Natural non-dcpo Domains and f-Spaces

Vladimir Sazonov

Department of Computer Science, the University of Liverpool,
Liverpool L69 3BX, U.K.

Abstract

As Dag Normann has recently shown, the fully abstract model for **PCF** of hereditarily-sequential functionals is not ω-complete (in contrast to the old fully abstract continuous dcpo model of Milner). This is also applicable to a potentially wider class of models such as the recently constructed by the author fully abstract (universal) model for $\mathbf{PCF}^+ = \mathbf{PCF} + \mathbf{pif}$ (parallel **if**). Here we will present an outline of a general approach to this kind of 'natural' domains which, although being non-dcpos, allow considering 'naturally' continuous functions (with respect to existing directed 'pointwise', or 'natural' least upper bounds). There is also an appropriate version of 'naturally' algebraic and 'naturally' bounded complete 'natural' domains which serves as the non-dcpo analogue of the well-known concept of Scott domains, or equivalently, the complete f-spaces of Ershov. It is shown that this special version of 'natural' domains, if considered under 'natural' Scott topology, exactly corresponds to the class of f-spaces, not necessarily complete.

Key words: domain theory, dcpo and non-dcpo domains, Scott topology, algebraic domains, f-spaces, **LCF**, **PCF**, full abstraction, sequentiality
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1 Introduction

The goal of this paper is to present an outline of so-called 'natural' [1] version of domain theory in general setting, where domains are not necessary directed complete partial orders (dcpos). 'Natural' (possibly non-dcpo) domains are a generalization of the concept of dcpo domains, and there is a good reason for introducing such a notion which first appeared in [16] in a special

Email address: Sazonov@liverpool.ac.uk (Vladimir Sazonov).

[1] Note that in this context the term 'natural' has nothing to do with the concept of 'natural transformation' in category theory.

form for describing order theoretic and topological structure of the unique fully abstract model $\{\mathbb{Q}_\alpha\}$ of hereditarily-sequential finite type functionals for **PCF** [1,6,9,16] [2]. As Dag Normann has recently shown [10], this model is not ω-complete (hence non-dcpo). This is also applicable to a potentially wider class of models such as the unique fully abstract model $\{\mathbb{W}_\alpha\}$ of (hereditarily) wittingly consistent functionals for $\mathbf{PCF}^+ = \mathbf{PCF} + \mathbf{pif}$ (parallel **if**); cf. [16] and Note 1 below. [3] Note that until the above mentioned negative result in [10] and further positive results in [16] the domain theoretical structure of such models was essentially unknown. This structure was described in [16] in terms of 'natural' (non-dcpo) domains, in fact, of their special version of 'naturally' algebraic and 'naturally' bounded complete 'natural' domains. This is the non-dcpo analogue of the well-known concept of Scott domains (see e.g. [2]), or equivalently, the complete f_0-spaces of Ershov [3]. Moreover, it is shown that this special version of natural domains, if considered under 'natural' Scott topology, exactly corresponds to the general class of f-spaces, not necessarily complete. This is, in fact, a representation theorem for f-spaces.

The point of using the term 'natural' for these kinds of domains is that in the case of non-dcpos the ordinary definitions of continuity and finite (algebraic) elements via arbitrary directed least upper bounds (lubs) prove to be inappropriate. A new, restricted concept of 'natural' lub is necessary, and it leads to a generalized theory applicable also to non-dcpos. More informally, if some directed least upper bounds do not exist in a partial ordered set D then this can serve as an indication that even some existing least upper bounds can be considered as 'unnatural' in a sense. Although 'natural' lubs for functional domains can also be characterised technically as 'pointwise' (in the well-known sense), using the latter term for generalizing the concepts of continuous functions or finite elements as defined in terms of pointwise lubs is, in fact, somewhat misleading. The term 'pointwise continuous' is in this sense awkward and not intended to be considered literally as 'continuous for each argument value', but rather as 'continuous with respect to the pointwise lubs' which is lengthy. Thus, the more neutral and not so technical term 'natural' is used instead of 'pointwise' to characterise our generalizations of the concepts of continuity, Scott topology and algebraicity (finite elements). Moreover, for general non-functional non-dcpo domains the term 'pointwise' lub does not seem to have the straightforward sense what leads again to the necessity of a neutral term. However we should also note the terminological peculiarity of the

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[2] As to the language **PCF** for sequential finite type functionals see [8,11,13,18]. Note also that the technical part of [16] — the source of considerations of the present paper — is heavily based on [12,13,15].

[3] Wittingly consistent functionals were first introduced in [15] (alongside with sequential functionals) in the framework of the typed full continuous model $\{\mathbb{D}_\alpha\}$ for **PCF** (**LCF**) [18] and its type-free version $\mathbb{D}_\infty \cong [\mathbb{D}_\infty \to \mathbb{D}_\infty] \cong [\mathbb{D}_\infty^\omega \to \mathbb{D}_\iota]$ (arising from a standard inverse limit construction by Scott).

term ‘natural’. For example, the existence of ‘naturally finite but not finite’ elements in such ‘natural’ domains is quite possible (see Hypotheses 1 below concerning sequential functionals). Although the main idea of the current approach has already appeared in [16], it was applied there only in a special situation of typed non-dcpo models with ‘natural’ understood as (hereditarily) ‘pointwise’.

Here we will show that a general non-dcpo domain theory of this kind can be developed almost as smoothly as the usual dcpo domain theory which it generalizes. Now, a posteriori, it might seem that it was a self-evident solution to take a restricted notion of ‘natural’ or pointwise lub (to get a good general description of the fully abstract models of hereditarily-sequential / wittingly consistent functionals for **PCF** and, respectively, $\mathbf{PCF}^{+}$). Indeed, this choice can be suggested by traditional dcpo domain theory where the lub of any set of continuous functions is, in fact, inevitably pointwise. Happily, this approach makes things go well. However, there are also some technicalities related to the necessity of generalising an appropriate version of the Algebraicity Lemma of Milner (our Lemma 5.1 below) and applying a rather involved theory of computational strategies.

Also note that the fact that ‘naturally’ algebraic ‘naturally’ bounded complete ‘natural’ domains prove to be equivalent to (or, more precisely, representations of) long-known topological f-spaces [3] does not diminish the value of the approach via ‘natural’ domains because this approach gives a means of describing these fully abstract models (and possibly those which might appear in other considerations). Thus, the present approach to non-dcpo domain theory is complementary to the topological one advocated in [5]. It also appears that $\{\mathbb{Q}_\alpha\}$ and $\{\mathbb{W}_\alpha\}$ (not considered here in detail as having too complicated definitions; see, however, Note 1 below) present first sufficiently non-trivial and non-artificially obtained examples of non-dcpo f-spaces, thereby giving a new evidence of the importance of this old general concept. [4] It seems that the role and potential of non-dcpo domains was underestimated in the literature, probably because of the lack of convincing and appropriately understood examples like $\mathbb{Q}$ and $\mathbb{W}$ arising by independent considerations.

Organisation. We start in Section 2 with a general theory of natural non-dcpo domains, with the concepts of natural continuity and natural Scott topology and showing that natural domains constitute a Cartesian closed category

[4] Note that, in general, an arbitrary non-dcpo f-space can be obtained from a complete (dcpo) f-space quite easily, just by omitting some arbitrarily chosen non-f-elements [3]. However in practice, such as with the models $\mathbb{Q}$ and $\mathbb{W}$ above, this may be not the most appropriate way in comparison with the approach via natural domains (say, with a particular version of a pointwise lub) where the structure of f-space is ‘naturally’ derived rather than just given.

in two ways, respectively, for monotonic and naturally continuous morphisms. Section 3 is devoted to naturally finite elements and naturally algebraic and naturally bounded complete natural domains, in particular, also for the functional domains $[D \to E]$. However, it is argued that although naturally finite elements have finite tabular form we cannot expect in general that they would behave fully computationally effectively in the non-dcpo case. Section 4 is related with the fact that the natural domain $[D \to E]$ consisting of *all* naturally continuous functions between natural domains D and E plays not the most important role here as in the ordinary domain theory. E.g. we can be interested only in sequential or any other kind of functions. Therefore we present some semi-formal considerations on the case of arbitrary 'typical' $F \subseteq [D \to E]$ induced by the old paper of Milner [8] devoted, however, originally only to the case of dcpos. These considerations are summarised in Section 5 quite formally as a generalization of the Algebraicity Lemma of Milner [8] to the case of non-dcpos and to the 'natural' case which can be used, as in [16], to show that typed λ-models like those of sequential functionals $\{\mathbb{Q}_\alpha\}$ and wittingly consistent functionals $\{\mathbb{W}_\alpha\}$ are naturally continuous natural domains and satisfy the conditions of natural algebraicity and natural bounded completeness. On the other hand, some hypotheses are presented "showing" that the situation with these λ-models is probably more intriguing and less regular. We demonstrate in Section 6 that naturally algebraic naturally bounded complete natural domains serve as representations (or are topologically equivalent to the class) of arbitrary f-spaces. Finally, Section 7 concludes the paper.

2 Natural domains

A non-empty partially ordered set (poset) $\langle I, \leq \rangle$ is called *directed* if for all $i, j \in I$ there is a $k \in I$ such that $i, j \leq k$. By saying that a (non-empty) family of elements x_i in a poset $\langle D, \sqsubseteq \rangle$ is *directed*, we mean that I, the range of i, is a directed poset, and, moreover, the map $\lambda i.x_i : I \to D$ is *monotonic* in i, that is, $i \leq j \Rightarrow x_i \sqsubseteq x_j$. However in general, if it is not said explicitly or does not follow from the context, x_i may denote a not necessarily directed family. Moreover, we will usually omit mentioning the range I of i, relying on the context. Different subscript parameters i and j may range, in general, over different index sets I and J. As usual $\bigsqcup X$ denotes the ordinary least upper bound (lub) of a subset $X \subseteq D$ in a poset D which may exist or not. That is, this is a partial map $\bigsqcup : 2^D \dot{\to} D$ with 2^D denoting the powerset of D. If D has a least element, it is denoted as $\bot_D$ or $\bot$ and called *undefined*.

