15

Hedonic Games

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15.1 Introduction

Coalitions are a central part of economic, political, and social life, and coalition formation has been studied extensively within the mathematical social sciences. Agents (be they humans, robots, or software agents) have preferences over coalitions and, based on these preferences, it is natural to ask which coalitions are expected to form, and which coalition structures are better social outcomes. In this chapter, we consider coalition formation games with hedonic preferences, or simply hedonic games. The outcome of a coalition formation game is a partitioning of the agents into disjoint coalitions, which we will refer to synonymously as a partition or coalition structure.

The defining feature of *hedonic preferences* is that every agent only cares about which agents are in its coalition, but does not care how agents in other coalitions are grouped together (Drèze and Greenberg, 1980). Thus, hedonic preferences completely ignore inter-coalitional dependencies. Despite their relative simplicity, hedonic games have been used to model many interesting settings, such as research team formation (Alcalde and Revilla, 2004), scheduling group activities (Darmann et al., 2012), formation of coalition governments (Le Breton et al., 2008), clusterings in social networks (see e.g., Aziz et al., 2014b; McSweeney et al., 2014; Olsen, 2009), and distributed task allocation for wireless agents (Saad et al., 2011).

Before we give a formal definition of a hedonic game, we give a standard hedonic game from the literature that we will use as a running example (see e.g., Banerjee et al. (2001)).

Example 15.1 The hedonic game has three agents, 1, 2, and 3, and their preferences over coalitions are as follows.

- All agents prefer coalitions of size two to coalitions of size one or three.
- All agents prefer coalitions of size one to coalitions of size three.

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- Agent 1 prefers to be with agent 2 than to be with agent 3.
- Agent 2 prefers to be with agent 3 than to be with agent 1.
- Agent 3 prefers to be with agent 1 than to be with agent 2.

A coalition is as non-empty subset of $\{1,2,3\}$. The outcomes of the game are the partitions of the set of all agents into coalitions: $\{\{1,2,3\}\}$, $\{\{1,2\},\{3\}\}$, $\{\{1,3\},\{2\}\}$, $\{\{2,3\},\{1\}\}$ and $\{\{1\},\{2\},\{3\}\}$. Agent 1 is indifferent between the last two outcomes, because its own coalition is the same in both.

We now formally define a hedonic game. The description of a hedonic game can be exponentially large in the number of agents, since for every agent it must describe the preferences of this agent over all possible coalitions to which the agent may belong.

Definition 15.2 Let N be a finite set of agents. A *coalition* is a non-empty subset of N. Let $\mathcal{N}_i = \{S \subseteq N : \ i \in S\}$ be the set of all coalitions (subsets of N) that include agent $i \in N$. A *coalition structure* is a partition π of agents N into disjoint coalitions. A *hedonic coalition formation game* is a pair $(N, \succsim)$, where $\succsim$ is a *preference profile* that specifies for every agent $i \in N$ a reflexive, complete, and transitive binary relation $\succsim_i$ on $\mathcal{N}_i$. We call $\succsim_i$ a *preference relation.*

Given a coalition structure, a deviation by a single agent is a move by this agent to leave its current coalition and join a different (possibly empty) coalition. If this different coalition is empty, by deviating, the agent has broken away from a non-singleton coalition to go alone. We say that an agent has an *incentive to deviate* if there exists a deviation of this agent to a new coalition that it prefers to its old coalition; we call such a deviation a *profitable deviation.* Sometimes, we restrict which deviations are allowed, for example, based on what effect the deviation has on coalitions that lose members via the deviation.

A coalition structure is *stable* with respect to a class of allowable single-agent deviations if no agent has a profitable allowable deviation (Bogomolnaia and Jackson, 2002). Stability is the main criterion that has been used to analyze which coalition structures will form. Later in this chapter, in addition to single-agent deviations, we will also consider deviations by groups of agents.

Before we formally define notions of stability in Section 15.2, we discuss stability under deviations by single agents in the context of Example 15.1. In this example, no coalition structure is stable under single-agent deviations, abbreviated in the subsequent argument to simply "not stable". Suppose that all three agents are together in a single coalition. All three agents have an incentive to deviate and "go alone", so this coalition structure is not stable. Now consider the coalition structure $\{\{1,2\},\{3\}\}$. This coalition structure is not stable, since agent 2 would prefer to deviate and join agent 3. Likewise, the other two coalition structures that consist of one coalition of size two and one coalition of size one, are also not stable, since exactly one of the players in the coalition of size two wants to join the agent in the singleton coalition. Finally, the coalition structure where all three agents are alone is not stable, since all three agents prefer being with another agent than to being alone. Technically, this shows that the example does not admit a *Nash stable* partition, which is a solution concept that we will define formally in Section 15.2.

In order to achieve some weaker notion of stability, one can consider situations where the types of allowable deviations are more restricted. For example, for every coalition S in a coalition structure, we give every member of S the power to veto the deviation of a member of S who would like to leave S, and to veto a deviation by an agent from another coalition that would like to join S. In our example, every partition of the three agents into two non-empty coalitions is stable under the allowed deviations. For example, the partition $\{\{1\}, \{2, 3\}\}$ is stable because the only agent that has an incentive to deviate is agent 3, and although agent 1 would be happy for agent 3 to join it, agent 2 would not want 3 to leave and would veto this move. Technically, we have shown that the example admits a *contractually individually stable* partition, which is another solution concept that we will define formally in Section 15.2.

A significant amount of research has been undertaken to understand what stability requirements and what restrictions on preferences guarantee that stable partition exists (see e.g., Alcalde and Revilla, 2004; Aziz and Brandl, 2012; Banerjee et al., 2001; Bogomolnaia and Jackson, 2002; Dimitrov et al., 2006; Dimitrov and Sung, 2007; Karakaya, 2011). In this chapter we consider a number of standard stability requirements and restrictions on preferences from a computational viewpoint. We consider questions such as the following ones. What is the most expressive or compact way to represent hedonic games? What is the computational complexity of deciding if a stable partition exists, or finding a stable partition if one is known to exist? Two other surveys on the computational aspects of hedonic games are presented by Hajduková (2006) and Cechlárová (2008).

15.1.1 Relationship with Cooperative Game Theory

Coalitions have always played a central role in *cooperative game theory*. Initially, coalition formation was understood in the context of *transferable utility cooperative games*, which are defined by a valuation function that assigns a value to every coalition, and it is assumed that the value of a coalition can be split between its members in every possible way. A common assumption in transferable utility cooperative games is that the valuation function is super-additive, which means that the union of two disjoint coalitions has a value greater than or equal to sum of the values of the separate coalitions. Under this assumption, the formation of the coalition structure consisting of the grand coalition with all agents together maximizes the total value achieved. In these settings, the question of which coalition structures form is not relevant, and the focus has been on the question of how the values of smaller coalitions should determine the division of the value of the grand coalition among its members (Curiel, 1997; Chalkiadakis et al., 2011; Deng and Fang, 2008; Elkind et al., 2013; Peleg and Sudhölter, 2007). In case a transferable utility cooperative game is *not* super-additive, then a natural approach that has been considered in the literature is to first compute a coalition structure that maximises social welfare, i.e., the sum of values of coalitions (Aziz and de Keijzer, 2011; Michalak et al., 2010; Rahwan et al., 2009b,a; Sandholm et al., 1999).

Hedonic games are a sub-class of *non-transferable utility cooperative games.* A non-transferable utility game describes the possibilities open to a coalition as a set of outcomes, where each outcome specified the payoff to each agent in the coalition. These possible outcomes can be thought of as different ways for the coalition to organise itself, which in turn can result in different utilities for the members of the coalition. An outcome is represented as a payoff vector that assigns a payoff to every member of the coalition. In a hedonic game, *the set of payoff vectors for every possible coalition that may form can be viewed as being a singleton.* For more details on cooperative games and in particular non-transferable utility cooperatives games and the eventual formalisation of hedonic games, we refer to the introductory section of the survey of Hajduková (2006).

