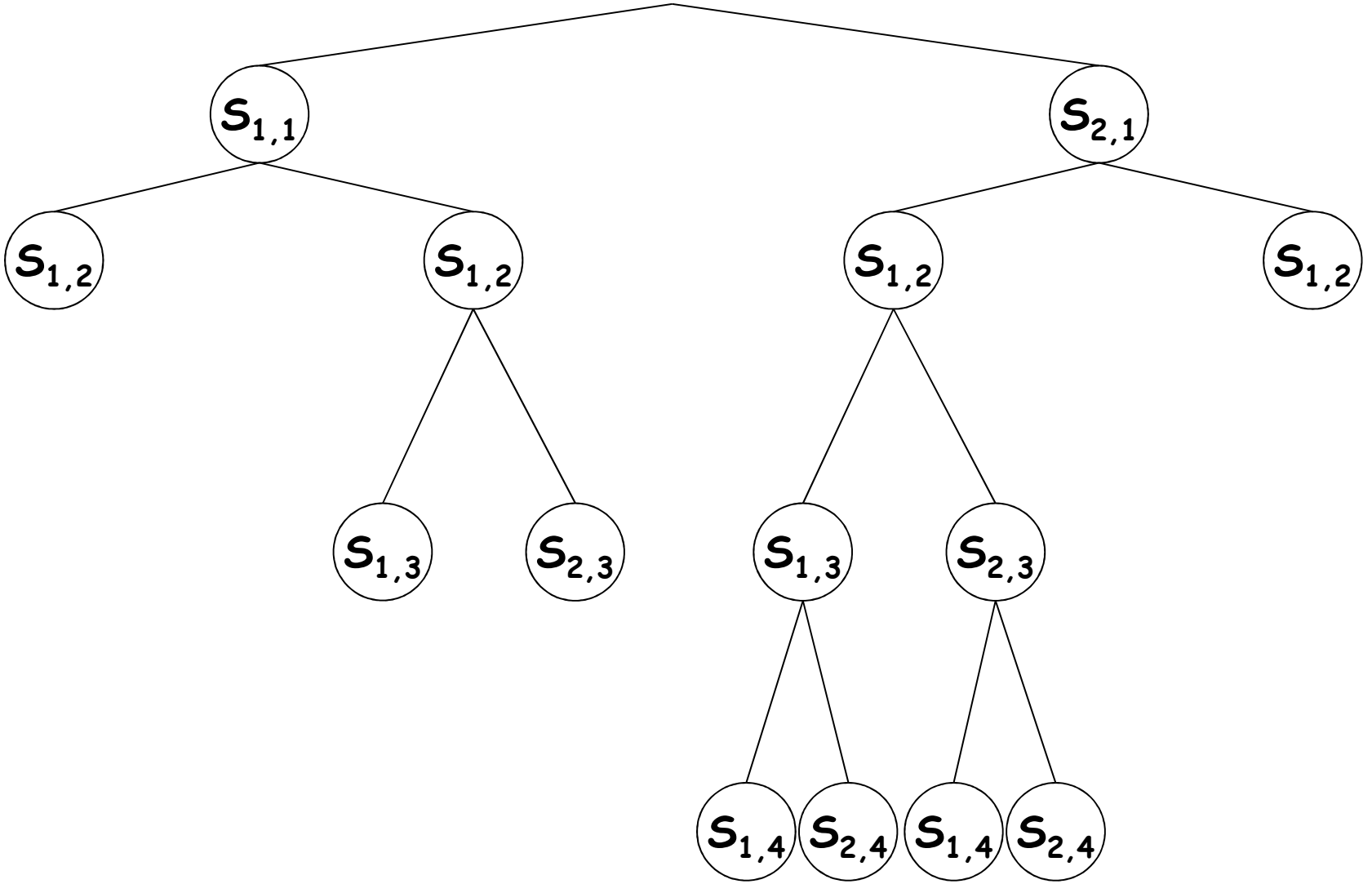
Similarly, …



All possible choices



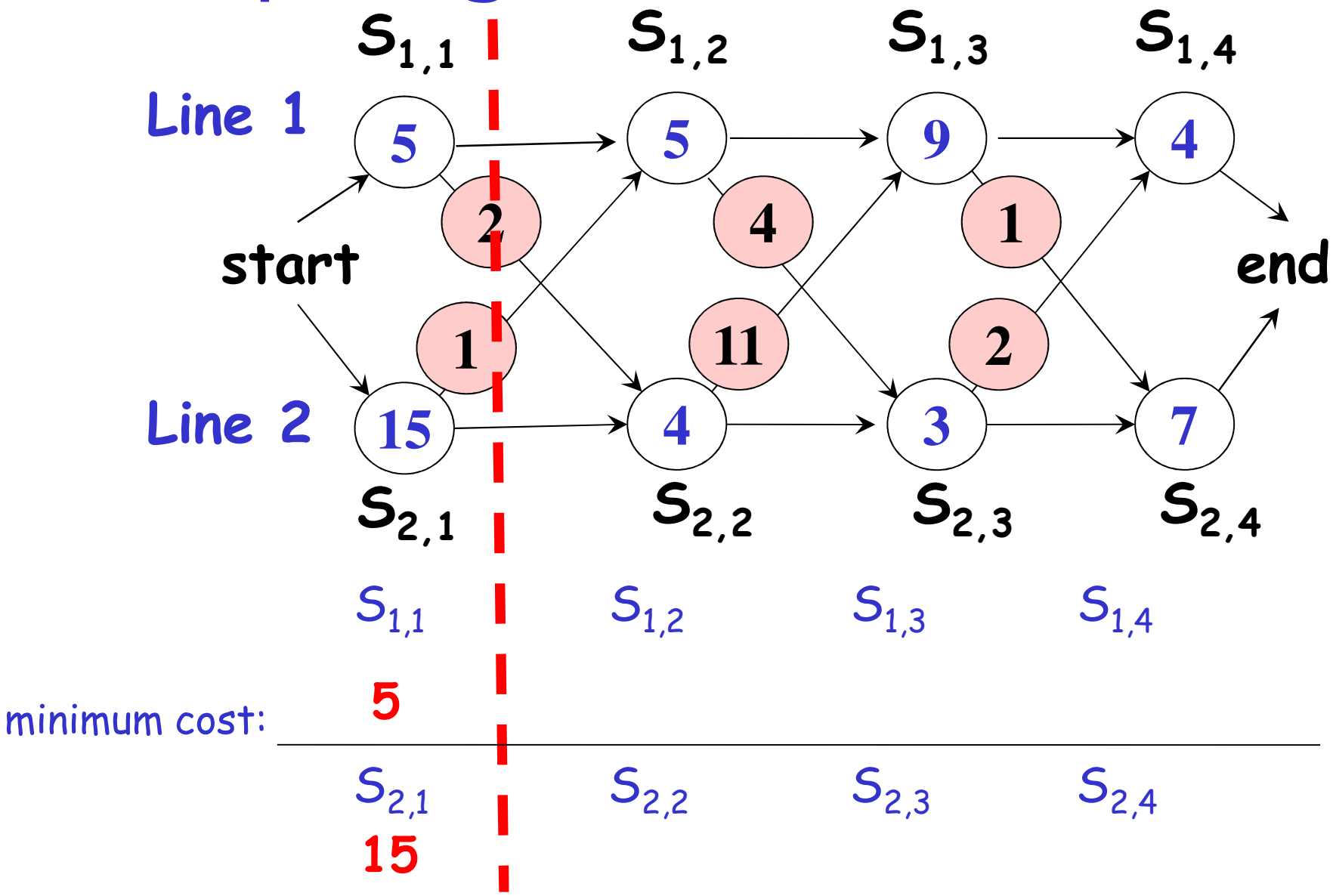
All possible choices



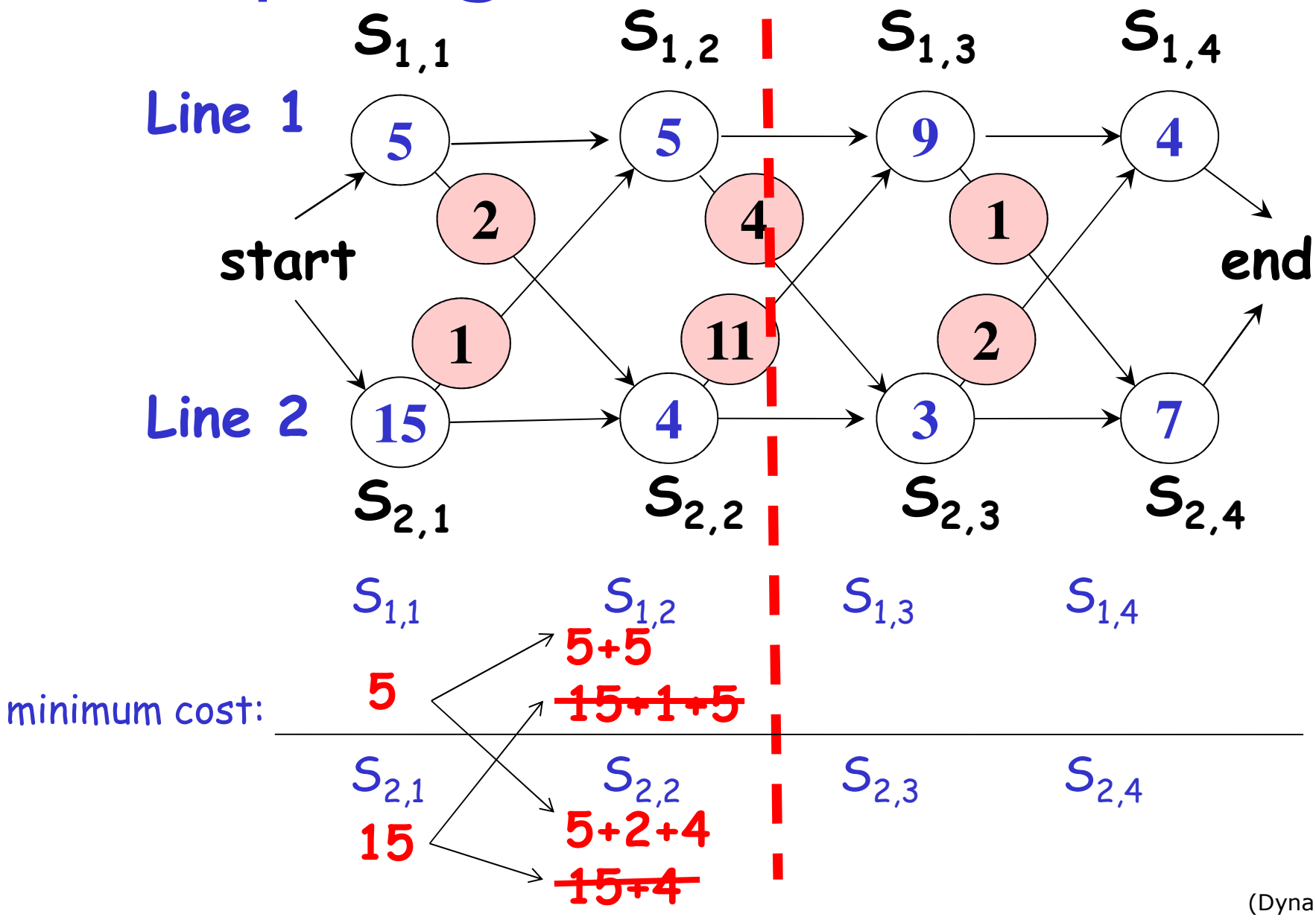
Good news: Dynamic Programming

- We **don't** need to try all possible choices.
- We can make use of **dynamic programming**:
  1. If we can compute the fastest ways to get thro' station $S_{1,n}$ and $S_{2,n}$, then the faster of these two ways is the overall fastest way.
  2. To compute the fastest ways to get thro' $S_{1,n}$ (similarly for $S_{2,n}$), we need to know the fastest way to get thro' $S_{1,n-1}$ and $S_{2,n-1}$
  3. In general, we want to know the fastest way to get thro' $S_{1,j}$ and $S_{2,j}$, for all j.