Definition 2.1

(a) Any non-empty poset $\langle D, \sqsubseteq^D \rangle$ (not necessarily a dcpo) is also called a *domain.*

(b) Recall that a *directly complete partial order* (or dcpo domain) is required to be closed under taking directed least upper bounds $\bigsqcup x_i$.[5]

(c) A *natural pre-domain* is a domain D (in general, non-dcpo) with a partially defined operator of *natural lub* $\biguplus : 2^D \dot{\rightarrow} D$ satisfying the first of the following *four* conditions. It is called a *natural domain* if all these conditions hold:

($\biguplus$1) $\biguplus \subseteq \bigsqcup$. That is, for all sets $X \subseteq D$, if $\biguplus X$ exists (i.e. X is in the domain of $\biguplus$) then $\bigsqcup X$ exists too and $\biguplus X = \bigsqcup X$.

($\biguplus$2) If $X \subseteq Y \subseteq D$, $\biguplus X$ exists, and Y is upper bounded by $\biguplus X$ then $\biguplus Y$ exists too (and is equal to $\biguplus X$).

($\biguplus$3) $\biguplus \{x\}$ exists (and is equal to x).

($\biguplus$4) Let $\{y_{ij}\}_{i \in I, j \in J}$ be an arbitrary non-empty two-parametric family of elements[6] in D. Then the equalities

$$\biguplus_i \biguplus_j y_{ij} = (\biguplus_j \biguplus_i y_{ij} =) \biguplus_{ij} y_{ij} = \biguplus_i y_{ii}$$

hold under the following assumptions:

(1) Assuming all the required internal natural lubs $\biguplus_j y_{ij}$ in $\biguplus_i \biguplus_j y_{ij}$ and one of the external natural lubs $\biguplus_i \biguplus_j y_{ij}$ or $\biguplus_{ij} y_{ij}$ exist, then both exist and the corresponding equality above holds. (The case of $\biguplus_j \biguplus_i y_{ij}$ is symmetrical.)[7]

(2) For the last equality to hold (irrespectively of (1)), the family y_{ij} is required to be directed (and monotonic) in each parameter i and j ranging over the same I ($I = J$), and the existence of any natural lub in this equality implies the existence of the other.

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[5] In general, by $\bigsqcup_i z_i$ we mean $\bigsqcup \{z_i \mid i \in I\}$, and analogously for $\biguplus$ below. We also omit the usual requirement that a dcpo should contain a least element $\bot$.

[6] Although the natural lub is defined in terms of sets $X \subseteq D$, it is simpler and most natural to formulate this clause in terms of families of elements of D. This is just the way how it arose in [16] and works below in this paper. Thus we do not strictly stick here to a pure second-order language. Each family of elements defines a set of elements it ranges over, and it is this what is used. Probably this clause might be formulated in a pure second-order manner, but we did not bother to do that. It is meaningful, natural and works well, anyway (at least in the general framework of ZFC where arbitrary families of elements can be freely considered). Further, in the equalities stated in this clause what matters is only which of the expressions are defined, the equalities themselves following trivially. However, when using this clause we mostly will need just equalities between these lubs.

[7] It follows that for the equality $\biguplus_i \biguplus_j y_{ij} = \biguplus_j \biguplus_i y_{ij}$ to hold it suffices to require that all the internal and either one of the external natural lubs or the joint lub $\biguplus_{ij} y_{ij}$ exist.

This finishes the formal definition. The second part of ($\uplus$4) (directed case) evidently follows from ($\uplus$1), ($\uplus$2), and the following *optional clause* which might be postulated as well.

($\uplus$5) If $X \subseteq Y \subseteq D$, $\biguplus Y$ exists, and X is cofinal with Y[8] then $\biguplus X$ exists too (and $= \biguplus Y$).

(We do not include this clause in the formal definition because ($\uplus$1)–($\uplus$4) are mostly sufficient for our purposes.) In particular, any pre-domain with unrestricted $\biguplus \rightleftharpoons \bigsqcup$ is a natural domain. As an extreme case any *discrete* D with $\sqsubseteq$ coinciding with $=$ and $\biguplus \rightleftharpoons \bigsqcup$ is a natural domain. A related example is any *flat* domain — an extension $D_\perp$ of any discrete D by a least element $\perp$ such that $x \sqsubseteq y \rightleftharpoons x = y \in D \vee x = \perp$ for all $x, y \in D_\perp$. But, as in the case of [16], it may happen that only under a restricted $\biguplus \subseteq \bigsqcup$ a natural domain has some additional nice properties such as 'natural' algebraicity properties discussed below in Section 3. Note that a natural domain is actually a second-order structure $\langle D, \sqsubseteq^D, \biguplus^D \rangle$ in contrast to the ordinary dcpo domains represented as a first-order poset structure $\langle D, \sqsubseteq^D \rangle$ probably satisfying some additional requirements of continuity or algebraicity.

Definition 2.2 *Direct product* of natural (pre-) domains $D \times E$ (or more generally, $\prod_{k\in K} D_k$) is defined by letting $\langle x, y\rangle \sqsubseteq^{D\times E} \langle x', y'\rangle$ iff $x \sqsubseteq^D x' \& y \sqsubseteq^E y'$, and additionally $\biguplus_i \langle x_i, y_i\rangle \rightleftharpoons \langle \biguplus_i x_i, \biguplus_i y_i\rangle$ for any family $\langle x_i, y_i\rangle$ of elements in $D \times E$ whenever each natural lub $\biguplus_i x_i$ and $\biguplus_i y_i$ exists.

Proposition 2.1 *The direct product of natural (pre-) domains is a natural (pre-) domain as well.* □

The poset of all monotonic maps $D \to E$ between any domains ordered pointwise ($f \sqsubseteq^{(D\to E)} f' \rightleftharpoons fx \sqsubseteq^E f'x$ for all $x \in D$) is denoted as $(D \to E)$. We will usually omit the superscripts to $\sqsubseteq$.

Definition 2.3

(a) A monotonic map $f : D \to E$ between natural pre-domains is called *naturally continuous*[9] if $f(\biguplus_i x_i) = \biguplus_i f(x_i)$ for any directed natural lub $\biguplus_i x_i$, assuming it exists (that is, if $\biguplus_i x_i$ exists then $\biguplus_i f(x_i)$ is required to exist and

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[8] i.e. $\forall y \in Y \exists x \in X. y \sqsubseteq x$

[9] Using the adjective 'natural' here and in other definitions below is, in fact, rather annoying. We would be happy to avoid it at all, but we need to distinguish all these 'natural' non-dcpo versions of the ordinary definitions for dcpos relativized to the natural lub $\biguplus$ from similar definitions relativized to the ordinary lub $\bigsqcup$. In principle, if the context is clear, we could omit 'natural', and use this term only when necessary. Another way is to write '$\biguplus$-continuous' vs. '$\bigsqcup$-continuous', etc. to make the necessary distinctions.

satisfy this equality). The set of all (monotonic and) naturally continuous maps $D \to E$ is denoted as $[D \to E]$.

(b) Given an arbitrary family $f_i : D \to E$ of monotonic maps between natural pre-domains, define a *natural lub* $f = \biguplus_i f_i : D \to E$ — also a monotonic map — pointwise, as

$$fx \rightleftharpoons \biguplus_i (f_i x),$$

assuming the latter natural lub exists for all $x \in D$; otherwise $\biguplus_i f_i$ is undefined.

Proposition 2.2 *For any family of naturally continuous maps $f_i : D \to E$ between natural pre-domains the natural lub $f = \biguplus f_i$, if it exists, is a naturally continuous map as well, assuming E is a natural domain.*

Proof. Use the first part of ($\biguplus$4): $f \biguplus_j x_j \rightleftharpoons \biguplus_i (f_i \biguplus_j x_j) = \biguplus_i \biguplus_j (f_i x_j) = \biguplus_j \biguplus_i (f_i x_j) \rightleftharpoons \biguplus_j f x_j$, for x_j any directed family in D having a natural lub (with all other natural lubs evidently existing). □

Definition 2.4 For any non-empty set $F \subseteq (D \to E)$ of monotonic functions between natural pre-domains and a family $f_i \in F$, if the natural lub $\biguplus_i f_i$ exists and is also an element of F then it is denoted as $\biguplus_i^F f_i$; otherwise, $\biguplus_i^F f_i$, is considered as undefined. When defined, $\biguplus_i^F f_i = \bigsqcup_i^F f_i = \bigsqcup_i^{(D \to E)} f_i$. Here $\bigsqcup^F$ denotes the lub relativized to the poset F with the pointwise partial order $\sqsubseteq^F \rightleftharpoons \sqsubseteq^{(D \to E)} \upharpoonright F$. In particular, this gives rise to natural pre-domain $\langle F, \sqsubseteq^F, \biguplus^F \rangle$.

Evidently, $F \subseteq F' \Longrightarrow \bigsqcup_i^{F'} f_i \sqsubseteq \bigsqcup_i^F f_i$ when both lubs exist. In contrast with $\bigsqcup^F$, the natural lub $\biguplus_i^F f_i = \biguplus_i f_i$ is essentially independent on F, except it is required to be in F. We will omit the superscript F when it is evident from the context. Further, it is easy to show (by pointwise considerations) that

Proposition 2.3 *For D and E natural pre-domains, any $F \subseteq (D \to E)$ is (trivially) a natural pre-domain under $\sqsubseteq^F$ and $\biguplus^F$. It is also a natural domain if E is, and, in particular, $(D \to E)$ and $[D \to E]$ are natural domains in this case with $[D \to E]$ closed under (existing, not necessarily directed) natural lubs in $(D \to E)$.*

Proof. Assuming that E is a natural domain, show that F (i.e. $\langle F, \sqsubseteq^F, \biguplus^F \rangle$) is a natural domain too.