Hedonic games are a strict generalization of various well-studied matching models such as the marriage market (Gale and Shapley, 1962), roommate market, and many-to-one market (see, e.g., Roth and Sotomayor, 1990). These are all models with hedonic preferences but where not all partitions are permitted. We refer the reader to Chapter 14 (Klaus et al., 2015), which is on Matching under Preferences.

15.1.2 Roadmap

The rest of the chapter is organised as follows. In Section 15.2, we present the standard solution concepts used for hedonic games. In Section 15.3, we consider the most common representations of hedonic games as well as standard restrictions on preferences. In Section 15.4, we give an overview of computational aspects of hedonic games. Finally, we conclude the chapter with suggestions for further reading in Section 15.5.

15.2 Solution Concepts

In this section, we describe the most prominent solution concepts for hedonic games. We use the following notation throughout the chapter. For two coalitions $S, T \in \mathcal{N}_i$ that contain agent i, we use $S \succ_i T$ to denote that i strictly prefers S over T, $S \sim_i T$ to denote that i is indifferent between S and T, and $S \succsim_i T$ to denote that either $S \succ_i T$ or $S \sim_i T$ holds. Given a coalition structure π we use $\pi(i)$ to denote the (unique) coalition of π that includes agent i.

The first solution concept that we consider is *individual rationality.* A partition π

is *individually rational (IR)* if every agent is at least as happy in its coalition $\pi(i)$ as it would be alone, i.e., for all $i \in N$, $\pi(i) \succsim_i \{i\}$. This is really a minimal requirement for a solution to be considered stable, and many solution concepts that we consider will satisfy individual rationality.

Next, we introduce the concept of a *perfect partition*. This solution concept is stronger than *all* other solution concepts that we will consider in this chapter. A partition is *perfect* if each agent belongs to one of its most preferred coalitions (Aziz et al., 2013b). Thus, no matter what deviations we allow, a perfect partition is always stable. However, in general a perfect partition does not exist. In our running example (Example 15.1, each agent has a unique most preferred coalition ({1,2}, {2,3}, and {3,1} for agents 1,2, and 3 respectively), but these are not consistent with any partition, and so there is no perfect partition.

We next discuss *Pareto optimality*, which is a concept that has been used throughout the economics literature. A partition π is *Pareto optimal (PO)* if there is no partition π' with $\pi'(j) \succsim_j \pi(j)$ for all agents j and $\pi'(i) \succ_i \pi(i)$ for at least one agent i. Pareto optimality can also be considered a group-based stability concept in the sense that if a partition is not Pareto optimal, then there exists another partition that each agent weakly prefers and at least one agent strictly prefers.

The remaining solution concepts that we consider are all based on stability with respect to deviations by agents or groups of agents, which we deal with separately.

15.2.1 Solution Concepts based on Group Deviations

The core is one of the most fundamental solution concepts in cooperative game theory. In the context of coalition formation games, we say that a coalition $S \subseteq N$ *strongly blocks* a partition π, if every agent $i \in S$ strictly prefers S to its current coalition $\pi(i)$ in the partition π. A partition which admits no strongly blocking coalition is said to be in the *core (C)*, or *core stable*.

Returning to Example 15.1, it is easy to check that no core stable partition exists. If all agents are together in a single coalition, then every coalition of size two is a strongly blocking coalition. If all agents are alone, then again every coalition of size two is a strongly blocking coalition. Thus, if there is a core stable partition, exactly one agent is alone. Suppose agent 1 is alone. Then agent 1 and agent 3 together form a strongly blocking coalition. Due to the cyclic nature of the preference of agents over coalitions of size two, we can similarly rule out a core stable partition in which either agent 2 or 3 is alone.

A weaker definition of blocking (that gives rise to a more stringent solution concept) has also been considered. We say that a coalition $S \subseteq N$ *weakly blocks* a partition π, if each agent $i \in S$ weakly prefers S to $\pi(i)$ and there exists at least one agent $j \in S$ who strictly prefers S to its current coalition $\pi(j)$. A partition which admits no weakly blocking coalition is said to be in the *strict core (SC)*, which is sometimes referred to as the strong core (Bogomolnaia and Jackson, 2002).

One can also define other stability concepts based on more complex deviations by coalitions of agents. For partition π, $\pi' \neq \pi$ is called *reachable* from π by movements of players $S \subseteq N$, denoted by $\pi \xrightarrow{S} \pi'$, if $\forall i, j \in N \setminus S, i \neq j : \pi(i) = \pi(j) \Leftrightarrow \pi'(i) = \pi'(j)$. A subset of players $S \subseteq N, S \neq \emptyset$ *strong Nash blocks* π if a partition $\pi' \neq \pi$ exists with $\pi \xrightarrow{S} \pi'$ and $\forall i \in S : \pi'(i) \succ_i \pi(i)$. If a partition π is not strong Nash blocked by any set $S \subseteq N$, π is called *strong Nash stable (SNS)* (Karakaya, 2011). The stability concept strong Nash stability can be suitably weakened or strengthened to obtain other stability concepts such as *strict strong Nash stability (SSNS)* and *strong individual stability (SIS)*. We refer to Aziz and Brandl (2012) for the definitions.

15.2.2 Solution Concepts based on Single-Agent Deviations

In this sub-section, we define a number of solution concepts based on single-agent deviations. The most basic is *Nash stability* which is named after Nash equilibrium. A partition is Nash stable if no agent would gain by unilaterally moving to a different (possibly empty) coalition. The other solution concepts restrict which deviations are allowed based on the preferences of other agents in the coalitions that may lose or gain an agent through the deviation. We say that a partition π is:

- *Nash stable (NS)* if no agent can benefit by moving from its coalition to another (possibly empty) coalition, i.e., for all $i \in N$, $\pi(i) \succsim_i S \cup \{i\}$ for all $S \in \pi \cup \{\emptyset\}$.
- *Individually stable (IS)* if no agent can benefit by moving from its coalition to another (possibly empty) coalition while not making the members of that coalition worse off, i.e., for all $i \in N$, if there exists a coalition $S \in \pi \cup \{\emptyset\}$ with $S \neq \pi(i)$ s.t. $S \cup \{i\} \succ_i \pi(i)$ then there exists a $j \in S$ with $S \succ_j S \cup \{i\}$.
- *Contractual Nash stable (CNS)* if no agent i can benefit by moving from its coalition $\pi(i)$ to another (possibly empty) coalition $S \in \pi \cup \{\emptyset\}$ with $S \neq \pi(i)$ while not making the members of $\pi(i)$ worse off (Sung and Dimitrov, 2007b). Formally, for all $i \in N$, if there exists a coalition $S \in \pi \cup \{\emptyset\}$ with $S \neq \pi(i)$ s.t. $S \cup \{i\} \succ_i \pi(i)$ then there exists a $j' \in \pi(i)$ with $\pi(i) \succ_{j'} \pi(i) \setminus \{i\}$.
- *Contractually individually stable (CIS)* if no agent can benefit by moving from its coalition to another existing (possibly empty) coalition while making no member of either coalition worse off. Formally, for all $i \in N$, if there exists a coalition $S \in \pi \cup \{\emptyset\}$ with $S \neq \pi(i)$ s.t. $S \cup \{i\} \succ_i \pi(i)$ then there exists a $j \in S$ with $S \succ_j S \cup \{i\}$ or there exists a $j' \in \pi(i)$ with $\pi(i) \succ_{j'} \pi(i) \setminus \{i\}$.

All of the preceding concepts (except contractual Nash stability) have been defined by Bogomolnaia and Jackson (2002). In Figure 15.1, we show relationships between these and other solution concepts. For example, the figure shows that if a partition is perfect, then it satisfies all the stability concepts defined above. The hedonic game in Example 15.1 admits no IS partition (and therefore no NS) but has multiple CIS partitions. For example, the partition $\{\{1\}, \{2, 3\}\}$ is CIS, but not IS, because agent 3 has incentive to deviate to agent 1 and form coalition $\{1, 2\}$.