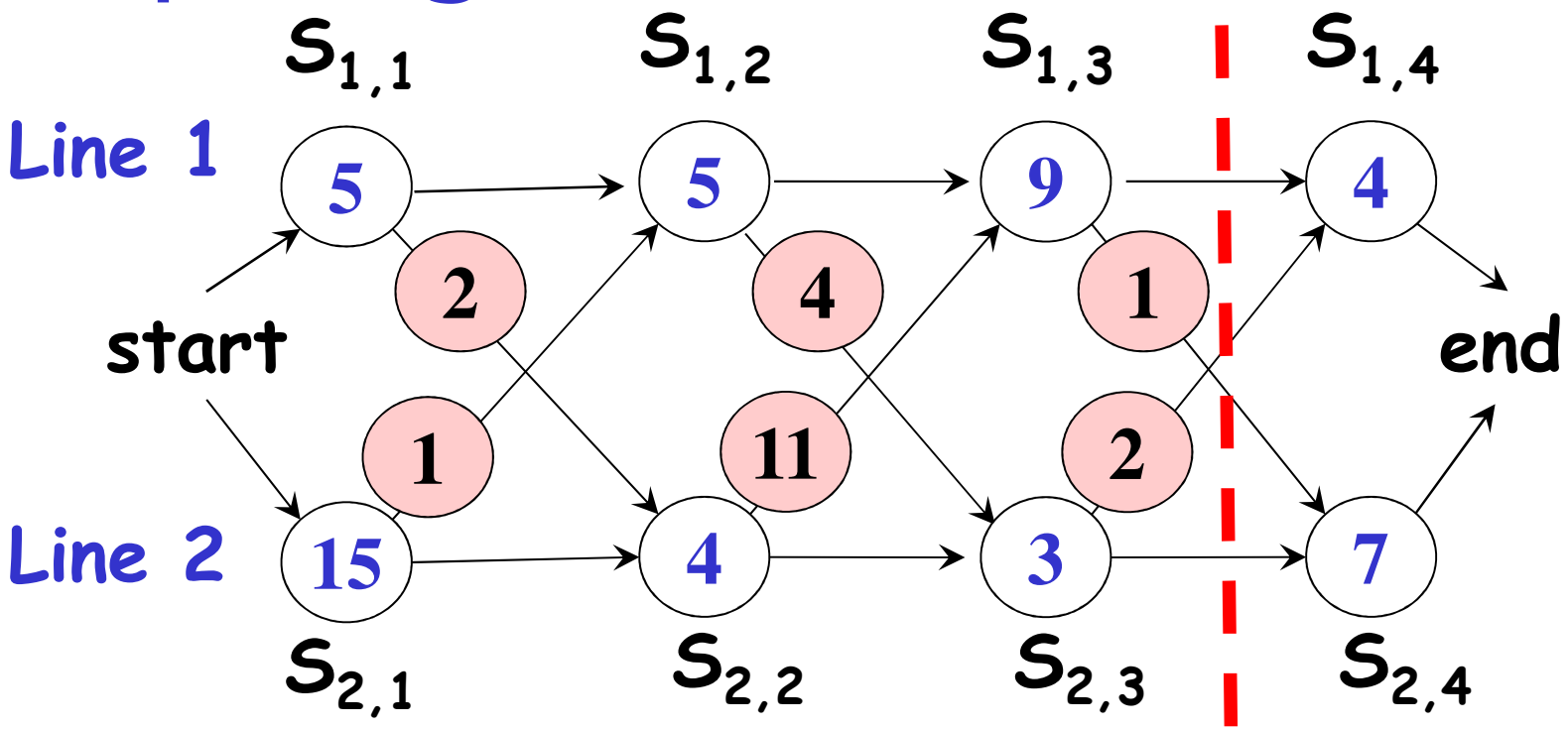
Example again



Example again



Example again

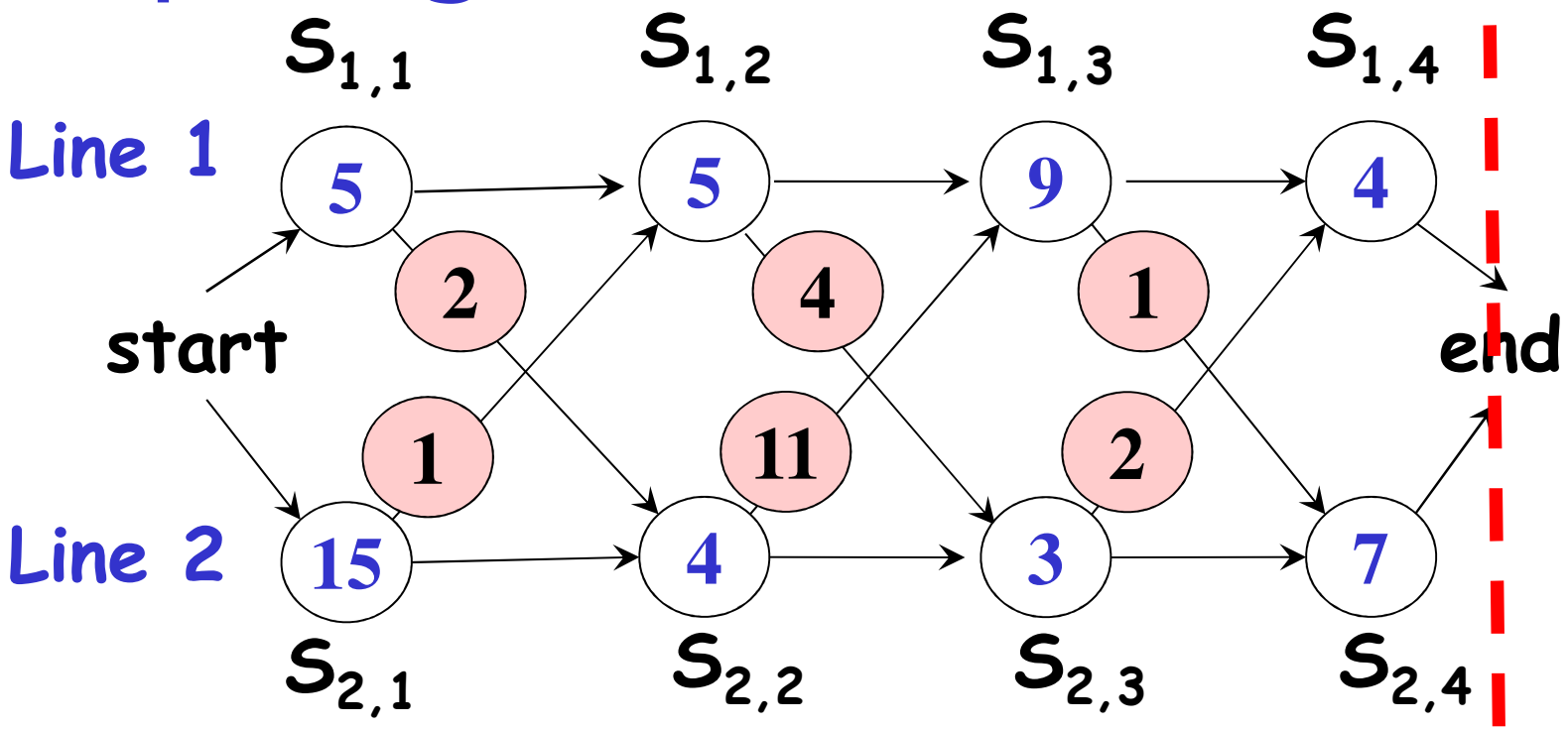


minimum cost:

$S_{1,1}$	$S_{1,2}$	$S_{1,3}$	$S_{1,4}$
5	5+5=10	10+9	
	~~15+1+5~~	~~11+11+9~~	

$S_{2,1}$	$S_{2,2}$	$S_{2,3}$	$S_{2,4}$
15	5+2+4=11	~~10+4+3~~	
	~~15+4~~	11+3	

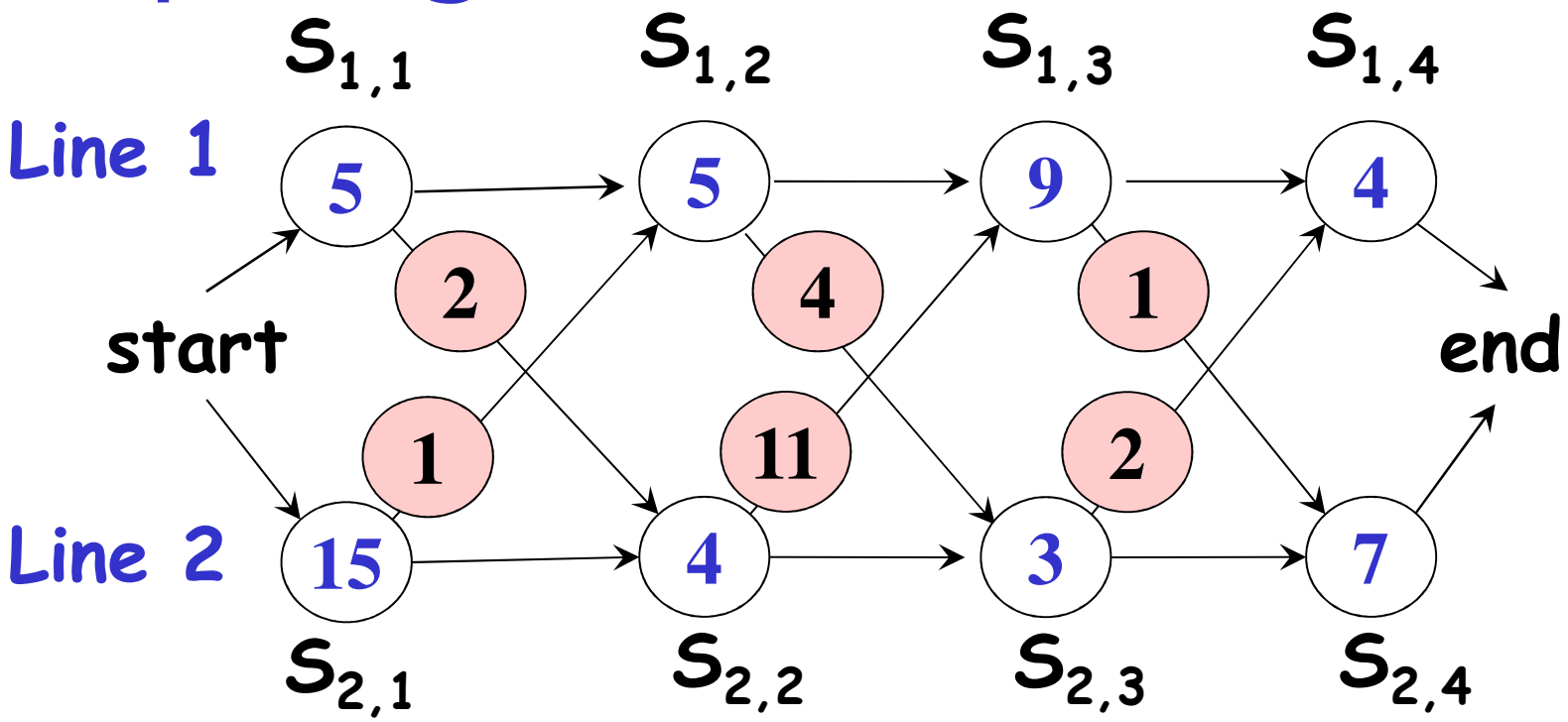
Example again



minimum cost:

$S_{1,1}$	$S_{1,2}$	$S_{1,3}$	$S_{1,4}$	
	5+5=10	10+9=19	~~19+4~~	
5	~~15+1+5~~	~~11+11+9~~	14+2+4	20
$S_{2,1}$	$S_{2,2}$	$S_{2,3}$	$S_{2,4}$	
	5+2+4=11	~~10+4+3~~	~~19+1+7~~	~~21~~
15	~~15+4~~	11+3=14	14+7	

Example again



minimum cost:

$S_{1,1}$	$S_{1,2}$	$S_{1,3}$	$S_{1,4}$	
5	5+5=10 ~~15+1+5~~	10+9=19 ~~11+11+9~~	~~19+4~~ 14+2+4	20
$S_{2,1}$	$S_{2,2}$	$S_{2,3}$	$S_{2,4}$	
15	5+2+4=11 ~~15+4~~	~~10+4+3~~ 11+3=14	~~19+1+7~~ 14+7	~~21~~

A dynamic programming solution

What are the sub-problems?