($\biguplus$1) is trivial.

($\biguplus$2) For a family of monotonic functions $\{f_j \in F\}_{j \in J}$ and $I \subseteq J$, assume that $\biguplus_{i \in I} f_i \in F$ and $f_j \sqsubseteq \biguplus_{i \in I} f_i$ for all $j \in J$. It follows that for all $j \in J$ and $x \in D$, $f_j x \sqsubseteq \biguplus_{i \in I}(f_i x)$. Therefore, by using ($\biguplus$2) for E, $\biguplus_{j \in J}(f_j x)$ exists for all x in the natural domain E, and hence $\biguplus_{j \in J} f_j$ does exist too in $(D \to E)$ and therefore coincides with $\biguplus_{i \in I} f_i \in F$, as required.

($\uplus 3$) For any f, $(\biguplus\{f\})x = \biguplus\{fx\} = fx$. Thus, $\biguplus\{f\} = f$, as required.

($\uplus 4$) For arbitrary family of functions $f_{ij} \in F$ ($\uplus 4$) reduces to the same in E for $y_{ij} = f_{ij}x$ with arbitrary $x \in D$ as follows.

(1) Indeed, assume all the required internal natural lubs $\biguplus_j f_{ij}$ and one of the external natural lubs $\biguplus_i \biguplus_j f_{ij}$ or $\biguplus_{ij} f_{ij}$ exist and belong to F. Then for all $x \in D$ the corresponding assertion holds for $\biguplus_j f_{ij}x$ and $\biguplus_i \biguplus_j f_{ij}x$ or $\biguplus_{ij} f_{ij}x$, and therefore $\biguplus_i \biguplus_j f_{ij}x = \biguplus_{ij} f_{ij}x$ in E. This pointwise identity implies both existence of the required natural lubs in F and equality between them $\biguplus_i \biguplus_j f_{ij} = \biguplus_{ij} f_{ij}$.

(2) For directed f_{ij}, $i, j \in I$, and one of the natural lubs $\biguplus_j f_{ij}$ or $\biguplus_j f_{ii}$ existing, we evidently have for all $x \in D$ that $f_{ij}x$ is directed in each parameter i and j, and $\biguplus_j f_{ij}x = \biguplus_j f_{ii}x$ holds in E, and therefore both the required lubs exist in F and the equality $\biguplus_j f_{ij} = \biguplus_j f_{ii}$ holds. □

If natural domains D and E are dcpos (with $\biguplus = \bigsqcup$) then the same holds both for $(D \to E)$ and $[D \to E]$, and the latter domain coincides with that of all (usual) continuous functions with respect to arbitrary directed lubs. This way natural domain theory generalizes that of dcpo domains, and we will see that other important concepts of domain theory over dcpos have their counterparts in natural domains with all the ordinary considerations extending quite smoothly to the 'natural' non-dcpo case.

These considerations allow us to construct inductively some natural domains of finite type functionals by taking, for each type $\sigma = \alpha \to \beta$, an arbitrary subset $F_{\alpha\to\beta}$ of monotonic (or only naturally continuous) mappings $F_\alpha \to F_\beta$. More general, we can assume that only some embeddings $F_{\alpha\to\beta} \hookrightarrow (F_\alpha \to F_\beta)$ are given. If we additionally require that these F_σ are closed under λ-definability then the family $\{F_\alpha\}$ is called *typed monotonic order extensional λ-model*. The extensionality condition (corresponding to the above embeddings) means that for all α, β and $f, f' \in F_{\alpha\to\beta}$,

$$f \sqsubseteq f' \iff \forall x \in F_\alpha (fx \sqsubseteq f'x).$$

We require additionally that each F_σ has a least element $\bot_\sigma$ satisfying

$$\bot_{\alpha\to\beta}x = \bot_\beta \text{ for all } x \in F_\alpha.$$

This way, for example, the λ-model of hereditarily-sequential finite type functionals can be obtained. In [16] this was done inductively over level of types with an appropriate definition of sequentially computable functionals as elements of non-dcpo domains

$$\mathbb{Q}_{\alpha_1,\ldots,\alpha_n\to\iota} \subseteq (\mathbb{Q}_{\alpha_1}, \ldots, \mathbb{Q}_{\alpha_n} \to \mathbb{Q}_\iota) \tag{1}$$

over the basic flat domain $\mathbb{Q}_\iota = \mathbf{N}_\bot$, $\mathbf{N} = \{0, 1, 2, \ldots\}$. It was proved only a posteriori and quite non-trivially that all sequential functionals are naturally continuous ($\mathbb{Q}_{\alpha_1,\dots,\alpha_n\to\iota} \subseteq [\mathbb{Q}_{\alpha_1},\dots,\mathbb{Q}_{\alpha_n} \to \mathbb{Q}_\iota]$ and $\mathbb{Q}_{\alpha\to\beta} \hookrightarrow [\mathbb{Q}_\alpha \to \mathbb{Q}_\beta]$), and satisfy further 'natural' algebraicity properties discussed in Section 3 below. It was while determining the domain theoretical nature of $\mathbb{Q}_\alpha$ that the idea of natural domains emerged; and, although this idea proved to be quite simple, it was unclear at that moment whether anything reasonable could be obtained at all. What is new here is a general, abstract presentation of natural domains that does not rely, as in [16], on a type structure like that of $\{\mathbb{Q}_\alpha\}$ with $\uplus^\alpha$ for each $\mathbb{Q}_\alpha$ defined in the hereditarily-pointwise way (cf. Definitions 2.3 (b) and 2.4 above).

Note 1 (Digression on $\{\mathbb{Q}_\alpha\}$ and $\{\mathbb{W}_\alpha\}$) Unfortunately, it would take too much space to consider here the construction of the λ-model $\{\mathbb{Q}_\alpha\}$ — the source of general considerations of this paper. (See also [1,6,9] where the same model was defined in a different way and where its domain theoretical structure was not described; it was even unknown whether it is different from the older dcpo model of Milner [8] which was shown later by Normann [10].)

Although it is not formally necessary for this paper [10], we can present here (rather roughly and imprecisely) the ideas of [16]. The domain $\mathbb{Q}_{\alpha_1,\dots,\alpha_n\to\iota}$ in (1) consists of functionals computable by so called sequential (deterministic) strategies. To compute a functional $qx_1\dots x_n$ on its arguments, the strategy asks sequentially (step-by-step) queries of an appropriate form on the values of the arguments $x_1\dots x_n$ (which are also finite type sequentially computable functionals, by induction). Each query depends on the answers obtained from the previous queries. At some moment (if the process terminates at all) the strategy may decide that the answers retrieved are sufficient to assert that the value of $qx_1\dots x_n$ of the basic type of natural numbers is, say, 5. (Strictly speaking, an Oracle answering these queries is used and it is defined recursively via a fixed point.) As we take only sequentially computable functionals, i.e. not all abstractly computable/continuous ones, the resulting $\{\mathbb{Q}_\alpha\}$ should hardly be a dcpo (and it is indeed non-dcpo according to Normann [10]). It is essential that the definition of sequential computability proceeds inductively, by level of types, so that in a definite sense we avoid taking the quotient (except in proving some properties of the model $\{\mathbb{Q}_\alpha\}$) used in other approaches based on game strategies [1,6,9]. Fortunately, $\{\mathbb{Q}_\alpha\}$ also proves to be a system of natural domains satisfying good non-dcpo domain theoretic properties such as natural continuity, natural algebraicity, etc. (see below). For things to go smoothly in non-dcpos $\mathbb{Q}_\alpha$ it proved fruitful to use hereditarily pointwise lubs $\uplus^\alpha$ for each type α (coinciding with the ordinary lub $\bigsqcup$ for the basic type ι). By the way, it is easy to present an example of two sequential functions whose standard lub exists in corresponding $\mathbb{Q}_\alpha$ (and is a constant zero function), but whose natural (pointwise) lub does not exist in $\mathbb{Q}_\alpha$ as it would be a parallel function (see Example 2.4 in [16]). It is interesting that the above-mentioned

[10] so the reader can freely ignore the end of this note on $\{\mathbb{Q}_\alpha\}$ and $\{\mathbb{W}_\alpha\}$

result and example of Normann is nowhere used in [16] in any technical sense. The only thing used is that we *do not know* whether $\mathbb{Q}_\alpha$ are dcpos and thus need to work more carefully (with pointwise rather than with the ordinary lubs).

Analogously the typed lambda model $\{\mathbb{W}_\alpha\}$ is based on a special form of *wittingly consistent* non-deterministic computational strategies (such as the evident strategy computing **pif** — parallel **if**). The point is that non-deterministic strategies in general can be contradictory. Some non-deterministic computations (unlike deterministic sequential strategies) can lead to different results. But we consider only single-valued functionals. Fortunately, the evident non-deterministic strategy computing **pif**, as well as another strategy computing parallel existential quantification functional $\exists : (\iota \to \iota) \to \iota$ do not lead to contradiction, but some "inconsistent" strategies can, and the latter are excluded from consideration. However, the case of **pif** considerably differs from that of $\exists$. The natural strategy computing **pif** belongs to a special class of non-deterministic strategies called *wittingly consistent* whose behaviour, although non-deterministic, never leads to a contradiction because of a special guarantee: for wittingly consistent strategies, if there are two formal computational histories leading to a contradiction then this must be only due to some contradictory answers from the Oracle (which is impossible if the computation is a real one over some lambda-model). Note that no such wittingly consistent strategy can compute $\exists$ so that this functional lies outside of $\mathbb{W}_{(\iota\to\iota)\to\iota}$ (and thus non-definable in $\mathbf{PCF}^+$, also over $\{\mathbb{D}_\alpha\}$ where $\exists$ exists [11,14,15]; note that [11] used a different technique). Moreover, it is easy to present an increasing sequence $\exists_n$ of wittingly consistent restricted versions of $\exists$ such that $\exists = \bigsqcup_n \exists_n$ [16]. Thus, $\mathbb{W}_{(\iota\to\iota)\to\iota}$ is not a dcpo. This example is similar, but easier than that presented by Normann for the lambda-model of sequential functionals $\{\mathbb{Q}_\alpha\}$. Again, $\mathbb{W}_\alpha$ prove to be natural domains satisfying good continuity and other domain theoretical properties, like $\mathbb{Q}_\alpha$. (See Note 3 below.)