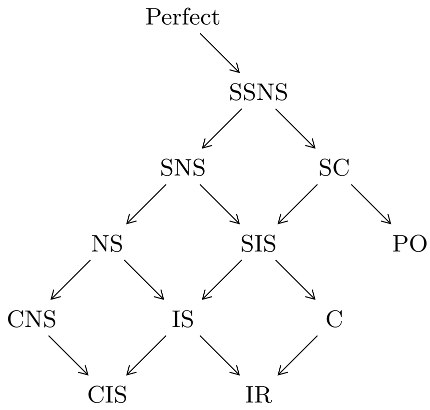

Figure 15.1 Logical relationships between stability concepts for hedonic games. For example, every NS partition is also IS. NS, SC, PO, C and IR are classic stability concepts. IS was formulated by Bogomolnaia and Jackson (2002); CNS by (Sung and Dimitrov, 2007b); SNS (strong Nash stability) by (Karakaya, 2011); and perfect partitions by (Aziz et al., 2013b). SSNS (strict strong Nash stability) and SIS (strong individual stability) were introduced by Aziz and Brandl (2012).

15.3 Preference Restrictions and Game Representations

When hedonic games are represented, we would ideally prefer a representation that not only allows agents to be as flexible as possible in describing their preferences but which also requires less space to succinctly store the preferences of the agents. In this section, we discuss different representations of hedonic games and, in particular the tradeoff that occurs between the expressiveness of a representation and its size; typically a fully expressive representation needs space that is exponentially large in the number of agents.

15.3.1 Fully Expressive Representations

A represenation is *fully expressive* if it can represent any hedonic game. We describe two types of fully expressive representations of hedonic games. The first, representation by *Individually Rational Lists of Coalitions*, is non-succinct and involves a complete enumeration of relevant preferences. The second, *Hedonic Coalition Nets*, allows for succinct representation in many cases. Without any restrictions, a representation of an agent's preferences involves comparisons over $2^{|N|-1}$ different coalitions.

Individually Rational Lists of Coalitions Most reasonable solution concepts require that an agent will be in a coalition that is *individually rational (IR)*, i.e., the agent prefers to be in that coalition over being alone. Therefore, instead of expressing preferences over all coalitions that include an agent, the agent may express preferences only over individually rational coalitions. Such a representation is called an *Individually Rational Lists of Coalitions* (*IRLC*) (Ballester, 2004). Of course such representations are essentially still complete enumerations and can be exponentially large in the number of agents.

Example 15.3 (An example of an IRLC) Let us examine a preference relation of agent 1 in the hedonic game defined in Example 15.1. Agent 1's preference list contains all coalitions that are at least as preferred as the singleton coalition:

$$\succsim_1\colon \quad \{1,2\} \succ_1 \{1,3\} \sim_1 \{1\}$$

Hedonic Coalition Nets. Hedonic coalition nets are a representation of hedonic games that are both *expressive*, i.e., able to capture potentially non-succinct IRLC, and also capable of allowing compact representations of games that have structured preferences. They were introduced by Elkind and Wooldridge (2009). In such a representation, each agent's preference relation is represented by a collection of rules of the form $\phi \mapsto_i b$ where ϕ is a predicate over coalitions and b is a real number. The value of an agent for a coalition is obtained by adding the values on the right hand side of those rules that are satisfied by the coalition. Hedonic coalition nets are inspired by *marginal contribution nets*, which were introduced in the context of transferable utility cooperative games by Ieong and Shoham (2005).

Hedonic coalition nets use the framework of propositional logic. Let Φ be a vocabulary of Boolean variables, and let $\mathcal{L}_\Phi$ be the set of propositional logic formulae over Φ. Given a truth assignment to the variables $\varphi \subseteq \Phi$, a *valuation* ξ is the subset of the variables such that we have $x \in \xi$ iff x is true. Given a valuation ξ and a formula $\phi \in \mathcal{L}_\varphi$, we say that $\xi \models \phi$ if and only if ϕ is true under valuation ξ.

In a hedonic coalition net, $\Phi = N$, each agent corresponds to a propositional variable and every coalition S defines a valuation over $\mathcal{L}_N$ where the variable $i \in N$ is set to true if $i \in S$ and set to false if $i \notin S$. A *rule* for agent $i \in N$ is $\phi \mapsto_i \beta$ where $\phi \in \mathcal{L}_N$ is a formula and $\beta \in \mathbb{R}$ is the *value* associated with the rule. A hedonic coalition net is a structure $(N, R_1, \ldots, R_n)$ where R_i specifies a set of rules for each $i \in N$. It specifies the utility of a coalition $S \in \mathcal{N}_i$ for an agent i in the following way:

$$u_i(S) = \sum_{\substack{\phi \mapsto_i \beta \ \in \ R_i \\ S \models \phi}} \beta.$$

In other words, for every coalition $S \in \mathcal{N}_i$, the utility of agent i for being in S is the sum of the values corresponding to those rules ϕ in R_i that the coalition S satisfies. Given a hedonic coalition net, the corresponding hedonic game is $(N, \succsim)$ such that for all $i \in N$ and for all $S, S' \in \mathcal{N}_i$, we have that $S \succsim_i S'$ if and only if $u_i(S) \geq u_i(S')$.

Hedonic coalition nets are fully expressive because they can represent an IRLC. In an IRLC with n agents, the preference list of agent i is represented as

$$S_1 *^1 S_2 *^2 \cdots *^{r-1} S_r$$

where $r \leq 2^{n-1}$, $*^j \in \{\succ_j, \sim_j\}$, $S_j \in \mathcal{N}_i$, and $S_r = \{i\}$. Based on an IRLC, we can construct a hedonic coalition net where there are r rules and the value of the r'th rule is $x_r = 0$, and for $j = r-1, \ldots, 1$, we set $x_j = x_j$ if $S_j \sim_i S_{j+1}$ and $x_j = x_j + 1$ if $S_j \succ_i S_{j+1}$. Now, for $j \in \{1, \ldots, r\}$, the j'th rule in R_i is

$$\left(\bigwedge_{k \in S_j} k\right) \wedge \left(\bigwedge_{l \in N \setminus S_j} \neg l\right) \mapsto_i x_j.$$

In other words, if agents in S_j are in the coalition and agents not in S_j are not in the coalition, then agent i gets utility x_j (exactly one rule is satisfied by a given coalition).

Example 15.4 (An example of hedonic coalition nets) Let us consider how a hedonic coalition net can represent the preferences of agent 1 in Example 15.3.

$$\begin{aligned} \neg i_1 &\mapsto -\infty & i_2 &\mapsto 3 \\ i_2 &\mapsto -\epsilon & i_3 \wedge \neg i_2 &\mapsto 2 \\ i_3 &\mapsto -\epsilon & i_4 \wedge \neg i_3 \wedge \neg i_2 &\mapsto 1 \\ i_4 &\mapsto -\epsilon \end{aligned}$$

The symbol ϵ represents an arbitrarily small positive real number. The symbol ∞ represents a sufficiently large number. Let us see how agent 1 compares coalition $\{1, 2\}$ with $\{1, 3, 4\}$. For coalition $\{1, 2\}$, the rules $i_2 \mapsto -\epsilon$ and $i_2 \mapsto 3$ are satisfied and the total utility of agent 1 for coalition $\{1, 2\}$ is $3 - \epsilon$. In fact, the rules $i_2 \mapsto -\epsilon$ and $i_2 \mapsto 3$ can simply be combined into a single rule $i_2 \mapsto 3 - \epsilon$. For coalition $\{1, 3, 4\}$, the rules $i_3 \wedge \neg i_2 \mapsto 2$, $i_3 \mapsto -\epsilon$ and $i_4 \mapsto -\epsilon$ are satisfied, so the total utility of agent 1 for coalition $\{1, 3, 4\}$ is $2 - 2\epsilon$.

In contrast to IRLC, hedonic coalition nets can be used to succinctly represent various classes of hedonic games such as additively separable hedonic games and games with $\mathcal{B}$-preferences, which we will introduce in Section 15.3.4.