- given j, what is the fastest way to get thro' $S_{1,j}$
- given j, what is the fastest way to get thro' $S_{2,j}$

Definitions:

- **$f_1[j]$** = the fastest time to get thro' $S_{1,j}$
- **$f_2[j]$** = the fastest time to get thro' $S_{2,j}$

The final solution equals to **min { $f_1[n]$, $f_2[n]$ }**

Task:

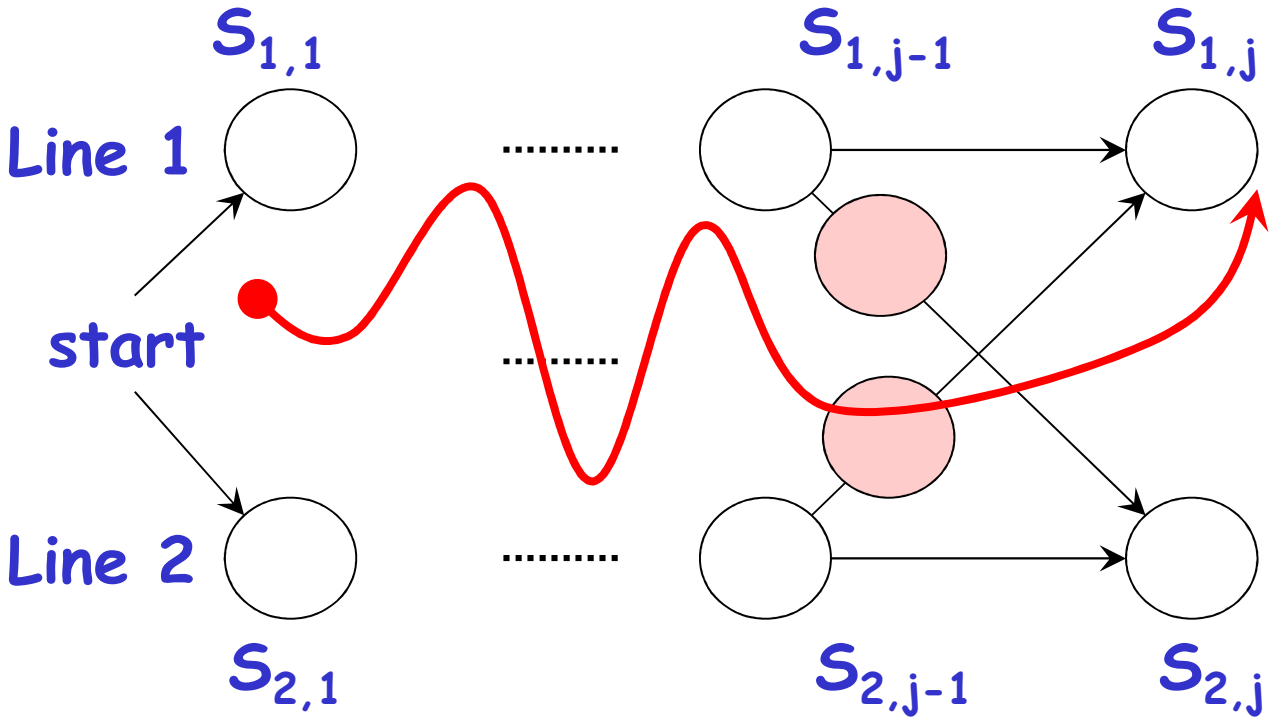
- Starting from $f_1[1]$ and $f_2[1]$,
  compute $f_1[j]$ and $f_2[j]$ incrementally

Solving the sub-problems (1)

Q1: what is the fastest way to get thro' $S_{1,j}$?

A: either

- the fastest way thro' $S_{1,j-1}$, then **directly** to $S_{1,j}$, or
- the fastest way thro' $S_{2,j-1}$, a **transfer** from line 2 to line 1, and then through $S_{1,j}$



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A: either

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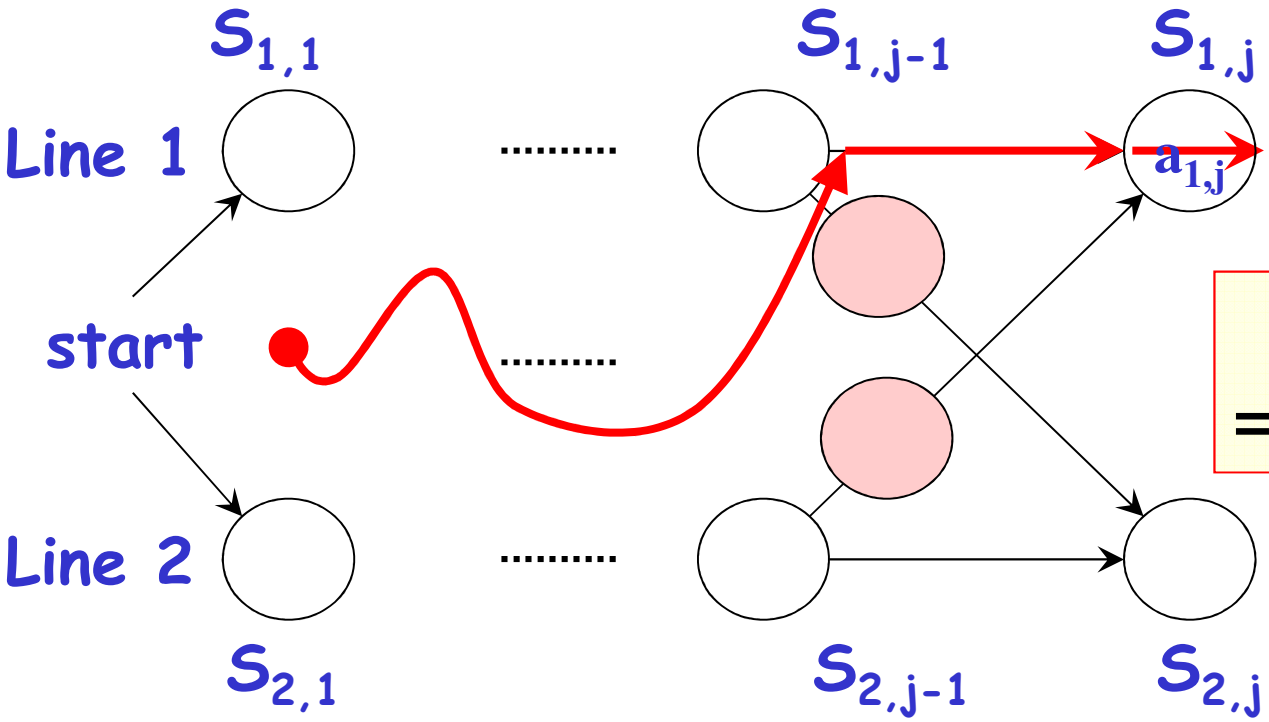
- the fastest way thro' $S_{2,j-1}$, a transfer from line 2 to line 1, and then through $S_{1,j}$



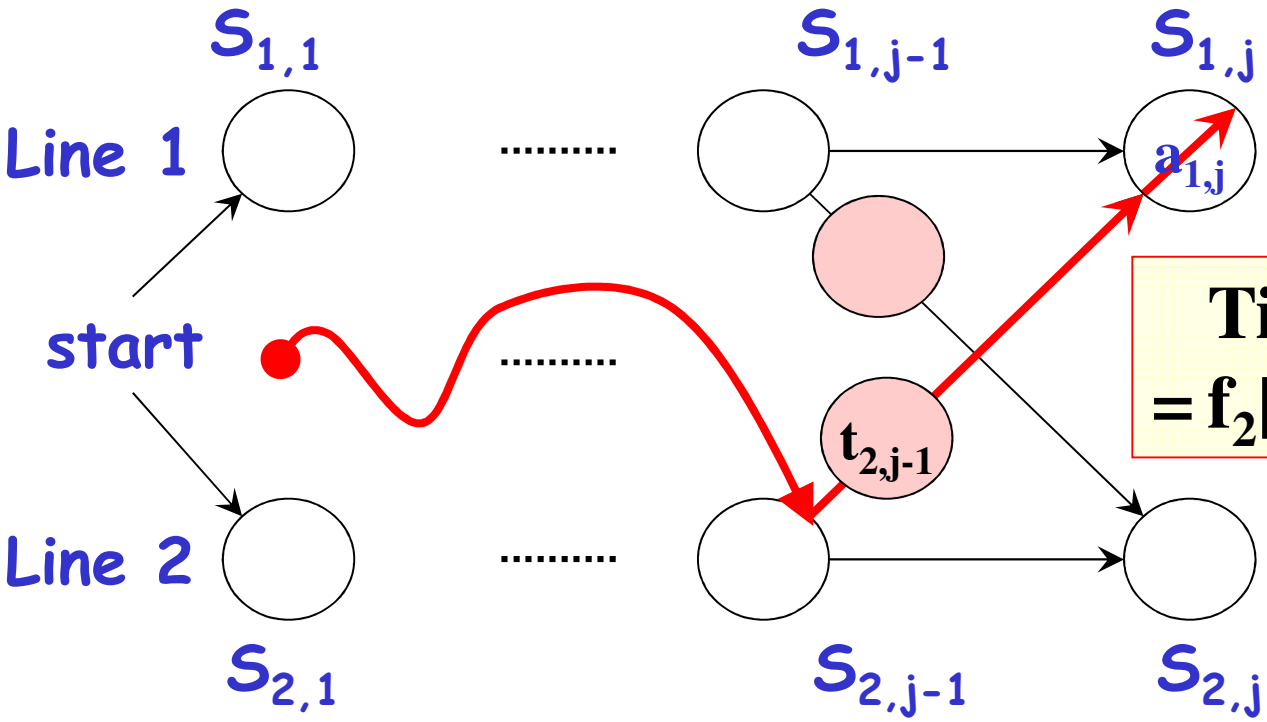
Time required
$= f_1[j-1] + a_{1,j}$

Solving the sub-problems (1)

Q1: what is the fastest way to get thro' $S_{1,j}$?