Proposition 2.4 *Let D, E be natural pre-domains and F a natural domain. A two place monotonic function $f : D \times E \to F$ is naturally continuous iff it is so in each argument.*

Proof. 'Only if' is trivial and uses ($\biguplus$3) for F. Conversely, for arbitrary directed families x_i and y_i having natural lubs we have

$$\begin{aligned} f(\biguplus_i \langle x_i, y_i\rangle) &\rightleftharpoons f(\langle \biguplus_i x_i, \biguplus_i y_i\rangle) = \biguplus_i \biguplus_j f(\langle x_i, y_j\rangle) = \biguplus_{ij} f(\langle x_i, y_j\rangle) \\ &= \biguplus_i f(\langle x_i, y_i\rangle), \end{aligned}$$

as required, by applying the natural continuity of f in each argument and using ($\biguplus$4) for F. □

The following Proposition makes the class of natural domains with monotonic, resp., naturally continuous morphisms a Cartesian closed category (ccc) in two corresponding ways.

Proposition 2.5 *There are the natural (in the sense of category theory) order isomorphisms over natural domains preserving additionally in both directions all the existing natural lubs, not necessarily directed* [11] *,*

$$(D \times E \to F) \cong (D \to (E \to F)), \tag{2}$$
$$[D \times E \to F] \cong [D \to [E \to F]]. \tag{3}$$

Moreover, each side of the second isomorphism is a subset of the corresponding side of the first, with embedding making the square diagram commutative.

Proof. Indeed, the isomorphism (2) and its inverse are defined for any $f \in (D \times E \to F)$ and $g \in (D \to (E \to F))$, as usual, by

$$\begin{aligned} f^* &\rightleftharpoons \lambda x.\lambda y.f(x,y) \in (D \to (E \to F)), \\ \hat{g} &\rightleftharpoons \lambda(x,y).gxy \in (D \times E \to F). \end{aligned}$$

Then $\lambda f.f^*$ preserves (in both directions) all the existing natural lubs $(\biguplus_i f_i)^* = \biguplus_i f_i^*$. Indeed,

$$\begin{aligned} (\biguplus_i f_i)^* xy &\rightleftharpoons (\biguplus_i f_i)(x,y) \rightleftharpoons \biguplus_i f_i(x,y) \rightleftharpoons \biguplus_i ((f_i^* x)y) \rightleftharpoons (\biguplus_i (f_i^* x))y \\ &\rightleftharpoons ((\biguplus_i f_i^*)x)y \rightleftharpoons (\biguplus_i f_i^*)xy \end{aligned}$$

holds for all $x \in D$ and $y \in E$ where if the first natural lub exists then all the others exist too, and conversely. Here we used only the definitions of * and $\biguplus$ for functions.

The second isomorphism (3) is just the restriction of the first. For its correctness we should check that f^* (resp. $\hat{g}$) is naturally continuous if f (resp. g) is:

$$f^* \biguplus_i x_i \rightleftharpoons \lambda y.f(\biguplus_i x_i, y) = \lambda y.\biguplus_i f(x_i, y) \rightleftharpoons \biguplus_i \lambda y.f(x_i, y) \rightleftharpoons \biguplus_i f^* x_i$$

by using additionally Proposition 2.4 in the second equality. Similarly,

$$\begin{aligned} \hat{g}(\biguplus_i x_i, \biguplus_i y_i) &\rightleftharpoons g(\biguplus_i x_i)(\biguplus_i y_i) = \biguplus_i g x_i (\biguplus_i y_i) = \biguplus_i \biguplus_j g x_i y_j \\ &= \biguplus_i g x_i y_i \rightleftharpoons \biguplus_i \hat{g}(x_i, y_i) \end{aligned}$$

by using ($\biguplus$4) for F. □

[11] and, of course, preserving the ordinary lubs

Definition 2.5 An upward closed set U in a natural pre-domain D is called *naturally Scott open* if for all directed families x_i having the natural lub

$$\biguplus_i x_i \in U \Longrightarrow x_i \in U \text{ for some } i.$$

Such subsets constitute the *natural Scott topology* on D.

This is a straightforward generalization of the ordinary *Scott topology* on any poset defined in terms of the usual lub $\bigsqcup$ of directed families. Evidently, each Scott open set (in the standard sense) is naturally Scott open, and therefore the latter sets constitute a T_0-topology.

Proposition 2.6

(a) *Any natural pre-domain $\langle D, \sqsubseteq^D, \biguplus^D \rangle$ is a T_0-space under its natural Scott topology whose standardly generated partial ordering coincides with the original ordering $\sqsubseteq^D$ on D.*

(b) *Continuous functions $f \in [D \to E]$ between pre-domains defined as preserving the existing natural lubs are also continuous relative to the natural Scott topologies in D and E.*

(c) *But the converse holds only in the weakened form: continuity of a map f in the sense of natural Scott topologies implies $f(\biguplus_i x_i) = \bigsqcup_i f(x_i)$ for any directed family x_i with existing $\biguplus_i x_i$.* [12]

Proof.

(a) If $x \sqsubseteq y$ and $x \in U$ for any naturally Scott open $U \subseteq D$ then $y \in U$ because U is upward closed. Conversely, assume $x \not\sqsubseteq y$, and define $U_y \rightleftharpoons \{z \in D \mid z \not\sqsubseteq y\}$. This set is evidently upward closed. Let $\biguplus_i x_i \in U_y$ for a directed family. Then it is impossible that all $x_i \notin U_y$, i.e. $x_i \sqsubseteq y$, because then we should have $\biguplus_i x_i \sqsubseteq y$ — a contradiction. Therefore U_y is a naturally Scott open set (in fact, even Scott open in the standard sense) such that $x \not\sqsubseteq y$, $x \in U_y$ but $y \notin U_y$, as required.

(b) Assume monotonic $f : D \to E$ preserves natural directed lubs and $U \subseteq E$ is naturally Scott open in E. Then $f^{-1}(U)$ is evidently upward closed in D as U is such in E. Further, let $\biguplus_i x_i \in f^{-1}(U)$, i.e. $f(\biguplus_i x_i) = \biguplus_i f(x_i) \in U$ and hence $f(x_i) \in U$ and $x_i \in f^{-1}(U)$ for some i. Therefore $f^{-1}(U)$ is naturally Scott open. That is, f is continuous in the sense of natural Scott topologies in D and E.

(c) Conversely, assume $f : D \to E$ is continuous in the sense of natural Scott topologies in D and E, and $\biguplus_i x_i$ exists in D for a directed family. Let

[12] In the special case of $\biguplus \rightleftharpoons \bigsqcup$ and standard Scott topologies we have, as usual, the full equivalence of the two notions of continuity of maps with $f(\biguplus_i x_i) = \biguplus_i f(x_i)$. We will see below that the full equivalence of these two notions of continuity holds also for naturally algebraic and naturally bounded complete natural pre-domains.

us show that $f(\biguplus_i x_i) = \bigsqcup_i f(x_i)$. The inequality $f(\biguplus_i x_i) \sqsupseteq f(x_i)$ follows by monotonicity of f. Assume y is an upper bound of all $f(x_i)$ in E but $f(\biguplus_i x_i) \not\sqsubseteq y$. Define like above the Scott open set $V_y \rightleftharpoons \{z \in E \mid z \not\sqsubseteq y\}$. Then $f^{-1}(V_y)$ is naturally Scott open containing $\biguplus_i x_i$ and therefore some x_i, implying $f(x_i) \in V_y$, i.e. $f(x_i) \not\sqsubseteq y$ — a contradiction. This means that $f(\biguplus_i x_i) = \bigsqcup_i f(x_i)$. □

3 Naturally finite elements

Definition 3.1 A *naturally finite* element d in a natural pre-domain D is such that for any directed natural lub (assuming it exists) if $d \sqsubseteq \biguplus X$ then $d \sqsubseteq x$ for some x in X. If arbitrary directed lubs $\bigsqcup X$ are considered in arbitrary (either dcpo or non-dcpo) domain D then d is called just *finite*. Let $D^{[\omega]}$ denote the set of naturally finite elements of D.

The part of the definition on (simply) finite elements is most reasonable in the case of dcpos. For non-dcpos (if $\biguplus = \bigsqcup$ is not assumed), 'finite' could be read for definiteness as 'non-natural finite'.

Definition 3.2 A natural pre-domain D is called *naturally (ω-) algebraic* if (it has only countably many naturally finite elements and) each element in D is a natural lub of a (non-empty) directed set of naturally finite elements.

If D is dcpo with $\biguplus = \bigsqcup$ then the above reduces to the traditional concept of *(ω-) algebraic dcpo*. It follows for naturally algebraic pre-domains D satisfying additionally ($\biguplus$2) (or for natural domains), that

$$x = \biguplus \hat{x} \tag{4}$$

where $\hat{x} \rightleftharpoons \{d \sqsubseteq x \mid d \text{ is naturally finite}\}$ for any $x \in D$.

Definition 3.3 If any two upper bounded elements c, d have least upper bound $c \sqcup d$ in D then D is called *bounded complete*, and it is called *finitely bounded complete* if, in the above, only finite c, d (and therefore $c \sqcup d$) are considered.

This is the traditional definition adapted to the case of an arbitrary poset D. If D is an algebraic dcpo then it is bounded complete iff it is finitely bounded complete. In fact, for dcpos bounded completeness is equivalent to existence of a lub for any bounded set, not necessarily finite. Algebraic and bounded complete dcpos with least element $\perp$ are also known as *Scott domains* [2] or as the *complete f_0-spaces* of Ershov [3]. For the 'natural', non-dcpo version of these domains we need

Definition 3.4 A natural pre-domain D is called *naturally bounded complete* if any two naturally finite elements upper bounded in D have a lub (not necessarily natural lub, but evidently naturally finite element).