The use of weighted logics for succinct representations is prevalent in computational social choice (see Chapter 12 (Bouveret et al., 2015)). More generally, succinct representations are discussed in several chapters (for example Chapters 9 (Lang and Xia, 2015), 12 (Bouveret et al., 2015), and 14 (Klaus et al., 2015)).

15.3.2 Additively Separable Hedonic Games

Separability of preferences is a property of certain preferences that allows for a succinct representations. The main idea of separability is that adding a liked (unliked) agent to a coalition makes the coalition more (less) preferred.

Definition 15.5 (Separability) A game $(N, \succsim)$ is called *separable* if for every agent $i \in N$, coalition $S \in \mathcal{N}_i$, and agent j not in S, we have the following:

- $S \cup \{j\} \succ_i S$ if and only if $\{i, j\} \succ_i \{i\}$;
- $S \cup \{j\} \prec_i S$ if and only if $\{i, j\} \prec_i \{i\}$; and
- $S \cup \{j\} \sim_i S$ if and only if $\{i, j\} \sim_i \{i\}$.

Additive separable preferences are a particularly appealing (strict) subclass of separable preferences. In an *additively separable hedonic game (ASHG)* $(N, \succsim)$, each agent $i \in N$ has value $v_i(j)$ for agent j being in the same coalition as i and for any coalition $S \in \mathcal{N}_i$, i gets utility $\sum_{j \in S \setminus \{i\}} v_i(j)$ for being in S. The utility that an agent gets for being alone in a singleton coalition is assumed to be 0. For coalitions $S, T \in \mathcal{N}_i$, we have $S \succsim_i T$ if and only if $\sum_{j \in S \setminus \{i\}} v_i(j) \geq \sum_{j \in T \setminus \{i\}} v_i(j)$. Therefore an ASHG can be represented by a weighted directed graph in which every vertex corresponds to an agent and weight of an arc (i, j) represents $v_i(j)$. Additively separable preferences are *symmetric* if $v_i(j) = v_j(i)$ for every two agents $i, j \in N$.

A *non-symmetric* ASHG need not have a Nash stable partition (Bogomolnaia and Jackson, 2002), and deciding whether there is one is NP-complete (Ballester, 2004). A symmetric ASHG is represented by an *undirected* weighted graph, and always has a Nash stable partition as we will explain after an example.

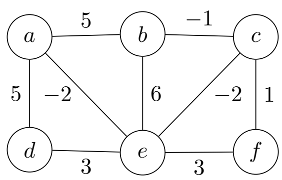

Figure 15.2 Example of an additively separable symmetric hedonic game.

Example 15.6 Figure 15.2 gives an example of a symmetric additively separable hedonic game. Consider the partition $\{\{a, b, d\}, \{c, e, f\}\}$. The utilities of the agents a, b, c, d, e, f are $10, 5, -1, 5, 1, 4$, respectively. Agents a, b, d, f have no profitable single-agent deviations, c has a profitable deviation to go alone and start a singleton coalition, and e has a profitable deviation to join the other coalition. This is a contractually individually stable partition. The partition $\{\{a, b, d\}, \{c\}, \{e, f\}\}$ is an individual stable partition, and $\{\{a, b, d, e, f\}, \{c\}\}$ is Nash stable.

For an additively separable *symmetric* hedonic game, the existence of a Nash stable (NS) partition (and therefore also a CNS, CIS, and IS partition) is guaranteed by an argument based on a 'potential function', which was noted by (Bogomolnaia and Jackson, 2002). We will construct an *exact potential function*. For every partition the exact potential function assigns a potential value equal to half the sum of agents' utilities under this partition. A deviation by a single agent from its coalition to another will change the value of this potential function by exactly the change in the utility of this deviating agent (every edge gained or lost in the sum of agents' utilities counts twice). Since the potential function is bounded, and there are finitely many partitions, and every profitable deviation by a single agent improves the value of the potential function, we have the following: starting from any partition, every maximally long sequence of profitable deviations by agents terminates with a Nash stable partition. Note that symmetry is key for this potential function argument to work.

15.3.3 Games Based on the Best or Worst Agents

We now describe classes of hedonic games in which the agents' preferences over coalitions are induced by their *ordinal* preferences over the other individual agents. Two of the most natural ways to extend preferences is based on the most preferred or least preferred agent in the coalition. For a subset J of agents, we denote by $\max_{\succsim_i}(J)$ and $\min_{\succsim_i}(J)$ the sets of the most and least preferred agents in J by i, respectively. In $\mathcal{B}$ (where $\mathcal{B}$ stands for best) and $\mathcal{W}$ (where $\mathcal{W}$ stands for worst) games each agent's appreciation of a coalition depends on the most preferred (best) or least preferred (worst) agent in the coalition. Note that roommate markets can also be considered in the framework with the additional constraint that coalitions of size three or more are not feasible.

In *hedonic games with $\mathcal{B}$-preferences* (which we will refer to as $\mathcal{B}$-hedonic games), $S \succ_i T$ if and only if, we have either:

(i) for each $s \in \max_{\succsim_i}(S \setminus \{i\})$ and $t \in \max_{\succsim_i}(T \setminus \{i\})$, $s \succ_i t$, or
(ii) for each $s \in \max_{\succsim_i}(S \setminus \{i\})$ and $t \in \max_{\succsim_i}(T \setminus \{i\})$, $s \sim_i t$ and $|S| < |T|$.

So an agent's appreciation of a coalition depends on its most favoured agents in the coalition. If two coalitions have equally preferred agents, then a smaller coalition is strictly preferred (Cechlárová and Hajduková, 2002; Cechlárová and Romero-Medina, 2001).

Example 15.7 (A $\mathcal{B}$-hedonic game) Let $(N, \succsim)$ be a game with $N = \{1, 2, 3, 4\}$ and let the agents have preferences over other agents as follows:

$$\begin{aligned} 2 \succ_1 3 \succ_1 4 \succ_1 1 \\ 3 \succ_2 4 \succ_2 1 \succ_2 2 \\ 1 \succ_3 2 \succ_3 4 \succ_3 3 \\ 3 \succ_4 2 \succ_4 1 \succ_4 4 \end{aligned}$$

For the $\mathcal{B}$-hedonic game, the preferences of each agent over other agents are extended over the preferences over sets of agents. The preferences of the agents are as follows (the preference of agent 1 is the same as in Example 15.3). For brevity, we omit the commas separating agents in each coalition.

$$\begin{aligned} \{12\} \succ_1 \{123\} \sim_1 \{124\} \succ_1 \{1234\} \succ_1 \{13\} \succ_1 \{134\} \succ_1 \{14\} \succ_1 \{1\} \\ \{23\} \succ_2 \{234\} \sim_2 \{123\} \succ_2 \{1234\} \succ_2 \{24\} \succ_2 \{124\} \succ_2 \{12\} \succ_2 \{2\} \\ \{13\} \succ_3 \{123\} \sim_3 \{134\} \succ_3 \{1234\} \succ_3 \{23\} \succ_3 \{234\} \succ_3 \{34\} \succ_3 \{3\} \\ \{34\} \succ_4 \{134\} \sim_4 \{234\} \succ_4 \{1234\} \succ_4 \{24\} \succ_4 \{124\} \succ_4 \{14\} \succ_4 \{4\} \end{aligned}$$

In *hedonic games with $\mathcal{W}$-preferences* (which we will refer to as $\mathcal{W}$-hedonic games), $S \succsim_i T$ if and only if for each $s \in \min_{\succsim_i}(S \setminus \{i\})$ and $t \in \min_{\succsim_i}(T \setminus \{i\})$, we have $s \succsim_i t$. So an agent's appreciation of a coalition depends on its least preferred favoured agents in the coalition (Cechlárová and Hajduková, 2004b; Cechlárová and Romero-Medina, 2001).

