Lower and Upper Approximations for Depleting Modules of Description Logic Ontologies

William Gatens and Boris Konev and Frank Wolter
University of Liverpool, UK

Abstract. It is known that no algorithm can extract the minimal depleting $\Sigma$-module from ontologies in expressive description logics (DLs). Thus research has focused on algorithms that approximate minimal depleting modules ‘from above’ by computing a depleting module that is not necessarily minimal. The first contribution of this paper is an implementation (AMEX) of such a depleting module extraction algorithm for expressive acyclic DL ontologies that uses a QBF solver for checking conservative extensions relativised to singleton interpretations. To evaluate AMEX and other module extraction approaches we propose an algorithm approximating minimal depleting modules ‘from below’ (which also uses a QBF solver). We present experiments based on NCI (the National Cancer Institute Thesaurus) that indicate that our lower approximation often coincides with (or is very close to) the upper approximation computed by AMEX, thus proving for the first time that an approximation algorithm for minimal depleting modules can be almost optimal on a large ontology. We use the same technique to evaluate locality-based module extraction and a hybrid approach on NCI.

1 Introduction

Module extraction is the task of computing, given an ontology and a signature $\Sigma$ of interest, a subset (called module) of the ontology such that for certain applications that use the signature $\Sigma$ only, the original ontology can be equivalently replaced by the module [16, 17, 15, 13]. In most applications of module extraction it is desirable to compute a small (and, if possible, even minimal) module. In logic-based approaches to module extraction, the most robust and popular way to define modules is via model-theoretic $\Sigma$-inseparability, where two ontologies are called $\Sigma$-inseparable iff the $\Sigma$-reductions of their models coincide. Then, a $\Sigma$-module of an ontology is defined as a $\Sigma$-inseparable subset of the ontology [8, 4]. It is often helpful and necessary to refine this notion of $\Sigma$-module by considering self-contained $\Sigma$-modules (modules that are inseparable from the ontology not only w.r.t. $\Sigma$ but also w.r.t. their own signature) and depleting $\Sigma$-modules (modules such that the remaining axioms in the ontology are inseparable from the empty ontology w.r.t. $\Sigma$ and the signature of the module) [3, 11, 9]. Note that every depleting module is a self-contained module and a module. In all three cases it is often not possible to compute $\Sigma$-modules: by results in [9, 12], for acyclic $\text{ALC}$-TBoxes and general $\mathcal{EL}$-TBoxes it is undecidable whether a given subset of a TBox is a (self-contained, depleting) $\Sigma$-module. The “maximal” description logics (DLs) for which efficient algorithms computing minimal self-contained and depleting $\Sigma$-modules have been developed are acyclic $\mathcal{ELI}$ [9] and DL-Lite [10, 11, 7].

1 For this reason, for module extraction in ontologies given in expressive DLs one has to employ approximation algorithms: instead of computing a minimal (self-contained, depleting) $\Sigma$-module, one computes some (self-contained, depleting) $\Sigma$-module and the main research problem is to minimise the size of the module (or, equivalently, to approximate minimal modules). Currently, the most popular and successful approximation algorithm is based on locality and computes the so-called $\mathcal{T}_{\bot}$-module [18] which is a (possibly not minimal) depleting module. The size of $\mathcal{T}_{\bot}$-modules and the performance of algorithms extracting $\mathcal{T}_{\bot}$-modules has been analysed systematically and in great detail [18]. However, for expressive DLs neither alternative implemented depleting module extraction algorithms nor any lower approximation algorithms for depleting modules were available. So it remained open how large and significant the difference between a $\mathcal{T}_{\bot}$-module and the minimal depleting module is and in how far it is possible to improve upon the approximation obtained by $\mathcal{T}_{\bot}$-modules.

The aim of this paper is to start to fill this gap and investigate how close one can approximate the ideal minimal depleting modules of an ontology by a sound extraction algorithm. To this end, we present algorithms and experiments that extract two approximations of depleting minimal modules of an ontology:

1. an upper approximation which is a depleting module that is possibly larger than the minimal one; and
2. a lower approximation which is contained in the minimal depleting module but which is not guaranteed to be a depleting module.

Clearly, if the upper and lower approximation coincide (or are very close to each other), then we know that the upper approximation coincides with (or comes very close to) the minimal depleting module. In detail, the contribution of this paper is as follows.

1. For the upper approximation, we have extended, optimised, and implemented the depleting module extraction algorithm introduced in [9] for acyclic $\text{ALCQI}$-TBoxes. The implementation (called AMEX) covers repeated concept inclusions (as present in NCI) and uses a QBF solver as an oracle.

AMEX is available from http://www.csc.liv.ac.uk/~wagatens/software/amex.html.