In such domains any set of the form $\hat{x}$ is evidently directed, if non-empty. (It is indeed non-empty in naturally algebraic pre-domains.)

Lemma 3.1 *For a naturally algebraic natural domain D the natural lub of an arbitrary family x_i can be represented as*

$$\biguplus_{i\in I} x_i = \biguplus \bigcup_{i\in I} \hat{x}_i \tag{5}$$

where both natural lubs either exist or not simultaneously.

Proof. The case of empty I is trivial. Otherwise, let $x_i^0 \sqsubseteq x_i$ denote an arbitrarily chosen naturally finite approximation of x_i, and let j range over the set $J = D^{[\omega]}$ of naturally finite elements of D. Define $x_{ij} \rightleftharpoons j$ if $j \sqsubseteq x_i$, and $\rightleftharpoons x_i^0$ otherwise, so that $\{x_{ij} \mid j \in J\} = \hat{x}_i$. Then $\biguplus_i x_i = \biguplus_i \biguplus \hat{x}_i = \biguplus_i \biguplus_j x_{ij} = \biguplus_{ij} x_{ij} = \biguplus \bigcup_i \hat{x}_i$ by (4) and the first part of ($\biguplus$4). $\square$

Therefore, any naturally algebraic natural domain D is, in fact, defined by the quadruple $\langle D, D^{[\omega]}, \sqsubseteq^D, \mathcal{L}\rangle$ where $\mathcal{L} \subseteq 2^{D^{[\omega]}}$ is the set of all sets of naturally finite elements having a natural lub. Indeed, we can recover $\biguplus_i x_i \rightleftharpoons \bigsqcup \bigcup_i \hat{x}_i$ by (5) whenever $\bigcup_i \hat{x}_i \in \mathcal{L}$. Moreover, in the case of naturally algebraic and naturally bounded complete natural domains D their elements x can be identified, up to the evident order isomorphism, with the *ideals* $\hat{x} \in \mathcal{L}$ (non-empty directed downward closed sets of naturally finite elements) ordered under set inclusion and having a natural lub. In particular,

$$x \sqsubseteq y \iff \hat{x} \subseteq \hat{y}. \tag{6}$$

Proposition 3.1

(a) For D and E naturally algebraic and naturally bounded complete natural pre-domains, a monotonic map $f : D \to E$ is naturally continuous (in the sense of preserving directed natural lubs) iff for all $x \in D$ and naturally finite $b \sqsubseteq fx$ there exists naturally finite $a \sqsubseteq x$ such that $b \sqsubseteq fa$. This means that natural continuity of functions between such domains is equivalent to topological continuity with respect the natural Scott topology[13] *because*

(b) Naturally Scott open sets in such domains are exactly arbitrary unions of the upper cones $\check{a} \rightleftharpoons \{x \mid a \sqsubseteq x\}$ for a naturally finite.

[13] This improves part (c) of Proposition 2.6 (see also Footnote 12).

Proof.

(a) Indeed, for f naturally continuous, $fx = \biguplus f(\hat{x})$, so $b \sqsubseteq fx$ implies $b \sqsubseteq fa$ for some $a \sqsubseteq x$ for naturally finite a, b.

Conversely, assume f satisfies the above b-a-continuity property and $x = \biguplus_i x_i$ be a natural directed lub in D. Let us show that $fx = \biguplus_i fx_i$. The inclusions $fx_i \sqsubseteq fx$ hold by monotonicity of f and imply $\bigcup_i \widehat{fx_i} \subseteq \widehat{fx}$. Now, it suffices to show, by (5) and (6) applied to E, the inverse inclusion $\widehat{fx} \subseteq \bigcup_i \widehat{fx_i}$. Thus, assume $b \sqsubseteq fx$ for a naturally finite b and hence $b \sqsubseteq fa$ for some naturally finite $a \sqsubseteq x = \biguplus_i x_i$ and, therefore, $a \sqsubseteq x_i$ for some i. Then $b \sqsubseteq fa \sqsubseteq fx_i$, as required.

(b) This follows straightforwardly from the definitions of naturally finite elements, naturally Scott open sets, and from the identity $x = \biguplus \hat{x}$ (with $\hat{x}$ directed). □

This proposition also means that non-dcpo domains considered are actually f-spaces [3]. (See Section 6 and Theorem 6.1 below.) Further generalizing the traditional dcpo case and working in line with the theory of f-spaces, we can show

Proposition 3.2 *If natural domains D and E are naturally (ω-)algebraic and naturally bounded complete then so are $D \times E$ and $[D \to E]$, assuming additionally in the case of $[D \to E]$ that E contains a least element $\bot_E$. Then such a restricted class of domains with $\bot$ and with naturally continuous maps as morphisms constitute a ccc.*

Proof. For $D \times E$ this is evident. Let us show this for $[D \to E]$. Indeed, let $a_0, \ldots, a_{n-1} \in D$ and $b_0, \ldots, b_{n-1} \in E$ be two arbitrary lists of naturally finite elements satisfying the

Consistency condition: for any $x \in D$ the set $\{b_i \mid a_i \sqsubseteq x, i < n\}$ is upper bounded in E, and hence its lub exists and is a naturally finite element.

(In general, assume that $a, b, c, d, \ldots$, possibly with subscripts, range over naturally finite elements.) Then define a *tabular* function $\left[\begin{smallmatrix} b_0,\ldots,b_{n-1} \\ a_0,\ldots,a_{n-1} \end{smallmatrix}\right] \in [D \to E]$ by taking for any $x \in D$

$$\left[\begin{smallmatrix} b_0,\ldots,b_{n-1} \\ a_0,\ldots,a_{n-1} \end{smallmatrix}\right] x \rightleftharpoons \bigsqcup\{b_i \mid a_i \sqsubseteq x, i < n\} \tag{7}$$

because this lub does always exist. (Here we use the fact that E contains a least element $\bot_E$ needed to get the lub defined if the set on the right-hand side is empty.) In particular, $\left[\begin{smallmatrix} b_0,\ldots,b_{n-1} \\ a_0,\ldots,a_{n-1} \end{smallmatrix}\right]$ is the least monotonic function $f : D \to E$ for which $b_i \sqsubseteq fa_i$ for all $i < n$, that is,

$$\left[\begin{smallmatrix} b_0,\ldots,b_{n-1} \\ a_0,\ldots,a_{n-1} \end{smallmatrix}\right] \sqsubseteq f \iff b_i \sqsubseteq fa_i \text{ for all } i < n. \tag{8}$$

Moreover, this is also a naturally continuous function. Indeed, for any directed family $\{x_k\}_{k\in K}$ in D with the natural lub existing

$$\begin{bmatrix} b_0,\ldots,b_{n-1} \\ a_0,\ldots,a_{n-1} \end{bmatrix} \biguplus_k x_k = \bigsqcup\{b_i \mid a_i \sqsubseteq \biguplus_k x_k\} = \begin{bmatrix} b_0,\ldots,b_{n-1} \\ a_0,\ldots,a_{n-1} \end{bmatrix} x_{k_0}$$

for some $k_0 \in K$ (due directedness of $\{x_k\}_{k\in K}$) so that, in fact, $\begin{bmatrix} b_0,\ldots,b_{n-1} \\ a_0,\ldots,a_{n-1} \end{bmatrix} x_k \sqsubseteq \begin{bmatrix} b_0,\ldots,b_{n-1} \\ a_0,\ldots,a_{n-1} \end{bmatrix} x_{k_0}$ for all $k \in K$ and hence, by ($\uplus$2) and ($\uplus$3) for E,

$$\begin{bmatrix} b_0,\ldots,b_{n-1} \\ a_0,\ldots,a_{n-1} \end{bmatrix} \biguplus_k x_k = \biguplus_k \begin{bmatrix} b_0,\ldots,b_{n-1} \\ a_0,\ldots,a_{n-1} \end{bmatrix} x_k.$$

It is also follows from (8) that $\begin{bmatrix} b_0,\ldots,b_{n-1} \\ a_0,\ldots,a_{n-1} \end{bmatrix} = \bigsqcup_{i<n} \begin{bmatrix} b_i \\ a_i \end{bmatrix}$. Moreover, this is a naturally finite element in $[D \to E]$. Thus, in the simplest case of $\begin{bmatrix} b \\ a \end{bmatrix}$

$$\begin{bmatrix} b \\ a \end{bmatrix} \sqsubseteq \biguplus_j f_j \overset{(8)}{\iff} b \sqsubseteq \biguplus_j f_j a \iff \exists j. b \sqsubseteq f_j a \iff \exists j. \begin{bmatrix} b \\ a \end{bmatrix} \sqsubseteq f_j$$

for any directed family of naturally continuous functions f_j with $\biguplus_j f_j$ and therefore $\biguplus_j f_j a$ existing. If $\begin{bmatrix} b_0,\ldots,b_{n-1} \\ a_0,\ldots,a_{n-1} \end{bmatrix} \sqsubseteq f$ and $\begin{bmatrix} d_0,\ldots,d_{m-1} \\ c_0,\ldots,c_{m-1} \end{bmatrix} \sqsubseteq f$ then evidently $\begin{bmatrix} b_0,\ldots,b_{n-1} \\ a_0,\ldots,a_{n-1} \end{bmatrix} \bigsqcup \begin{bmatrix} d_0,\ldots,d_{m-1} \\ c_0,\ldots,c_{m-1} \end{bmatrix} = \begin{bmatrix} b_0,\ldots,b_{n-1},d_0,\ldots,d_{m-1} \\ a_0,\ldots,a_{n-1},c_0,\ldots,c_{m-1} \end{bmatrix} \sqsubseteq f$. Thus, the set $\hat{f}$ of tabular approximations to any monotonic function f is directed. Moreover, any naturally continuous f is, in fact, the natural lub of this set:

$$f = \biguplus \hat{f} = \biguplus\{\varphi \mid \varphi \sqsubseteq f \& \varphi \text{ tabular}\} \tag{9}$$

because

$$\begin{aligned} fx &= \biguplus \widehat{fx} = \biguplus\{b \mid b \sqsubseteq fx\} = \biguplus\{b \mid \exists \text{ naturally finite } a \sqsubseteq x \ (b \sqsubseteq fa)\} \\ &= \biguplus\{\begin{bmatrix} b \\ a \end{bmatrix} x \mid \begin{bmatrix} b \\ a \end{bmatrix} \sqsubseteq f\} = \biguplus\{\varphi x \mid \varphi \sqsubseteq f \& \varphi \text{ tabular}\}. \end{aligned}$$