Other games can be defined based on preferences that depend on both the best and worst agents in the coalition (Hajduková, 2006) or in which the presence of an unacceptable agent in the coalition makes the coalition unacceptable (Aziz et al., 2013b, 2012). $\mathcal{B}$-hedonic games, in which agents express strict preferences over other agents, are known to admit a core stable partition (Cechlárová and Romero-Medina, 2001). For further details on games based on the best or worst agents, we refer the reader to Aziz et al. (2013b, 2012); Cechlárová and Romero-Medina (2001); Cechlárová and Hajduková (2004b); Hajduková (2006). $\mathcal{B}$-hedonic games with strict preferences form a subclass of a larger class of hedonic games that satisfy *top responsiveness.*

15.3.4 Top Responsiveness

Identifying sufficient and necessary conditions for the existence of stability in coalition formation has been a very active area of research. Perhaps the most celebrated result in this field is the existence of a (core) stable matching for the stable marriage problem, which is shown constructively via the *deferred-acceptance* algorithm (Gale and Shapley, 1962). For hedonic games, more generally, a number of preference restrictions have been identified for which a stable partition is guaranteed to exist for some notion of stability (see e.g., Alcalde and Revilla (2004); Aziz and Brandl (2012); Banerjee et al. (2001); Bogomolnaia and Jackson (2002); Dimitrov et al. (2006); Dimitrov and Sung (2007); Karakaya (2011).) In this sub-section, we present an example of hedonic games that satisfy *top responsiveness* (Alcalde and Revilla, 2004) that always admit a core stable partition.

Every coalition contains subset subcoalitions over which the agents also have preferences. Top responsiveness captures the situation in which each agent's appreciation of a coalition depends on the most preferred subset within the coalition. If two coalitions have the same most preferred subcoalition, then the coalition with the smaller size is preferred. Top responsiveness is based on *choice sets*, which are sets of agents that an agent wants to be with. We use $Ch(i, S)$ to denote the *choice sets* of agent i in coalition S. It is formally defined as follows:

$$Ch(i, S) := \{S' \subseteq S : (i \in S') \wedge (S' \succsim_i S'' \ \forall S'' \subseteq S)\}.$$

If $|Ch(i, S)| = 1$, we denote by $ch(i, S)$ the unique subset of S that is maximally preferred by agent i on under $\succsim_i$: A game satisfies *top responsiveness* if $|Ch(i, S)| = 1$ for each $i \in N$ and $S \in \mathcal{N}_i$ and the following conditions hold for each $i \in N$ and all $S, T \in \mathcal{N}_i$,

(*i*) $S \succ_i T$ if $ch(i, S) \succ_i ch(i, T)$;
(*ii*) $S \succ_i T$ if $ch(i, S) = ch(i, T)$ and $S \subset T$.

Example 15.8 (An example of top responsive preferences) The hedonic game specified in Example 15.7. satisfies top responsiveness. For example, for agent 1, its choice set of the grand coalition is $\{1, 2\}$. It likes each coalition that is a superset of $\{1, 2\}$ more than any coalition that is not a superset of $\{1, 2\}$. Moreover it prefers a smaller coalition that is a superset of $\{1, 2\}$ more than a bigger coalition.

In Section 15.4.3, we show how to exploit the top responsiveness property algorithmically.

15.4 Algorithms and Computational Complexity

In this section we give an overview of computational results concerning hedonic games. For a given solution concept α, such as core stability, we consider the following natural computational problems:

VERIFICATION: Given $(N, \succsim)$ and a partition π of N, does π satisfy α?
EXISTENCE: Given $(N, \succsim)$, does there exist a partition satisfying α?
CONSTRUCTION: Given $(N, \succsim)$, if a partition satisfying α exists, find one.

15.4.1 Hardness to Check Non-emptiness of Core

We next consider the problem EXISTENCE for core stability for a hedonic game given as an IRLC, which was shown by Ballester (2004) to be NP-complete.

Theorem 15.9 (Ballester (2004)) *For hedonic games in IRLC, the problem of checking whether there exists a core stable partition is NP-complete.*

Proof Given an IRLC, VERIFICATION for core stability can be solved in polynomial time because each agent explicitly lists all the individually rational coalitions, and these are also all the potentially blocking coalitions. Thus the problem of deciding whether a core stable partition exists is in NP.

We now show that it is NP-hard and thus NP-complete, via a reduction from EXACTCOVERBY3SETS.

EXACTCOVERBY3SETS (X3C).
Instance: Set X and set T that consists of 3-element subsets of X.
Question: Does there exist a subset of T that partitions X?

We construct a hedonic game where $N = \{x, x', x'' : x \in X\}$ and $\succsim$ is defined as follows. For each $i \in \{x, x', x''\}$, let $X_1^i, \ldots, X_m^i$ be the elements in T such that they include i.

- For each x, $X_1^x \sim_x \cdots \sim_x X_m^x \succ_x \{x, x''\} \succ_x \{x, x'\} \succ_x \{x\}$
- For each x', $\{x, x'\} \succ_{x'} \{x', x''\} \succ_{x'} \{x'\}$
- For each x'', $\{x', x''\} \succ_{x''} \{x, x''\} \succ_{x''} \{x''\}$

It can be shown that $(N, \succsim)$ admits a core stable partition if and only if the X3C instance is a yes instance. If the X3C instance is a yes instance, then there exists a partition π of N that puts each $x \in X$ in one of its most preferred coalitions. As for each x' and x'', they can be paired up in a coalition so that x'' is in its most preferred coalition, and x' is in its second most preferred coalition. Hence the partition is core stable.

If X3C is a no instance, then there exists no partition in which each x gets a most preferred coalition. Hence, in each partition, at least one $x \in X$ is in one of the following coalitions: $\{x, x''\}, \{x, x'\}, \{x\}$. For any partition containing coalition $\{x, x''\}$, the coalition $\{x', x''\}$ is blocking. For any partition containing coalition $\{x, x'\}$, the coalition $\{x, x''\}$ is blocking. For any partition containing coalition $\{x\}$, the coalition $\{x, x'\}$ is blocking. Hence each partition admits a blocking coalition and is not core stable. □

NP-hardness of checking existence of core is not restricted to IRLC but also holds for various other representations and classes of games. For a survey on this topic, we recommend the report of Woeginger (2012).

15.4.2 Symmetric Additively Separable Hedonic Games

In this section, we focus on the class of symmetric additively separable games. As described in Section 15.3.2, an argument that uses a potential function shows that every instance from this class possesses a Nash stable outcome. This places the computational problem of finding a Nash stable outcome for a game of this type, as well as the problem of finding any stable outcome based on more restrictive (polynomial-time checkable) notions of deviation, in the complexity class PLS, which stands for polynomial local search. In this section, we first give a brief overview of the complexity class PLS and PLS-reductions (Johnson et al., 1988). Then, we give an overview of some negative results (PLS-completeness) and positive results (polynomial-time algorithms) for the problem of finding stable outcomes for hedonic games in this class.

A problem in PLS comprises a finite set of candidate solutions. Every candidate solution has an associated non-negative integer cost, and a neighbourhood of candidate solutions. In addition, a PLS problem is specified by the following three polynomial-time algorithms that:

(*i*) construct an initial candidate solution;
(*ii*) compute the cost of any candidate solution in polynomial time;
(*iii*) given a candidate solution, provide a neighbouring solution with lower cost if one exists.

The goal in a PLS problem is to find a local optimum, that is, a candidate solution whose cost is smaller than all of its neighbours.

Suppose A and B are problems in PLS. Then A is PLS-reducible to B if there exist polynomial-time computable functions f and g such that f maps instances I of A to instances $f(I)$ of B, and g maps the local optima of instances $f(I)$ of B to local optima of instance I. A problem in PLS is PLS-complete if all problems in PLS are PLS-reducible to it. PLS captures the problem of finding pure Nash equilibria for many classes of games where pure equilibria are guaranteed to exist, such as congestion games Fabrikant et al. (2004).

On the one hand, it is very unlikely that a PLS problem is NP-hard since this would imply NP=coNP (Johnson et al., 1988). On the other hand, a polynomial-time algorithm for a PLS-complete problem would resolve a number of long open problems, e.g., since it would show that *simple stochastic games* can be solved in polynomial time (Yannakakis, 2008). Thus, PLS-complete problems are believed not to admit polynomial-time algorithms.