A: either

- the fastest way thro' $S_{1,j-1}$, then directly to $S_{1,j}$, or
- *the fastest way thro' $S_{2,j-1}$, a transfer from line 2 to line 1, and then through $S_{1,j}$*



Time required
$= f_2[j-1] + t_{2,j-1} + a_{1,j}$

Solving the sub-problems (1)

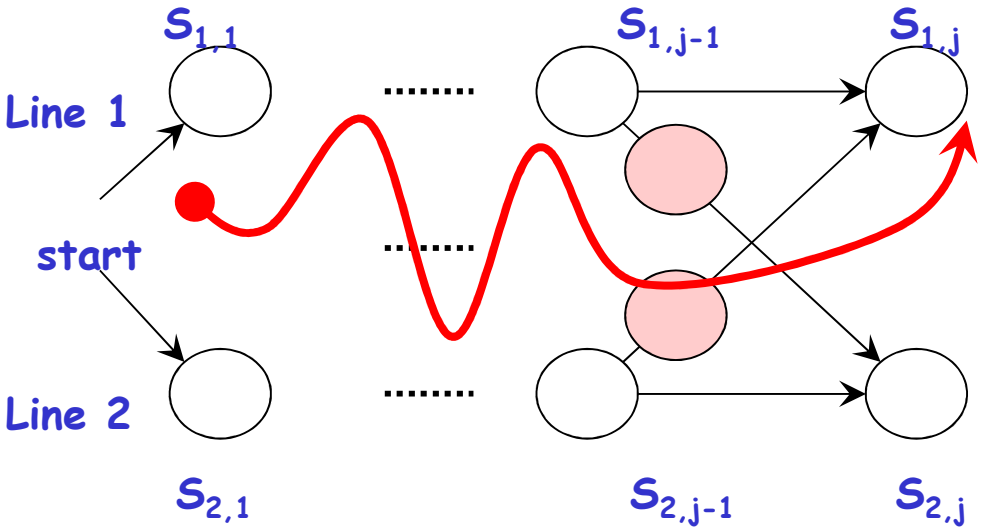
Q1: what is the fastest way to get thro' $S_{1,j}$?

A: either

- the fastest way thro' $S_{1,j-1}$, then directly to $S_{1,j}$, or
- the fastest way thro' $S_{2,j-1}$, a transfer from line 2 to line 1, and then through $S_{1,j}$

Conclusion: $f_1[j] = \min(f_1[j-1]+a_{1,j} , f_2[j-1]+t_{2,j-1}+a_{1,j})$

Boundary case: $f_1[1] = a_{1,1}$



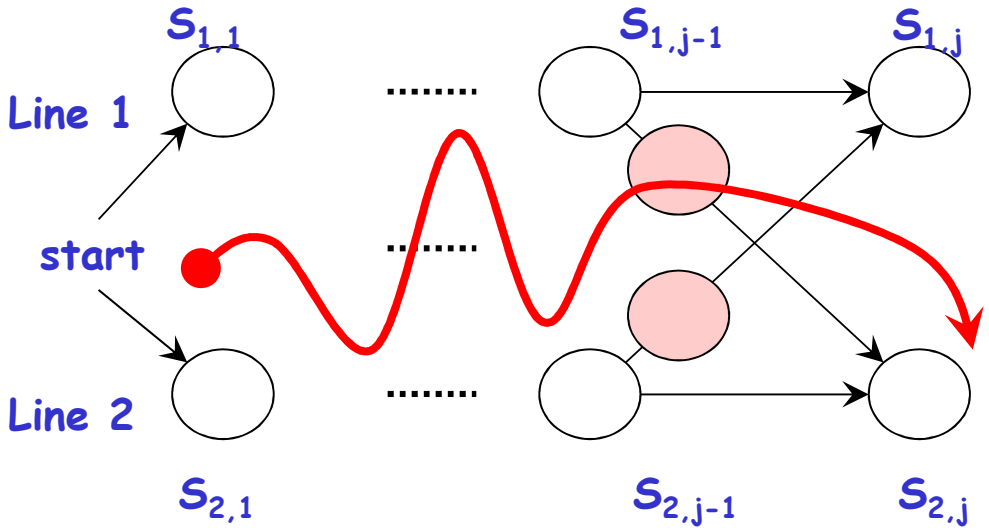
Solving the sub-problems (2)

Q2: what is the fastest way to get thro' $S_{2,j}$?

By exactly the same analysis, we obtain the formula for the fastest way to get thro' $S_{2,j}$:

$$f_2[j] = \min(f_2[j-1]+a_{2,j} \ , \ f_1[j-1]+t_{1,j-1}+a_{2,j})$$

Boundary case: $f_2[1] = a_{2,1}$

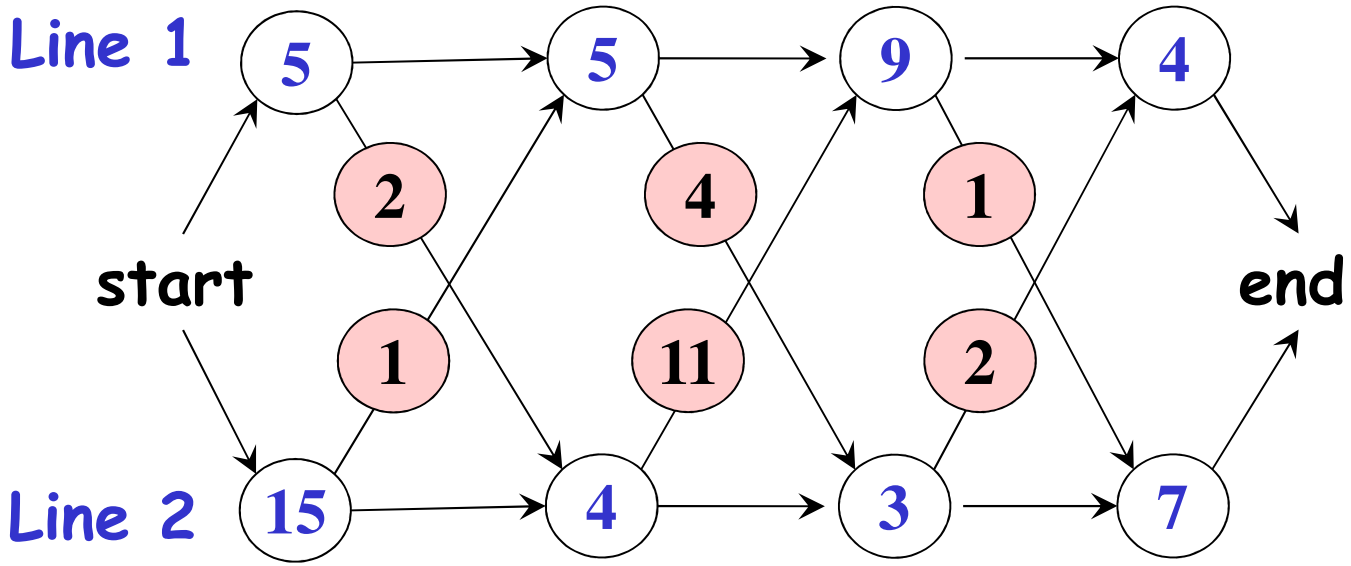


Summary

$$f_1[j] = \begin{cases} a_{1,1} & \text{if } j=1, \\ \min\left(f_1[j-1]+a_{1,j} \ , \ f_2[j-1]+t_{2,j-1}+a_{1,j} \right) & \text{if } j>1 \end{cases}$$