The last equality holds because for any tabular function $\varphi \in [D \to E]$ and $x \in D$ we have $\varphi x = \begin{bmatrix} b \\ a \end{bmatrix} x$ for some $\begin{bmatrix} b \\ a \end{bmatrix} \sqsubseteq \varphi$ (where, in accordance with (7), $\begin{bmatrix} b \\ a \end{bmatrix}$ does not necessary is one of the columns of the tabular representation of φ). It also follows from (9) that tabular elements of $[D \to E]$ are exactly the naturally finite ones. Moreover, this domain is naturally (ω-)algebraic and naturally bounded complete. □

Note 2 (on effectiveness) For any finite list of tabular elements $\varphi_1, \ldots, \varphi_k$ in $[D \to E]$, they are upper bounded in $[D \to E]$ iff the evident union of tables representing φ_i is consistent in the above sense. This reduces, essentially algorithmically, the problem of consistency of tables and upperboundedness for naturally finite elements in $[D \to E]$ to those in D and E. But if we would consider a subset of $F \subseteq [D \to E]$ (say, of sequential or other kind of restricted function(al)s, as in [16]) then no such algorithmic reduction for F is possible a priori, even if F is naturally algebraic and naturally bounded complete and its naturally finite elements are represented in the tabular way as above. (See also Section 4 on the typical functional domains F.) We refer the reader to [16, Section 2.4] for more detailed discussion on efficiency of naturally finite functionals in the fully abstract models $\{\mathbb{Q}_\alpha\}$ and $\{\mathbb{W}_\alpha\}$. A highly relevant undecidability result for $\mathbf{PCF}_2$ essentially dealing with (naturally) finite sequential functionals over the finite basic domain $\{\mathbf{true}, \mathbf{false}, \bot\}$ was also obtained by Loader [7] in the form: "The observational ordering of the finitary parts of PCF is undecidable". The main conclusion is that, unlike the ordinary ω-algebraic bounded complete dcpo domains (see also [5]), the role of naturally finite objects in non-dcpo case is not so straightforward and yet not so clear in defining what is effectively computable higher type functional. The usual requirement that the set of its naturally finite approximations is computably enumerable is not appropriate here. We should rather require effectivity of a 'strategy' computing this functional which does not immediately operate in terms of its naturally finite approximations. Only so called finitary strategies [16] serve as effective approximations of effectively computable strategies and as representations of naturally finite objects approximating the given computable functional. We cannot go into deeper details here.

4 Semi-formal considerations on "typical" $F \subseteq [D \to E]$

This section is mostly provoked by the work of Milner [8] which was devoted, however, only to the case of dcpos. Here most of our assertions will have a conditional character with intuitively appealing assumptions. Let $F \subseteq (D \to E)$ be a 'typical' natural domain of monotonic functions (for appropriate natural domains D and E, assuming E has $\bot_E$) with the natural lub $\biguplus^F$ understood pointwise. (See Proposition 2.3.) For example, F could consist of naturally continuous sequential or some other kind of 'computable' function(al)s. Postulating the additional requirement of natural continuity and ω-algebraicity property of a 'typical' function domain F looks quite reasonable from the computational perspective. Indeed, informally, only 'finite' fragments of the data — which may also be functional — do matter in a computation. But *the requirement of (natural) bounded completeness might seem questionable in general*. Why should the lub of two (naturally finite) sequential functionals exist at all and be sequentially computable, even if they have a joint upper bound? The same for any other style of 'computability'. However the following intuitive, semi-formal and sufficiently general argumentation in favour of natural bounded completeness can be given and then easily formalised for the case of finite type functionals in Lemma 5.1 of the next section.

The simplest, 'basic' domains D like flat ones may be reasonably postulated to be naturally bounded complete. Also, the greatest lower bound (glb) $x \sqcap y$ of any two elements can be considered computable / natural continuous. (Say, for flat domains we need only (sequential) conditional **if** and appropriately understood equality predicate $=$ to define $\sqcap$.) Then, assuming that F has the most basic computational closure properties, we can conclude that F should also be closed under the naturally continuous operation glb

$$f \sqcap g = (\lambda x \in D \,.\, fx \sqcap gx). \tag{10}$$

Moreover, it seems quite reasonable to assume that the set of naturally finite elements in any 'basic' D is a directed union, $D^{[\omega]} = \bigcup_k D^{[k]}$, of some finite sets $D^{[k]}$ of naturally finite objects which are suitably *finitely restricted* for each k where k (say, $0, 1, 2, \ldots$) may serve as a measure of restriction. For each $D^{[k]} \subseteq D$ we could expect that each $x \in D$ has a best naturally finite lower approximation $x^{[k]} = \Psi^{[k]}x \sqsubseteq x$ from $D^{[k]}$, assuming also $\Psi^{[k]}(\Psi^{[k]}x) = \Psi^{[k]}x$. Thus, $\Psi^{[k]}_D : D \to D$ is just a monotonic projection onto its finite range $D^{[k]}$. It easily follows that the family $\{x^{[k]}\}_k$ is directed for any $x \in D$. Also it is a reasonable assumption that such $\Psi^{[k]}_D$, for the basic domains, are computable (in fact, definable from **if** and $=$ in the case of flat domains) and therefore naturally continuous.

Then we can deduce that each finitely restricted element $x^{[k]}$ is naturally finite as follows: $x^{[k]} \sqsubseteq \biguplus Z$ for a directed set Z implies $x^{[k]} \sqsubseteq \biguplus\{z^{[k]} \mid z \in Z\} = z^{[k]} \sqsubseteq z$ for some z by monotonicity and natural continuity of $\Psi^{[k]}$, and because $D^{[k]}$ is finite.

Further, we could additionally assume that $x = \biguplus_k x^{[k]}$ holds for all x. This implies formally (from our assumptions) that naturally finite and finitely restricted (i.e., of the form $x^{[k]}$) elements in D are the same.

It follows that any two upper bounded finitely restricted elements $d, e \in D^{[k]}$ must have a (not necessarily natural) lub $d \sqcup e$ in D which is also finitely restricted. Indeed, it can be obtained as the greatest lower bound in D of a finite nonempty set:

$$d \sqcup e = \sqcap\{x^{[k]} \mid x \sqsupseteq d, e\}. \tag{11}$$

By induction, given any (not necessary 'basic') naturally ω-algebraic and naturally bounded complete domains D and E with such computable / natural continuous projections, we should conclude that the composition $\Psi^{[k]}_E \circ f \circ \Psi^{[k]}_D$, denoted as $\Psi^{[k]}_F f$ or $f^{[k]}$ ($f^{[k]}x \rightleftharpoons (fx^{[k]})^{[k]}$), is computable / naturally continuous, assuming $f \in F \subseteq [D \to E]$ is such. Assuming that 'typical' F has minimal reasonable closure properties, we can conclude that this composition should belong to F as well. But, once all $D^{[k]}$ and $E^{[k]}$ are finite sets consisting only of naturally finite elements, so defined $\Psi^{[k]}_F f$ is evidently just a naturally finite tabular function, which can be reasonably postulated as *k-restricted* in F, and $\Psi_F^{[k]} : F \to F$ is the corresponding directed family of projections having finite ranges $F^{[k]}$ consisting of some tabular k-restricted functions.

Projections $\Psi_F^{[k]}$ are naturally continuous and, moreover, preserve all existing natural lubs (not necessarily directed) assuming $\Psi_D^{[k]}$ and $\Psi_E^{[k]}$ do:

$$
\begin{aligned}
(\Psi_E^{[k]} \circ (\biguplus_i f_i) \circ \Psi_D^{[k]})x &= \Psi_E^{[k]}((\biguplus_i f_i)(\Psi_D^{[k]} x)) = \Psi_E^{[k]}(\biguplus_i (f_i(\Psi_D^{[k]} x))) \\
= \biguplus_i \Psi_E^{[k]}(f_i(\Psi_D^{[k]} x)) &= \biguplus_i ((\Psi_E^{[k]} \circ f_i \circ \Psi_D^{[k]})x) = (\biguplus_i (\Psi_E^{[k]} \circ f_i \circ \Psi_D^{[k]}))x.
\end{aligned}
$$

Moreover, having that $F \subseteq [D \to E]$ consists of only naturally continuous functions, $f = \uplus_k f^{[k]}$ should hold for all $f \in F$. Indeed, this follows from the same property in D and E: $fx = f(\uplus_k x^{[k]}) = \uplus_k (fx^{[k]}) = \uplus_k \uplus_m (fx^{[k]})^{[m]} \stackrel{(\biguplus 4)}{=} \uplus_k (fx^{[k]})^{[k]} = \uplus_k (f^{[k]} x)$.

Then we can conclude that the tabular functions (of the form $f^{[k]}$ for any $f \in F$) are exactly the naturally finite elements of the natural domain F, and F is naturally ω-algebraic.

Finally, having projections $\Psi_F^{[k]}$ and naturally continuous finite glb $\sqcap$ in F (definable by induction as in (10) and therefore existing in F by using reasonable closure properties), natural bounded completeness of F follows exactly as above in (11) for the case of 'basic' domains.