Observation 1 (Gairing and Savani (2010)) *For symmetric additively separable hedonic games,* Construction *for Nash Stability is PLS-complete.*

Proof We reduce from the PLS-complete problem PartyAffiliation, which is to compute an equilibrium of a party affiliation game. The input of PartyAffiliation is an undirected edge-weighted graph. A solution is a partition of the nodes into two parties such that for every node v the sum of edge weights of edges from v to other nodes in v's part is greater that the sum of edge weights to nodes in the other party. A party affiliation game is essentially a symmetric additively separable hedonic game where at most two coalitions are permitted.

Consider an instance $G = (V, E, w)$ of PartyAffiliation. We augment G by introducing two new agents, called *super nodes*. Every agent $i \in V$ has an edge, of weight $W > \sum_{e \in E} |w_e|$, to each of the super nodes. The two super nodes are connected by an edge of weight $-M$, where $M > |V| \cdot W$. Use the resulting graph to define a corresponding hedonic additively separable game and consider Nash stable outcomes. By the choice of M the two super nodes will be in different coalitions in any Nash stable outcome of the resulting hedonic game. Moreover, by the choice of W, each agent will be in a coalition with one of the super nodes. So, in each Nash stable outcome we have exactly two coalitions. The fact that edges to super nodes have all the same weight directly implies a one-to-one correspondence between the Nash stable outcomes in the hedonic game and in the party affiliation game. □

Every Nash stable outcome is also an individually stable outcome and thus Construction is no harder for individual stability than for Nash stability. It turns out that Construction for individual stability is still PLS-hard, though the simplest reduction we know that shows this result is much more involved than the proof of Observation 1 for Nash stability (Gairing and Savani, 2010, 2011). When deviations of players are restricted even further, and we move from individual stability to contractual individual stability, the problem Construction becomes efficiently solvable:

Observation 2 (Gairing and Savani (2010)) *For symmetric additively separable hedonic games,* Construction *for CIS can be solved in $\mathcal{O}(|E|)$ time. Moreover, local improvements converge in at most $2|V|$ steps.*

Proof Consider the following algorithm to solve the game $G = (V, E, w)$:

Delete all negative edges from G and put every connected component in a separate coalition.

Consider any coalition formed by this algorithm. If the coalition consists of only one agent, then this agent has no positive edges to any agent and staying alone is a (weakly) dominant strategy. Agents in larger coalitions are connected by a positive edge to some agent within the same coalition. Therefore, they are not allowed to leave the coalition. Thus, we have computed a CIS stable state. Finding the connected components of an undirected graph can be done by depth-first search in $\mathcal{O}(|E|)$ time.

Now consider local improvements. Observe that whenever an agent joins a nonempty coalition then this agent (and all agents to which it is connected by a positive edge in the coalition) will never move again. Moreover, an agent can only start a new coalition once. It follows that each agent can make at most two strategy changes. In total we have at most $2|V|$ local improvements. □

For further results on additively separable hedonic games, we refer the reader to Aziz et al. (2013a); Sung and Dimitrov (2010); Olsen (2009); Sung and Dimitrov (2007a).

15.4.3 Top Covering Algorithm

In Section 15.3.4, we discussed games that satisfy top responsiveness that are guaranteed to admit a core stable partition. Next, we define the *Top Covering Algorithm* to compute a core stable partition for hedonic games satisfying top responsiveness (Alcalde and Revilla, 2004; Dimitrov and Sung, 2006). For this we need the following definitions. For each $X \subseteq N$, we denote by $\backsim_X$ the relation on $X \times X$ where $i \backsim_X j$ if and only if $j \in ch(i, X)$. In this case j is called a *neighbour* of i in X. The connected component $CC(i, X)$ of i with respect to X is defined as follows:

$$CC(i, X) = \{k \in X : \exists j_1, \ldots, j_l \in X : i = j_1 \backsim_X \cdots \backsim_X j_l = k\}.$$

Based on the concept of connected components, we can specify the Top Covering Algorithm. The algorithm is formally specified as Algorithm 1 and is based on similar ideas as that of the *Top Trading Cycle Algorithm* for exchange of indivisible objects (Shapley and Scarf, 1974). We maintain a partition π and a set R^k as the set of remaining agents in round k. In each round, an agent i is selected from R^k for which the size of the connected component of i with respect to R^k is at most the the size of the connected component of some other agent $j \in R^k$ with respect to R^k. Such a connected component is a new coalition in partition π. The process is iterated until no more agents remain. For hedonic games that satisfy top responsiveness, the Top Covering Algorithm returns a core stable partition. Alcalde and Revilla (2004) also proved that, for top responsive preferences, the Top Covering Algorithm is strategyproof.

Example 15.10 (An example illustrating the Top Covering Algorithm) We run the Top Covering Algorithm on the hedonic game specified in Example 15.8. Example 15.8 pointed out that the hedonic game satisfies top responsiveness. First, we examine the connected components. Initially, $R^1 = \{1, 2, 3, 4\}$, $CC(1, \{1, 2, 3, 4\}) = CC(2, \{1, 2, 3, 4\}) = CC(3, \{1, 2, 3, 4\}) = \{1, 2, 3\}$ and $CC(4, \{1, 2, 3, 4\}) = \{1, 2, 3, 4\}$.
Since $|CC(1, \{1, 2, 3, 4\})| < |CC(4, \{1, 2, 3, 4\})|$, hence $S^1 = \{1, 2, 3\}$ and $R^2 = \{1, 2, 3, 4\} \setminus \{1, 2, 3\} = \{4\}$. Thus the coalition $S^1 = \{1, 2, 3\}$ is fixed and the next fixed coalition is $S^2 = \{4\}$. Hence the final partition $\pi = \{\{1, 2, 3\}, \{4\}\}$.

Recall from Section 15.3.4 that $\mathcal{B}$-hedonic games are one well-studied class of games that satisfy top responsiveness. For more details on computational aspects

Algorithm 1 Top Covering Algorithm

Input: A hedonic game $(N, \succsim)$ satisfying top responsiveness.
Output: A core stable partition π.

$R^1 \leftarrow N$; $\pi \leftarrow \emptyset$.
for $k = 1$ to $|N|$ **do**
  Select $i \in R^k$ such that $|CC(i, R^k)| \leq |CC(j, R^k)|$ for each $j \in R^k$.
  $S^k \leftarrow CC(i, R^k)$
  $\pi \leftarrow \pi \cup \{S^k\}$
  $R^{k+1} \leftarrow R^k \setminus S^k$
  if $R^{k+1} = \emptyset$ **then**
    return π
  end if
end for
return π

of games based on the best or worst agents, we refer the reader to Aziz et al. (2012); Cechlárová and Romero-Medina (2001); Cechlárová and Hajduková (2004b); Hajduková (2006).

15.4.4 Preference Refinement Algorithm

We outline the Preference Refinement Algorithm (PRA) to compute individually rational and Pareto optimal partitions (Aziz et al., 2013b). The idea of PRA is to relate the problem of computing a Pareto optimal partition to PerfectPartition — the problem of checking whether a perfect partition exists. First note that if there exists a polynomial-time algorithm to compute a Pareto optimal partition, then it returns a perfect partition if a perfect partition exists. For the opposite direction, we show that an oracle to solve PerfectPartition can be used by PRA to compute a Pareto optimal partition.

In PRA, the bottom preference $\succsim_i^{\perp}$ of each agent i is initially completely 'coarsened' so that each agent is indifferent among all acceptable coalitions. The top preference $\succsim_i^{\top}$ of each agent is set to $\succsim_i$. The preference profiles $\succsim^{\perp}$ and $\succsim^{\top}$ are updated during the running of PRA while ensuring that a perfect partition exists for $\succsim^{\perp}$. Since the partition of singletons is a perfect partition for the coarsest profile $(\succsim_1^{\perp}, \ldots, \succsim_n^{\perp})$, we know that a perfect partition exists. Before we formally specify PRA, we must define *coarsening*, *refinement*, and *cover* in preference relations.