$$f_2[j] = \begin{cases} a_{2,1} & \text{if } j=1, \\ \min\left(f_2[j-1]+a_{2,j} \ , \ f_1[j-1]+t_{1,j-1}+a_{2,j} \right) & \text{if } j>1 \end{cases}$$

$$f^* = \min(f_1[n] , f_2[n])$$



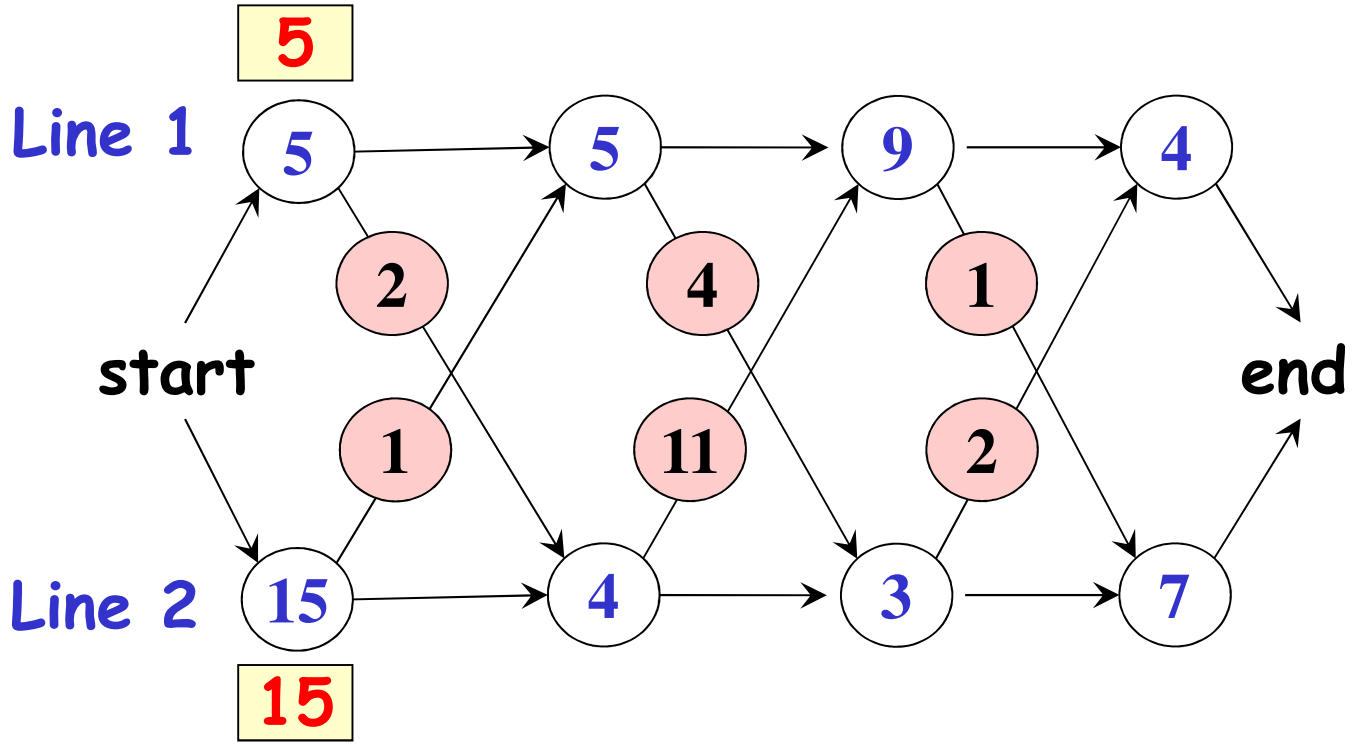
j	$f_1[j]$	$f_2[j]$
1		
2		
3		
4		

Summary

$$f_1[j] = \begin{cases} a_{1,1} & \text{if } j=1, \\ \min\left(f_1[j-1]+a_{1,j} \ , \ f_2[j-1]+t_{2,j-1}+a_{1,j} \right) & \text{if } j>1 \end{cases}$$

$$f_2[j] = \begin{cases} a_{2,1} & \text{if } j=1, \\ \min\left(f_2[j-1]+a_{2,j} \ , \ f_1[j-1]+t_{1,j-1}+a_{2,j} \right) & \text{if } j>1 \end{cases}$$

$$f^* = \min(f_1[n] , f_2[n])$$



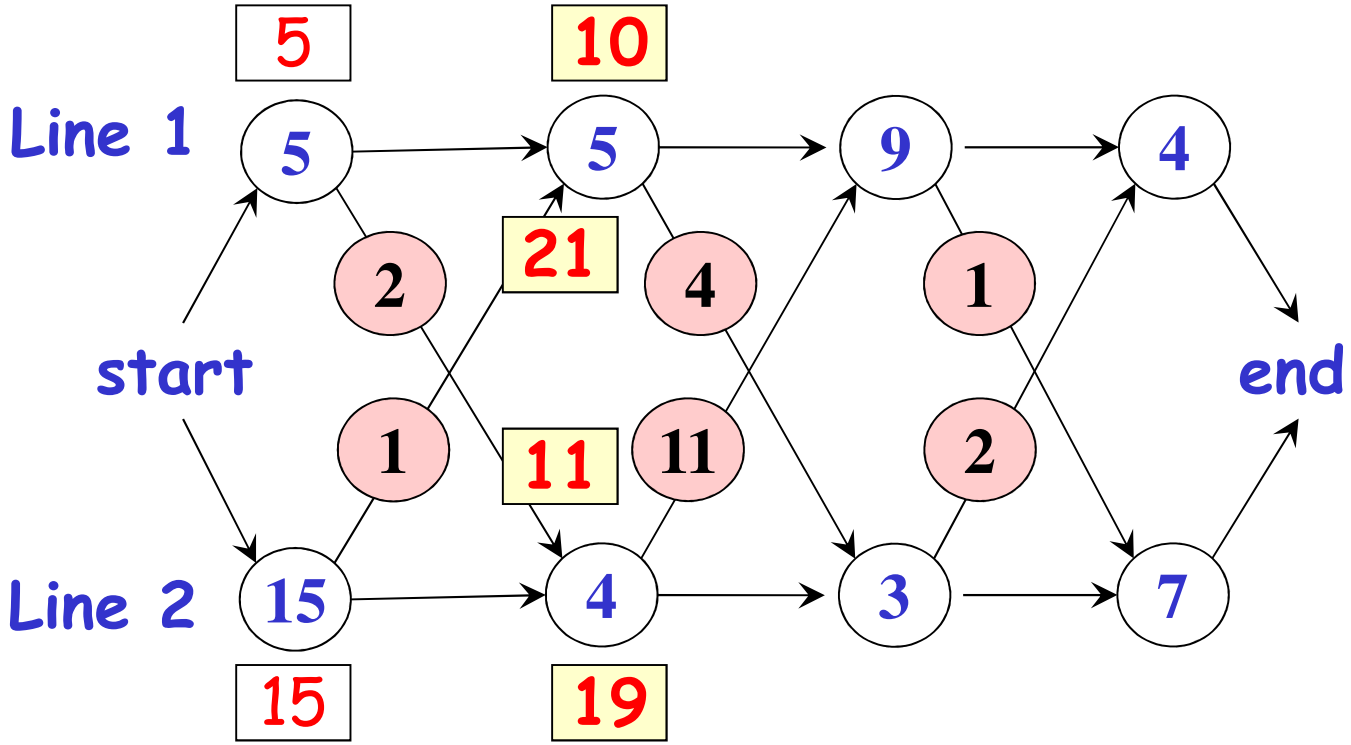
j	$f_1[j]$	$f_2[j]$
1	5	15
2		
3		
4		

Summary

$$f_1[j] = \begin{cases} a_{1,1} & \text{if } j=1, \\ \min\left(f_1[j-1]+a_{1,j} \ , \ f_2[j-1]+t_{2,j-1}+a_{1,j} \right) & \text{if } j>1 \end{cases}$$

$$f_2[j] = \begin{cases} a_{2,1} & \text{if } j=1, \\ \min\left(f_2[j-1]+a_{2,j} \ , \ f_1[j-1]+t_{1,j-1}+a_{2,j} \right) & \text{if } j>1 \end{cases}$$