It easily follows from the above considerations and assumptions that to specify a 'typical' naturally ω-algebraic and naturally bounded complete natural domain $F \subseteq [D \to E]$, with both $\sqsubseteq^F$ and $\uplus^F$ understood pointwise and restricted to F, we can fix an appropriate bounded complete subposet $\langle F^{[\omega]}, \sqsubseteq^{F^{[\omega]}} \rangle$ of tabular elements in $[D \to E]$ (with $\sqsubseteq^{F^{[\omega]}} = \sqsubseteq^{[D \to E]} \upharpoonright F^{[\omega]}$) containing $\bot_{[D \to E]} \rightleftharpoons \lambda x \in D \,.\, \bot_E$, and take F to be any extension of $F^{[\omega]}$ by some (if exist in $[D \to E]$) directed natural lubs of these tabular elements (or just appropriate ideals). However this style of specification of F might not work practically because $F^{[\omega]}$ may be unknown in advance. In [16] we used computational strategies (which in particular guaranteed the necessary computational and closure properties) instead of $F^{[\omega]}$ to define such functional domains.

5 Non-dcpo version of Algebraicity Lemma of Milner and domain theoretic properties of $\{\mathbb{Q}_\alpha\}$ and $\{\mathbb{W}_\alpha\}$

The above semi-formal considerations can be summarised quite formally as actually proving the following Lemma [16] which is a generalization of the Algebraicity Lemma of Milner in [8] to the case of natural non-dcpo domains, but formulated for simplicity only for the models with the numerical basic values $F_\iota = \mathbf{N}_\perp$. It clearly demonstrates that the non-dcpo generalizations introduced are quite adequate and natural.

Lemma 5.1 *Let $\{F_\alpha\}$ be any monotonic, order extensional λ-model with $F_\iota = \mathbf{N}_\perp$, all F_α being natural domains and $\biguplus^{F_\alpha}$ defined hereditarily-pointwise starting with $\biguplus^{F_\iota} = \bigsqcup^{F_\iota}$, which contains first order equality $=_\iota$ and the conditional operator* **if**. *Then*

(a) This model is naturally continuous (i.e. the application operator App $\rightleftharpoons \lambda fx\,.\,fx$ *is naturally continuous in both arguments for all types) if, and only if,*
 (*) for any type $\alpha = \alpha_1, \ldots, \alpha_n \to \iota$ and elements $f \in F_\alpha$ and $\bar{x} \in F_{\bar{\alpha}}$, $f\bar{x} = f\bar{d}$ holds for some finitely restricted $\bar{d} \sqsubseteq \bar{x}$;
(b) If the model is naturally continuous then (i) the naturally finite elements of each F_α are exactly the finitely restricted ones, (ii) $\{F_\alpha\}$ is naturally ω-algebraic, and (iii) it is naturally bounded complete;
(c) Same as (b), but with 'naturally' omitted.

Proof.

(b) follows easily from the above considerations on projections $\Psi_\alpha^{[k]}$ for all types, etc.
(c) It suffices to note that continuous structures (with respect to the ordinary lubs $\bigsqcup^{F_\alpha}$ and pointwise $\sqsubseteq^{F_\alpha}$) are also evidently naturally continuous, and the concepts of directed lubs, and hence of finite functionals in these models, are equivalent to their 'natural' (hereditarily-pointwise) versions. Indeed, for the directed lubs we have $(\bigsqcup_i f_i)x = \bigsqcup_i (f_i x)$ by $\bigsqcup$-continuity of fx in f. Thus, by induction on types, $\bigsqcup_i f_i = \biguplus_i f_i$ for directed families. Note that we do not assume here that F_α are dcpos.
(a) 'If' follows from natural finiteness of all finitely restricted elements (i.e. those having the form $x^{[k]}$). 'Only if' follows from (b). □

Note 3 (Domain theoretic properties of $\mathbb{Q}$ and $\mathbb{W}$) The clause (a) of this Lemma (not considered in [8]) is used in [16] to show that the two models of sequential functionals $\{\mathbb{Q}_\alpha\}$ and wittingly consistent functionals $\{\mathbb{W}_\alpha\}$ are naturally (with the natural lub understood as hereditarily-pointwise) continuous and satisfy the conditions (i)–(iii) from (b). That is, despite these are non-dcpo models, they have quite nice domain theoretic properties in terms of so simple concept of natural lubs. In fact, it was clear before any such considerations that some particular continuity properties do hold for these models. For example, it was evident that the least fixed point operator $\mathbf{Y}$ existing in $\mathbb{Q}$ should satisfy the property $\mathbf{Y}f = \bigsqcup_n f^n\bot$ so that $f(\bigsqcup_n f^n\bot) = \bigsqcup_n f^n\bot = \bigsqcup_n f^{n+1}\bot = \bigsqcup_n f(f^n\bot)$ meaning that f preserves at least the particular directed lub, but that everything is so nice as above (and the necessity to move from $\bigsqcup$ to $\biguplus$) was not recognised so easily.

In such applications of this Lemma the crucial point is that (*) in (a) implies all the essential domain theoretic properties holding for the model considered. Note that in the cases of $\{\mathbb{Q}_\alpha\}$ and $\{\mathbb{W}_\alpha\}$ proving (*), although an intuitively plausible assertion, was not a trivial enterprise (involving a complicated theory of computational strategies).

That $\{\mathbb{Q}_\alpha\}$ is not a dcpo (at the type level 3) was actually prowed by Normann [10]. For $\{\mathbb{W}_\alpha\}$ this was shown (for the level 2) in [16]. There evidently should be much more of negative examples of such character:

Hypotheses 1 ([16]) [14] *It seems quite plausible that in $\{\mathbb{Q}_\alpha\}$ there exist*

(1) a directed non-natural lub,
(2) a naturally finite, but not a finite functional (being a proper directed lub),
(3) a non-continuous (but naturally continuous) functional, and
(4) a naturally finite (and naturally continuous), but not a continuous functional.

We could also expect that

(5) a continuous (and therefore naturally continuous) lambda model exists whose higher type domains are not dcpos.

We see that these hypotheses reveal a terminological problem ("naturally finite, but not finite", etc.). Properly speaking, these are naturally finite functionals which are most naturally considered as full-fledged finite objects in the framework of $\mathbb{Q}$ or $\mathbb{W}$. The same holds for the concepts of natural continuity and natural Scott topology.

6 f-spaces

Consider any T_0-space D — a topological space in which any two different points x, y are separated one from another by some open set U: $x \in U \& y \notin U$ or $y \in U \& x \notin U$. Let $\sqsubseteq$ denote the partial ordering on D generated by the

[14] See also Footnote 15.

T_0-topology:

$$x \sqsubseteq y \rightleftharpoons \forall \text{ open } U(x \in U \Rightarrow y \in U).$$

Definition 6.1 (Ershov, [3]) An element a of a T_0-space D is called *f-element* if the set $\check{a} \rightleftharpoons \{x \in D \mid a \sqsubseteq x\}$ is open. Open sets of the form $\check{a}$ are called f-*sets*. The set of f-elements of D is denoted as $F(D)$. D is called f-*space* if

(1) f-sets constitute a base of the given topology on D (so that open sets in D are exactly arbitrary unions of f-sets), and
(2) intersection of any two f-sets, if non-empty, is also an f-set: $\check{a} \cap \check{b} = \check{c}$. (Then $c = a \sqcup b$ in D.)

f_0-*space* is an f-space D which has a least element $\bot_D$.

Evidently, the *continuous maps* between f-spaces, $f : D \to E$, are characterised by the '*b-a-condition*':

$$\forall x \in D \forall b \in F(D)(b \sqsubseteq fx \Longrightarrow \exists a \in F(D)(a \sqsubseteq x \& b \sqsubseteq fa)). \tag{12}$$

Let us present an evidently equivalent order theoretic version of the above definition.

Definition 6.2 ([3]) An *f-space* is a triple $\langle D, F(D), \sqsubseteq \rangle$ satisfying the following five conditions:

(1) $F(D) \subseteq D$ (the set of f-elements of D),
(2) $\sqsubseteq$ is a partial order on D,
(3) D is f-*bounded complete*: any two f-elements upper bounded in D have the least upper bound, and it is an f-element,
(4) for any $x \in D$ the (directed) set of its approximating f-elements $\hat{x} \rightleftharpoons \{a \in F(D) \mid a \sqsubseteq x\}$ is nonempty,
(5) for any $x, y \in D$, $x \sqsubseteq y \iff \hat{x} \subseteq \hat{y}$, or equivalently $x = \bigsqcup \hat{x}$.

In general, given any $Z \subseteq D$, define $\hat{Z} \rightleftharpoons \bigcup_{z \in Z} \hat{z}$.

Now, Propoposition 3.1 can be rewritten as

Theorem 6.1 *Each naturally algebraic naturally bounded complete natural pre-domain becomes an f-space under the topology consisting of naturally Scott open sets and with f-elements exactly those naturally finite. Moreover, continuous functions between such domains (in the sense of preserving natural directed lubs) are exactly those continuous under the natural Scott topology.* □

Note 4 (on natural domains vs. f-spaces) This gives another order theoretical approach to f-spaces via natural domains and naturally finite elements. The crucial point is that naturally finite elements are a derived concept in terms of the natural lub $\uplus$ whereas f-element in order theoretic Definition 6.2 are postulated as given. In real applications, such as in $\mathbb{Q}$ and $\mathbb{W}$ they are also not given in advance, but an appropriate version of $\uplus$ is easily definable (pointwise).

Now, let us show the inverse of Theorem 6.1, i.e. how any f-space D can be represented as a natural domain. If $Z \subseteq \hat{x}$ is any directed set of f-elements cofinal with $\hat{x}$ $(\forall a \in \hat{x} \exists z \in Z(a \sqsubseteq z))$ then evidently $\bigsqcup Z = \bigsqcup \hat{x} = x$. For a directed $Z \subseteq \hat{x}$ it is possible in principle [15] that $\bigsqcup Z = x$ but Z is not cofinal with $\hat{x}$. In this case we call the lub $\bigsqcup Z = x$ *non-natural*. Otherwise (the case of cofinality), the lub is called *natural* and denoted as $\biguplus Z$. More general,

Definition 6.3 An arbitrary (nonempty) directed set Z in an f-space D is said to have a *standard natural lub* x in D, $\biguplus Z = x$, if $\hat{Z} \rightleftharpoons \bigcup_{z \in Z} \hat{z}$ is cofinal subset of $\hat{x}$.