Let $\succsim = (\succsim_1, \ldots, \succsim_n)$ and $\succsim' = (\succsim_1', \ldots, \succsim_n')$. We say that $\succsim_i'$ refines $\succsim_i$ if $\succsim_i$ is exactly like $\succsim_i'$, except that in $\succsim_i'$ agent i may have strict preferences among some of his most preferred coalitions according to $\succsim_i$. Equivalently $\succsim_i$ is *coarser* than $\succsim_i'$. We say that $\succsim_i'$ is *strictly refines* $\succsim_i$ if $\succsim_i'$ refines $\succsim_i$ but $\succsim_i'$ does not refine $\succsim_i$. If a partition is perfect for some preference profile $\succsim$, then it is also perfect

Algorithm 2 Preference Refinement Algorithm (PRA).

Input: Hedonic game $(N, \succsim)$
Output: Pareto optimal and individually rational partition

1 $\succsim_i^{\top} \leftarrow \succsim_i$, for each $i \in N$
2 $\succsim_i^{\bot} \leftarrow \succsim_i \cup \{(X,Y) : X \succsim_i \{i\}$ and $Y \succsim_i \{i\}\}$, for each $i \in N$
3 **while** $\succsim_i^{\bot} \neq \succsim_i^{\top}$ for some $i \in N$ **do**
4 $\quad i \leftarrow$ `Choose`$(\{j \in N \colon \succsim_j^{\bot} \neq \succsim_j^{\top}\})$
$\quad$ {`Choose` specifies some way to choose an agent from a set of agents.}
5 $\quad \succsim_i' \leftarrow$ `Refine`$(\succsim_i^{\bot}, \succsim_i^{\top})$
6 $\quad$ **if** PerfectPartition$(N, (\succsim_1^{\bot}, \ldots, \succsim_{i-1}^{\bot}, \succsim_i', \succsim_{i+1}^{\bot}, \ldots, \succsim_n^{\bot})) = \emptyset$ **then**
7 $\quad\quad \succsim_i^{\top} \leftarrow \succsim_i''$ where $\succsim_i'$ covers $\succsim_i''$
8 $\quad$ **else**
9 $\quad\quad \succsim_i^{\bot} \leftarrow \succsim_i'$
10 $\quad$ **end if**
11 **end while**
12 **return** PerfectPartition$(N, \succsim^{\bot})$

for any profile in which the preferences are coarsened. The same holds for Pareto optimal partitions. We say that $\succsim_i$ *covers* $\succsim_i'$ if $\succsim_i$ is a minimal refinement of $\succsim_i'$ with $\succsim_i' \neq \succsim_i$, i.e., if $\succsim_i$ strictly refines $\succsim_i'$ and there is no $\succsim_i''$ such that $\succsim_i$ strictly refined $\succsim_i''$ and $\succsim_i''$ strictly refined $\succsim_i'$.

PRA can be viewed as gradually improving the minimum guaranteed welfare of agents while using binary search. When an agent i is chosen for whom $\succsim_i^{\bot}$ and $\succsim_i^{\top}$ do not coincide, $\succsim_i^{\bot}$ is temporarily set to some preference relation $\succsim_i'$ which is finer than $\succsim_i^{\bot}$ but coarser than $\succsim_i^{\top}$. If a perfect partition still exists for the given preference preference profile, then $\succsim_i^{\bot}$ is set to the updated preference. If no perfect partition exists, then we can restrict our attention to preferences of agent i that are not as fine as $\succsim_i'$. The main idea is that if no refinement of some preference profile with perfect partition π allow for a perfect partition, then π is Pareto optimal. PRA is specified more formally as Algorithm 2 where `Choose`$(\{j \in N \colon \succsim_j^{\bot} \neq \succsim_j^{\top}\})$ returns a player in the set $\{j \in N \colon \succsim_j^{\bot} \neq \succsim_j^{\top}\}$ and `Refine`$(\succsim_i^{\bot}, \succsim_i^{\top})$ returns a preference $\succsim_i'$ that strictly refines $\succsim_i^{\bot}$ and is a coarsening of $\succsim_i^{\top}$.

Example 15.11 (An example illustrating PRA) We run PRA on the hedonic game specified in Example 15.8. In the beginning, $\succsim_i^{\top} = \succsim_i$ and $\succsim_i^{\bot}$ specifies indifference between all coalitions.

Let us say that agent 4 is chosen in Step 4 of the algorithm and we consider its preferences $\succsim_4'$.

$$\{34\} \succ_4' \{134\} \sim_4' \{234\} \succ_4' \{1234\} \succ_4' \{24\} \succ_4' \{124\} \succ_4' \{34\} \succ_4' \{4\}$$

We check whether a perfect partition still exists or not for $(\succsim_1^{\bot}, \succsim_2^{\bot}, \succsim_3^{\bot}, \succsim_4')$. A perfect partition indeed exists: $\{\{3,4\},\{1\},\{2\}\}$. Thus $\succsim_4^\perp$ is set to $\succsim_4'$. Let us now take agent 2 and consider its preference $\succsim_2'$.

$$\{23\} \sim_2' \{234\} \sim_2' \{213\} \sim_2' \{1234\} \sim_2' \{24\} \sim_2' \{214\} \succ_2' \{12\} \succ_2' \{2\}$$

However, no perfect partition exists for $(\succsim_1^\perp, \succsim_2', \succsim_3^\perp, \succsim_4^\perp)$. Hence $\succsim_2^\top$ is changed as follows:

$$\{23\} \sim_2^\top \{234\} \sim_2^\top \{213\} \sim_2^\top \{1234\} \sim_2^\top \{24\} \sim_2^\top \{214\} \sim_2^\top \{12\} \succ_2^\top \{2\}$$

The process goes on until top and bottom preferences of the agents are as follows in the end:

$$\{12\} \succ_1^\perp \{123\} \sim_1^\perp \{124\} \succ_1^\perp \{1234\} \succ_1^\perp \{13\} \succ_1^\perp \{134\} \succ_1^\perp \{14\} \succ_1^\perp \{1\}$$
$$\{23\} \sim_2^\perp \{234\} \sim_2^\perp \{213\} \sim_2^\perp \{1234\} \sim_2^\perp \{24\} \sim_2^\perp \{214\} \sim_2^\perp \{12\}$$
$$\{13\} \sim_3^\perp \{123\} \sim_3^\perp \{134\} \sim_3^\perp \{1234\} \sim_3^\perp \{23\} \sim_3^\perp \{234\} \sim_3^\perp \{34\} \succ_3^\perp \{3\}$$
$$\{34\} \succ_4^\perp \{134\} \sim_4^\perp \{234\} \succ_4^\perp \{1234\} \succ_4^\perp \{24\} \succ_4^\perp \{124\} \succ_4^\perp \{14\} \succ_4^\perp \{4\}$$

The perfect partition for $(\succsim_1^\perp, \succsim_2^\perp, \succsim_3^\perp, \succsim_4^\perp) = (\succsim_1^\top, \succsim_2^\top, \succsim_3^\top, \succsim_4^\top)$ is $\{\{3,4\},\{1,2\}\}$. which is Pareto optimal for $\succsim$.

Any Pareto optimal and individually rational partition can be returned by PRA depending on how the refinements of preferences are carried out. The behaviour of PRA may depend on the specific settings of `Choose` and `Refine`. In particular the following are two interesting versions of PRA. In PRA_{SD}, `Choose` selects players according to a fixed order of the players and `Refine` returns a player's finest preference relation, i.e., generally $\texttt{Refine}(\succsim_i^\perp, \succsim_i^\top) = \succsim_i^\top$. In PRA_{Egal}, `Choose` selects a player that has been selected the fewest number of times during the execution of PRA. `Refine` is defined such that $\texttt{Refine}(\succsim_i^\perp, \succsim_i^\top) = \texttt{Cover}(Q_i^\perp)$. Both versions have their merits. If all coalitions are acceptable then PRA_{SD} is strategyproof. On the other hand, PRA_{Egal} satisfies the following property: for any $k \in \mathbb{N}$ for which there exists a Pareto optimal partition in which none of the players get one of their kth lowest-ranked or worse coalitions, PRA_{Egal} will return such a partition.