$$f^* = \min(f_1[n] , f_2[n])$$



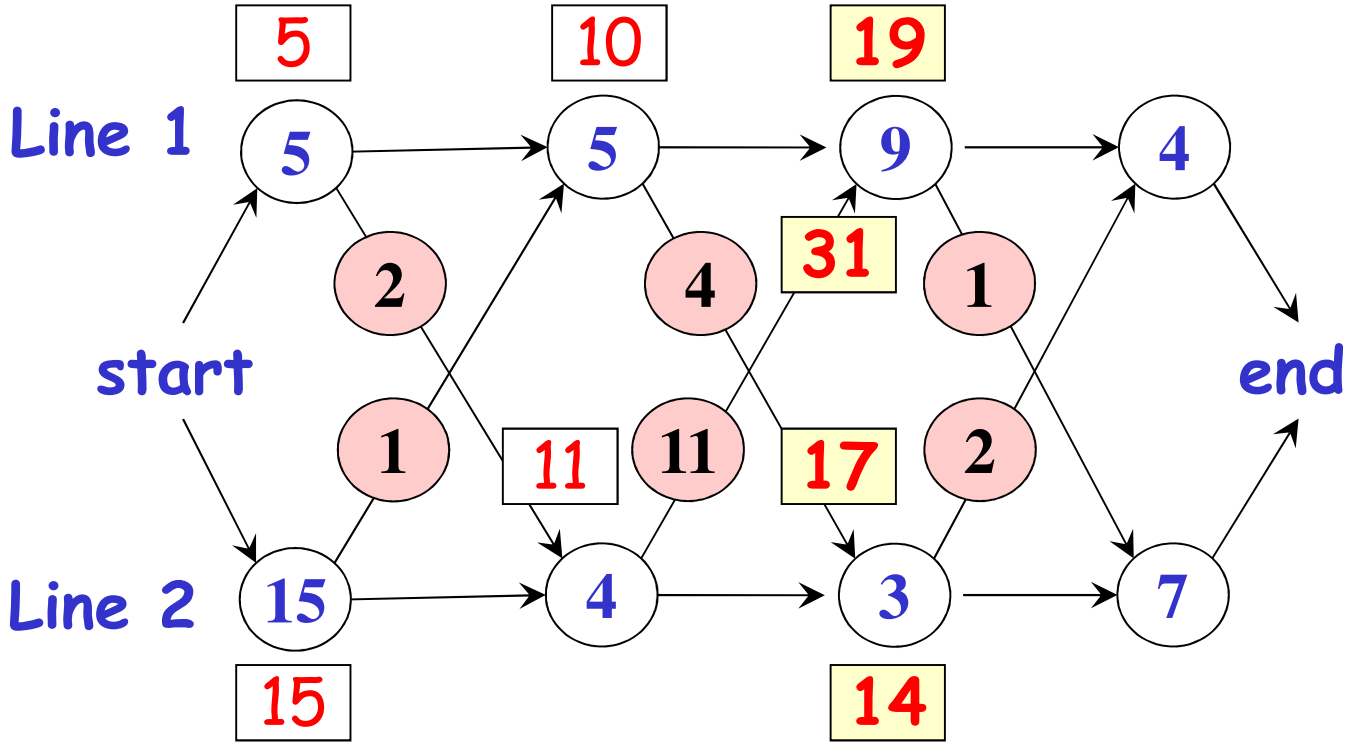
j	$f_1[j]$	$f_2[j]$
1	5	15
2	10	11
3		
4		

Summary

$$f_1[j] = \begin{cases} a_{1,1} & \text{if } j=1, \\ \min\left(f_1[j-1]+a_{1,j} \ , \ f_2[j-1]+t_{2,j-1}+a_{1,j} \right) & \text{if } j>1 \end{cases}$$

$$f_2[j] = \begin{cases} a_{2,1} & \text{if } j=1, \\ \min\left(f_2[j-1]+a_{2,j} \ , \ f_1[j-1]+t_{1,j-1}+a_{2,j} \right) & \text{if } j>1 \end{cases}$$

$$f^* = \min(f_1[n] , f_2[n])$$



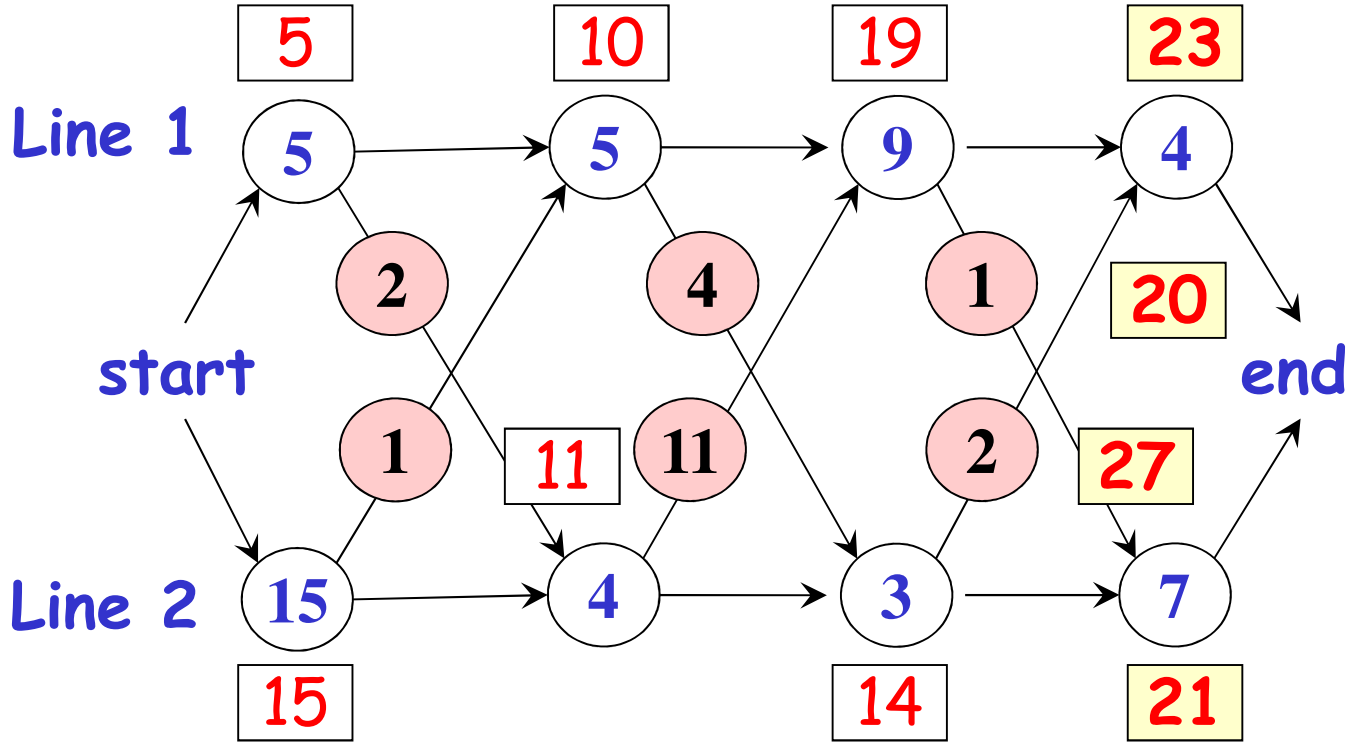
j	$f_1[j]$	$f_2[j]$
1	5	15
2	10	11
3	19	14
4		

Summary

$$f_1[j] = \begin{cases} a_{1,1} & \text{if } j=1, \\ \min\left(f_1[j-1]+a_{1,j} \ , \ f_2[j-1]+t_{2,j-1}+a_{1,j} \right) & \text{if } j>1 \end{cases}$$

$$f_2[j] = \begin{cases} a_{2,1} & \text{if } j=1, \\ \min\left(f_2[j-1]+a_{2,j} \ , \ f_1[j-1]+t_{1,j-1}+a_{2,j} \right) & \text{if } j>1 \end{cases}$$