In topological terms, the standard lub x of a directed set Z is the ordinary lub, $x = \bigsqcup Z$, which is actually a *limit point* of Z. In fact, $\hat{Z}$ above is also a directed set of f-elements, and $\hat{Z} = \hat{x}$ because $\hat{Z}$ is downward closed, and $\biguplus Z = \bigsqcup Z = x$. That is, we have for any directed set Z or directed family z_i:

$$\biguplus Z = x \iff \hat{Z} = \hat{x}, \quad \text{and} \quad \biguplus_i z_i = x \iff \bigcup_i \hat{z}_i = \hat{x}.$$

In this way directed natural lub corresponds to set theoretic union and in this sense it is indeed quite natural. Note that for any $x \in D$ we have $\hat{\hat{x}} = \hat{x}$, that is

$$x = \biguplus \hat{x}.$$

Even more general,

Definition 6.4 An arbitrary (not necessarily directed) set $Z \subseteq D$ in an f-space D is said to have the *standard natural lub* x, $\biguplus Z = x$, if

$$\bar{Z} \rightleftharpoons \{\bigsqcup F \mid F \subseteq \hat{Z}, F \neq \emptyset \text{ is finite}\} \tag{13}$$

is a cofinal subset of $\hat{x}$, or equivalently, $\hat{\bar{Z}} = \hat{x}$ (and therefore $\hat{Z}$ and Z are upperbounded by x, all participating $\bigsqcup F$ exist, and $\bar{Z}$ is directed) [16].

It follows that if the standard natural lub exists, it coincides with the ordinary lub, $x = \biguplus Z = \bigsqcup Z$, that is $\biguplus$ is just a restricted version of $\bigsqcup$. Evidently, $\hat{Z} \subseteq \bar{Z}$, and if Z is directed then $\hat{Z} = \bar{Z} = \hat{\bar{Z}}$. Therefore for directed Z both definitions 6.3 and 6.4 agree.

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[15] Extending our Hypotheses 1, we presume that such an example can be found in $\{\mathbb{Q}_\alpha\}$.

[16] Formally, (13) makes sense even if some $\bigsqcup F$ do not exist in which case such $\bigsqcup F$ generate no elements of this set.

Theorem 6.2 (on 'natural' representation of f-spaces) *The above definition of standard $\uplus$ makes any f-space D a naturally algebraic naturally bounded complete natural domain (satisfying $(\uplus 1)$–$(\uplus 4)$, and additionally $(\uplus 5)$) with the same partial order, f-elements exactly those naturally finite, and open sets exactly those naturally Scott open. Under this representation, topologically continuous functions over f-spaces are exactly those preserving standard natural directed lubs (i.e. preserving topological limits of directed families).*

Proof.

$(\uplus 1)$ is trivial.

$(\uplus 2)$ Use that $\hat{Z}$, $\bar{Z}$ and $\hat{\bar{Z}}$ are monotonic in $Z \subseteq D$ under set inclusion. Thus, if $\hat{\bar{X}} = \hat{x}$, and $Y \supseteq X$ is upperbounded by x then $\hat{\bar{Y}} = \hat{x}$.

$(\uplus 3)$ is evidently satisfied by $\widehat{\{x\}} = \hat{x}$.

$(\uplus 4)$ First note that for any (non-necessarily directed) family z_i for which $z = \biguplus_i z_i$ exists we have by definition

$$\hat{z} = \widehat{(\biguplus_i z_i)} = \overline{\widehat{(\bigcup_i \hat{z}_i)}},$$

so that for a any f-element, $a \in \hat{z} \iff a \sqsubseteq a_{i_1} \sqcup \ldots \sqcup a_{i_m}$ $(m > 0)$ for some $a_{i_r} \in \hat{z}_{i_r}$. Now, assume that either all the participating natural lubs in $\biguplus_i \biguplus_j y_{ij}$ or the lub $\biguplus_{ij} y_{ij}$ exist. Then, in each of these cases, the family y_{ij} is upperbouned, and we have, respectively, two equalities for sets of f-elements:

$$\widehat{(\biguplus_i \biguplus_j y_{ij})} = \overline{\widehat{(\bigcup_i \widehat{\biguplus_j y_{ij}})}} = \overline{\widehat{(\bigcup_i \overline{\widehat{(\bigcup_j \hat{y}_{ij})}})}}, \tag{14}$$

$$\widehat{(\biguplus_{ij} y_{ij})} = \overline{\widehat{(\bigcup_{ij} \hat{y}_{ij})}}. \tag{15}$$

Under any of the above assumptions, the rightmost expressions in (14) and (15) can be shown to be equal sets of f-elements. This is evidently what we need to show for the first part of $(\uplus 4)$. For the inclusion (14) $\subseteq$ (15) assume that a belongs to (14), i.e. $a \sqsubseteq a_{i_1} \sqcup \ldots \sqcup a_{i_m}$ (with the lub existing) for some $a_{i_r} \in \overline{\widehat{(\bigcup_j \hat{y}_{i_r j})}}$. The latter membership can be analogously rewritten as $a_{i_r} \sqsubseteq a_{i_r j_1} \sqcup \ldots \sqcup a_{i_r j_n}$ for some $a_{i_r j_s} \in \hat{y}_{i_r j_s}$. That is, our assumption on a implies $a \sqsubseteq (a_{i_1 j_1} \sqcup \ldots \sqcup a_{i_1 j_n}) \sqcup \ldots \sqcup (a_{i_m j_1} \sqcup \ldots \sqcup a_{i_m j_n})$ for some $a_{i_r j_s} \in \hat{y}_{i_r j_s}$ (with the lub existing as the family y_{ij} is upperbouned), that is a belongs to (15), as required. Conversely, if $a \sqsubseteq a_{i_1 j_1} \sqcup \ldots \sqcup a_{i_m j_n}$ for some $a_{i_r j_s} \in \hat{y}_{i_r j_s}$ then we can group the members of this least upper bound as above what leads to the inclusion (15) $\subseteq$ (14), and hence to the equality.

Finally, the second part of ($\uplus 4$) holds because of ($\uplus 5$) which can be shown as follows.

($\uplus 5$) If X is cofinal subset of Y then $\hat{X} = \hat{Y}$, and hence $\bar{X} = \bar{Y}$ and $\hat{\bar{X}} = \hat{\bar{Y}}$.

The rest of proof consists of the following steps:

- That 'f-elements' = 'naturally finite elements' evidently follows from $\biguplus_i z_i = x \iff \bigcup_i \hat{z}_i = \hat{x}$ for any directed family z_i, and the identity $x = \biguplus \hat{x}$.
- This and the definition of f-space evidently implies that D with the standardly defined $\biguplus$ is naturally algebric and naturally bounded complete.
- The required statement on open sets and topological and order theoretical continuity of functions follows from Proposition 3.1. □

Finally, Theorem 6.1 can be complemented by

Proposition 6.1 *For any naturally algebraic and naturally bounded complete natural domain $\langle D, \sqsubseteq, \biguplus \rangle$ (considered also as f-space with $F(D) = D^{[\omega]}$), the operator $\biguplus$ behaves on the directed families exactly as the standard natural lub. For non-directed families, $\biguplus$ is a restriction of the standard natural lub.*

Proof. Indeed, $\bigcup_i \hat{z}_i = \hat{x}$ implies $\biguplus_i z_i = x$ by using Lemma 3.1 (based on ($\uplus 4$)):

$$\biguplus_i z_i = \biguplus \bigcup_i \hat{z}_i = \biguplus \hat{x} = x$$

Conversely, if $\biguplus_i z_i = x$ holds for a directed family then evidently $\bigcup_i \hat{z}_i = \hat{x}$.

Now, assume $\biguplus_i z_i = x$ holds for an arbitrary family and show that the same holds for the standard natural lub, i.e. $\widehat{\overline{(\bigcup_i \hat{z}_i)}} = \hat{x}$. Indeed, using Lemma 3.1 again and ($\uplus 2$) gives

$$x = \biguplus_i z_i = \biguplus \bigcup_i \hat{z}_i \overset{(\uplus 2)}{=} \biguplus \overline{(\bigcup_i \hat{z}_i)}.$$

Then $\overline{(\bigcup_i \hat{z}_i)}$ is evidently directed and hence $\widehat{\overline{(\bigcup_i \hat{z}_i)}} = \hat{x}$, as required. □

7 Conclusion

Our presentation is that of the current state of affairs and has the peculiarity that really interesting concrete examples of non-dcpo domains (such as those of hereditarily-sequential and wittingly consistent higher type functionals [16]) from which this theory has, in fact, arisen require too much space to be presented here in full detail. The theory is general, but the non-artificial and instructive non-dcpo examples on which it is actually based are rather complicated and in a sense exceptional (dcpo case being more typical and habitual). However we can hope that there will be many more examples where this theory can be used, similarly to the case of dcpos.

One important topic particularly important for applications which was not considered here in depth and which requires further special attention is the possibility of the effective version of naturally algebraic, naturally bounded complete natural domains. Unlike the ordinary dcpo version (cf. also [5]), not everything goes so smoothly here as is noted in connection with the model of hereditarily-sequential functionals in Section 2.4 of [16]; see also Note 2 above. Recall also domain-theoretic Hypotheses 1 of a negative character which require a technical solution, probably non-trivial.

It is also interesting to adapt the theory of natural non-dcpo domains to the case of A-spaces of Ershov [4,5], which are non-dcpo versions of continuous lattices of Scott [17] with $\bot$ and $\top$ elements possibly omitted, likewise it was done above for f-spaces.

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