PRA can be adapted for various specific classes of hedonic games by formulating specific algorithms to solve PerfectPartition for those classes. For example, applying this idea yields a polynomial-time algorithm to compute a Pareto optimal partition for $\mathcal{W}$-hedonic games (Aziz et al., 2013b).

15.5 Further Reading

There are a number of topics that we have not touched on in this chapter. We have not discussed strategic issues in detail (see, e.g., Demange, 2009). One reason is that impossibility results hold even for restricted classes of hedonic games (Barberà and Gerber, 2007; Rodríguez-Álvarez, 2009).

Various classes of hedonic games can be represented by graphs. Some of them are special subclasses of additively separable hedonic games, which we have discussed. An additively separable hedonic game (N, v) is an *appreciation of friends* game if for all $i, j \in N$ such that $i \neq j$, $v_i(j) \in \{-1, +n\}$. It is an *aversion to enemies* game if for all $i, j \in N$ such that $i \neq j$, $v_i(j) \in \{-n, +1\}$. These games were introduced in Dimitrov et al. (2006).

There are other interesting graph based hedonic games that we did not discuss. *Social distance games* were introduced by Branzei and Larson (2011). Each social distance game is represented by an unweighted undirected graph. Agent i in coalition $C \subseteq N$ has utility for this coalition equal to $v_i(C) = \frac{1}{|C|} \sum_{j \in N \setminus \{i\}} \frac{1}{d_C(i,j)}$ where $d_C(i, j)$ is the shortest path distance between i and j in the subgraph induced by coalition C on the graph G. If i and j are disconnected in C, then $d_C(i, j) = \infty$. Another class of graph based game that has been recently proposed is that of *fractional hedonic games* (Aziz et al., 2014b). As in additively separable hedonic games, each agent i has a value function $v_i \colon N \to \mathbb{R}$, assigning a value to each agent $i \in N$ with $v_i(i) = 0$. A value function v_i can be extended to a value function over coalitions $S \subseteq N$ in such a way so that $v_i(S) = \frac{\sum_{j \in S} v_i(j)}{|S|}$. A hedonic game $(N, \succsim)$ is said to be a *fractional hedonic game* if for each agent i in N there is a value function v_i such that for all coalitions $S, T \subseteq N$, $S \succsim_i T$ if and only if $v_i(S) \geq v_i(T)$. Unlike additively separable hedonic games, even if the weights are all positive, the grand coalition need not be core stable for fractional hedonic games.

The class of *roommate games*, which are well-known from the literature on matching theory, can be defined as those hedonic games in which only coalitions of size one or two are feasible (see, e.g., Aziz, 2013). A *marriage game* is a roommate game in which the set N of agents can be partitioned into two sets *male* and *female* and an agent finds a member of the same sex unacceptable. For further reading on computational aspects of marriage, roommate and related games, we refer the reader to Ronn (1990); Irving (1985); Scott (2005); Irving (1994); Manlove (1999); Aziz (2013); Deineko and Woeginger (2013). Cechlárová and Hajduková (2004a) examined more complex preferences in which agents' appreciation of coalition depends on both the worst and best agents in the coalition.

There are also various classes of hedonic games in which agent's appreciation of a coalition depends on the size of the coalition. *Anonymous games* are a subclass of hedonic games in which an agent's preferences over coalitions *only* depends on the coalition sizes (see, e.g., Ballester, 2004). Anonymous games are closely related to congestion games (see, e.g., Milchtaich, 1996) in non-cooperative game theory. A setting that is related to anonymous games is that of *group activity selection game* in which each agent has preference over pairs of activity and number of agents participating in the activity. A number of variants of the games are defined by Darmann et al. (2012). Another class of hedonic games that is based on the number of agents is *Gamson's hedonic game*. This class of hedonic games is of considerable importance in modeling coalition formation in the parliament in which each political party wants to be in a majority coalition in which it has a maximum proportion of seats. Each agent $i \in N$ representing a party has weight $w(i)$. For each coalition $S \subset N$ such that $i \in S$, $v_i(S) = \frac{w(i)}{\sum_{j\in S} w(j)}$ if $\sum_{j\in S} w(j) > \sum_{j\in N} w(j)/2$ and zero otherwise (Le Breton et al., 2008; Deineko and Woeginger, 2014; Gamson, 1961).

Research Issues and Future Directions. An important area of future research is to model and capture realistic scenarios via hedonic games. There is a need to bring together the work on behavioural game theory and mathematical game theory. This may help identify other interesting classes of hedonic games and preference restrictions.

Another issue is that in many realistic scenarios, most agents are part of overlapping coalitions. Although there is interesting work on overlapping coalitions in transferable utility cooperative game theory (see, e.g., Chalkiadakis et al., 2010), there is scope for more work on overlapping coalitions in hedonic games. We have focussed on outcomes in which each agent is in one of the coalitions of the partition. The setting can be generalised to allow agents to be partial members of various coalitions. This could, for example, represent the proportion of time different coalitions are formed. Formally, a *fractional hypergraph matching* is a function w assigning non-negative weights to coalitions such that $\sum_{S\in\mathcal{N}_i} w(S) \leq 1$ for all $i \in N$. A fractional hypergraph matching is *stable* if for every $S \in 2^N$, there exists an $i \in S$ such that $\sum_{\substack{i\in T\in\mathcal{N}_i \\ T\succsim_i S}} w(T) = 1$. Aharoni and Fleiner (2003) used a connection with Scarf's Lemma to show that a stable fractional hypergraph matching is guaranteed to exist. In general, the complexity of computing a fractional stable matching is PPAD-complete (Kintali et al., 2009). There are other ways to define stability for fractional hypergraph matchings (Manjunath, 2013) and each of the concepts leads to corresponding computational problems.

Although hedonic games have been examined computationally, their algorithmic treatment has been somewhat piecemeal. The hope is to come up with general algorithms that are not tailor-made for a specific representation of hedonic games and can compute solutions of different classes of games. A plethora of intractability results indicates that a fixed parameter tractability approach (Niedermeier, 2006) may also be fruitful. Finding faster exact exponential algorithms is also a natural avenue (Fomin and Kratsch, 2010). Another research direction is to have logical representations of hedonic games and propose logical characterisations of solution concepts which would enable SAT solvers to compute stable partitions (see, e.g., Aziz et al., 2014a). Finally, given that it is computationally hard to find many types of (exactly) stable outcomes, it is natural to study the computational complexity of finding approximately stable partitions. A first step in this direction for cut and party affiliation games, which are closely related to additively separable hedonic games, was taken by Bhalgat et al. (2010).

Characterising conditions under which stable partitions are guaranteed to exist is one of the main research questions concerning hedonic games. Although various sufficient conditions have been identified in the literature, there is scope for a better understanding of sufficient and necessary conditions. Another interesting question is studying the conditions under which a game has a *unique* stable partition. Pápai (2004) characterised conditions under which a hedonic game has a unique core stable partition. There are various interesting questions regarding the complexity of checking whether the game has a unique core stable partition. It is not clear whether this question is easier or harder than checking the existence of a core stable partition.

Deviation dynamics in uncoordinated matching markets have been examined within computer science (see e.g., Ackermann et al., 2011). The rich landscape of hedonic games provides fertile ground for interesting research on dynamics of deviations.

Finally, the solutions of hedonic games based on graphs can be used as desirable ways to perform network clustering and community detection. Aziz et al. (2014b) suggested core stable and welfare-maximising partitions of the fractional hedonic game corresponding to the graph as an interesting way of clustering the vertices of the network. Further work (see e.g., Bilò et al., 2014) in this area may be of interest to other communities working in network analysis (McSweeney et al., 2014).

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