$$f^* = \min(f_1[n] , f_2[n])$$



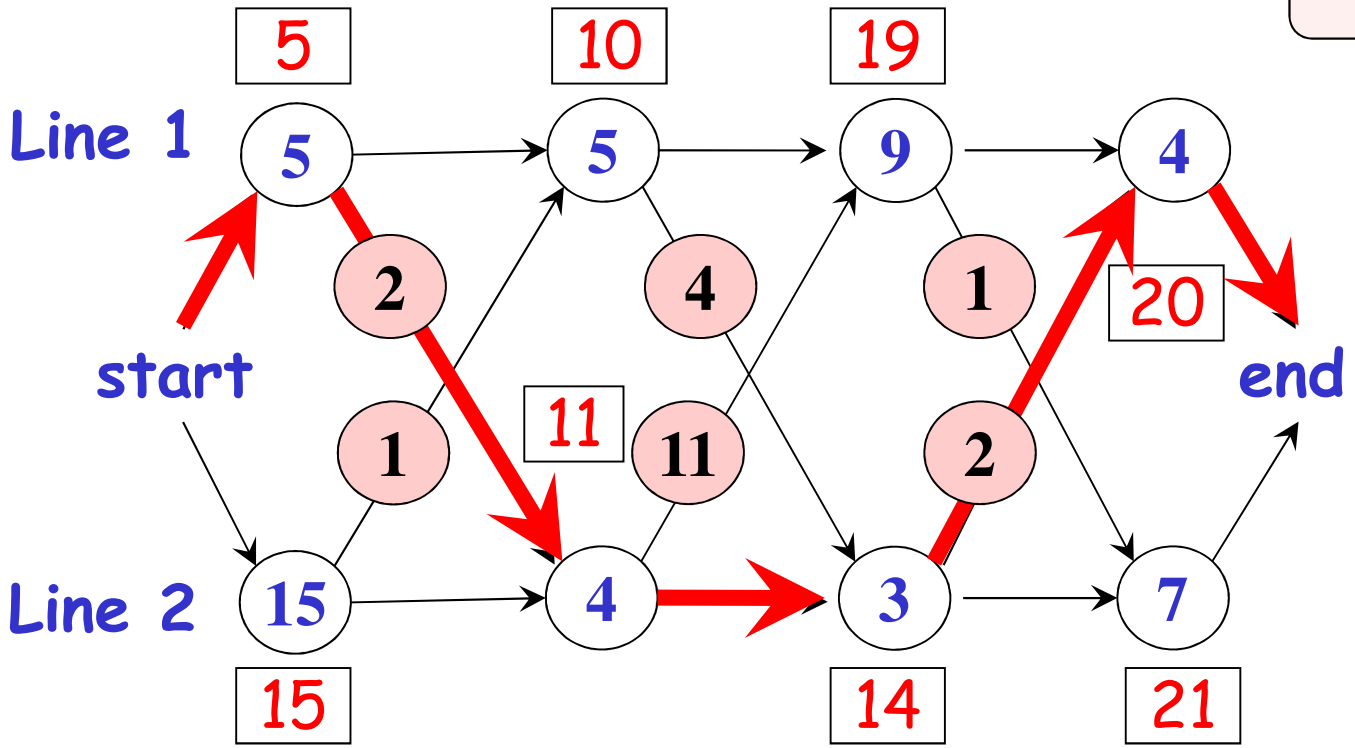
j	$f_1[j]$	$f_2[j]$
1	5	15
2	10	11
3	19	14
4	20	21

Summary

$$f_1[j] = \begin{cases} a_{1,1} & \text{if } j=1, \\ \min\left(f_1[j-1]+a_{1,j} \ , \ f_2[j-1]+t_{2,j-1}+a_{1,j} \right) & \text{if } j>1 \end{cases}$$

$$f_2[j] = \begin{cases} a_{2,1} & \text{if } j=1, \\ \min\left(f_2[j-1]+a_{2,j} \ , \ f_1[j-1]+t_{1,j-1}+a_{2,j} \right) & \text{if } j>1 \end{cases}$$

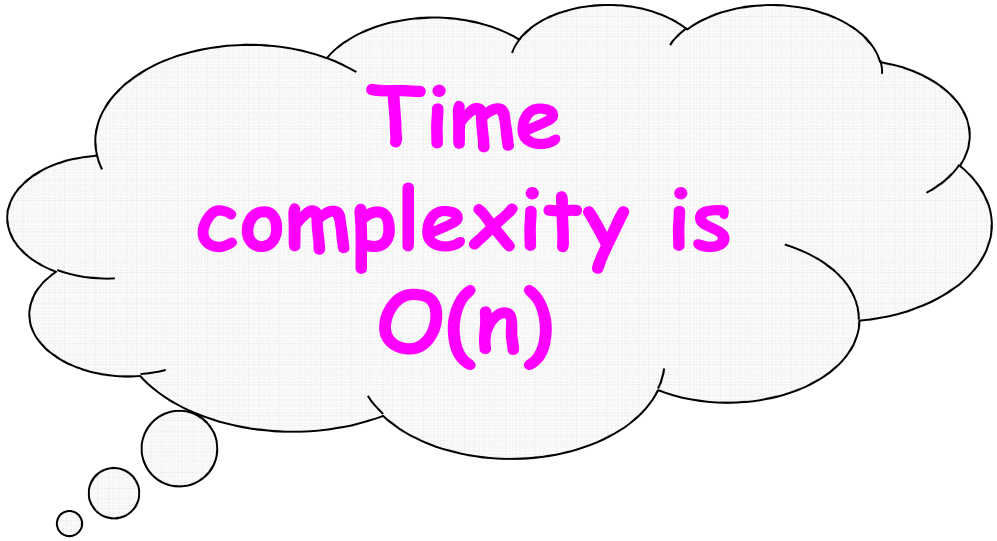
$$f^* = \min(f_1[n] , f_2[n])$$



$f^* = 20$

j	$f_1[j]$	$f_2[j]$
1	5	15
2	10	11
3	19	14
4	20	21

Pseudo code



set $f_1[1] = a_{1,1}$

set $f_2[1] = a_{2,1}$

for j = 2 to n **do**

begin

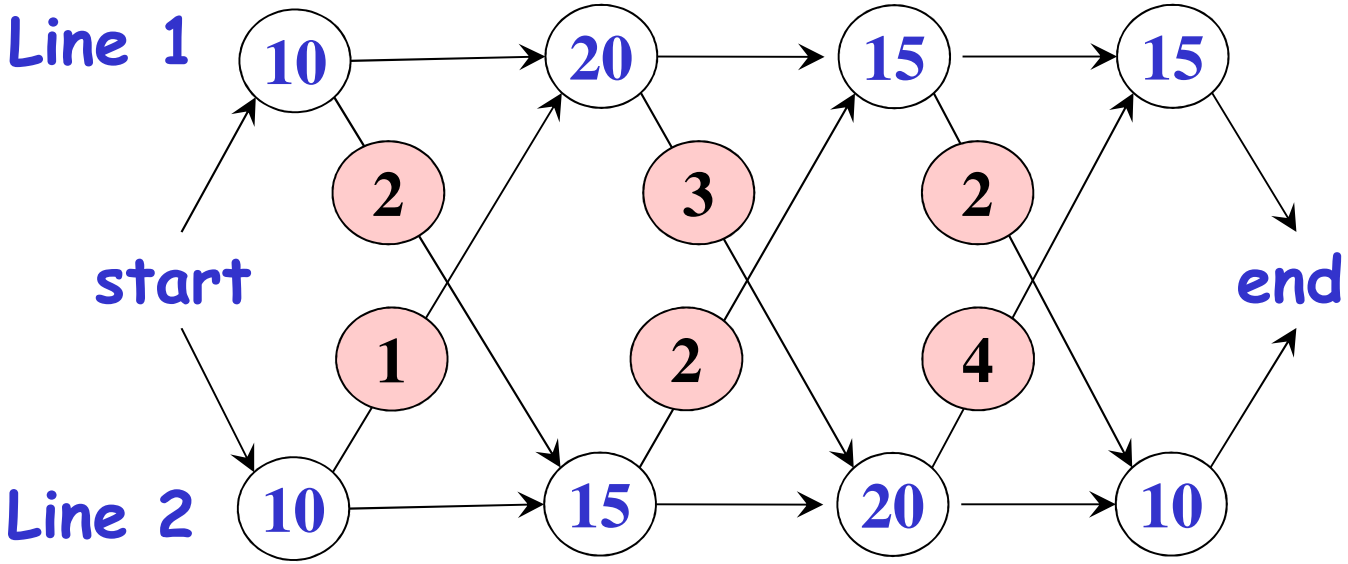
set $f_1[j] = \min (\mathbf{f_1[j-1]+a_{1,j}} \ , \ \mathbf{f_2[j-1]+t_{2,j-1}+a_{1,j}})$

set $f_2[j] = \min (\mathbf{f_2[j-1]+a_{2,j}} \ , \ \mathbf{f_1[j-1]+t_{1,j-1}+a_{2,j}})$

end

set $f^* = \min (f_1[\mathbf{n}] , f_2[\mathbf{n}])$

Exercise



j	$f_1[j]$	$f_2[j]$
1		
2		
3		
4		