2 An implementation of semantic locality-based $\Delta^\bot$-modules and a comparison between $\mathcal{T}_{\bot}$ and $\Delta^\bot$-modules has been presented in [18]; however, the authors found no significant difference between the two approaches.
For the lower approximation, we introduce an algorithm that extracts the minimal 1-depleting $\Sigma$-module of an arbitrary $\text{ALCQI}$-TBox; that is, the minimal subset $M$ of an ontology $T$ such that $T \setminus M$ is inseparable on singleton interpretations from the empty ontology w.r.t. $\Sigma$ and the signature of the module. Again, the implementation uses a QBF solver as an oracle.

Interestingly, QBF solvers have been used before in module extraction for DL-Lite [10, 11]. However, our application is completely different from their application in [10, 11]. This paper extends the workshop paper [5].

## 2 Preliminaries

We use standard notation from logic and description logic (DL), details can be found in [1]. In a DL, concepts are constructed from countably infinite sets $N_c$ of concept names and $N_r$ of role names using the concept constructors defined by the DL. For example, $\text{ALCQI}$-concepts are built according to the rule

$$C := A \mid \neg C \mid \geq n \cdot r.C \mid C \cap D \mid C \cup D,$$

where $A \in N_c, n$ is a natural number, and $r \in N_r$. As usual, we use the following abbreviations: $\bot$ stands for $A \cap \neg A$, $\top$ denotes $\neg \bot$, $\exists r.C$ denotes $\exists r.C$, $C \cup D$ denotes $\neg\neg(r \cdot C \cap \neg D)$, $\leq n \cdot r.C$ denotes $\neg\neg(n \cdot (n + 1) \cdot r.C)$, and $\neg\neg n \cdot r.C$ stands for $(\neg\neg n \cdot r.C) \cap \neg(\leq n \cdot r.C)$.

A general TBox $T$ is a finite set of axioms, where an axiom can be either a concept inclusion ($CI$) $C \sqsubseteq D$ or a concept equality (CE) $C \equiv D$, where $C$ and $D$ are concepts. A general TBox $T$ is acyclic if all its axioms are of the form $A \sqsubseteq C$ or $A \equiv C$, where $A \in N_c$, no concept name occurs more than once on the left-hand side and $A \notin \Sigma$, for any $A \in N_c$, where $\not\in \Sigma$ is the transitive closure of the relation $\not\in \subseteq N_c \times (N_c \cup N_r)$ defined by setting $A \not\in X$ iff there exists an axiom of the form $A \sqsubseteq C$ or $A \equiv C$ in $T$ such that $X$ occurs in $C$.

The semantics of DLs is given by interpretations $I = (\Delta^I, \cdot^I)$, where the domain $\Delta^I$ is a non-empty set and $\cdot^I$ is an interpretation function that maps each $A \in N_c$ to a subset $\Delta^I$ of $\Delta^I$ and each $r \in N_r$ to a binary relation $\cdot^I \subseteq \Delta^I \times \Delta^I$. The function $\cdot^I$ is inductively expanded to complex concepts in the standard way [1]. An interpretation $I$ satisfies a CI $C \sqsubseteq D$ (written $I \models C \sqsubseteq D$) if $C^I \subseteq D^I$, it satisfies a CE $C \equiv D$ (written $I \models C \equiv D$) if $C^I = D^I$. $T$ is a model of $I$ if it satisfies all axioms in $T$.

A signature $\Sigma$ is a finite subset of $N_c \cup N_r$. The signature sig($C$) (sig($\alpha$), sig($T$)) of a concept $C$ (axiom $\alpha$, TBox $T$, resp.) is the set of concept and role names that occur in $C$ (axiom $\alpha$, TBox $T$, resp.). If sig($C$) $\subseteq \Sigma$ we call $C$ a $\Sigma$-concept. The $\Sigma$-reduce $I|_{\Sigma}$ of an interpretation $I$ is obtained from $I$ by setting $\Delta^{I|_{\Sigma}} = \Delta^I$, and $\cdot^{I|_{\Sigma}} = \cdot^I$ for all $X \in \Sigma$, and $X^{I|_{\Sigma}} = \emptyset$ for all $X \notin \Sigma$. Let $T_1$ and $T_2$ be TBoxes and $\Sigma$ a signature. Then $T_1$ and $T_2$ are $\Sigma$-inseparable, in symbols $T_1 \equiv_{\Sigma} T_2$, if

$$\{I|_{\Sigma} \mid I \models T_1\} = \{I|_{\Sigma} \mid I \models T_2\}.$$

It is proved in [9] that TBoxes $T_1$ and $T_2$ are $\Sigma$-inseparable if, and only if, $T_1 \models \phi$ if and only if $T_2 \models \phi$ holds for any second-order sentence $\phi$ using symbols from $\Sigma$ only. Thus, $\Sigma$-inseparable TBoxes are indistinguishable not only in applications using entailed CIs between $\Sigma$-concepts but also in data access applications with data given in $\Sigma$. We use $\Sigma$-inseparability to define modules.

### 3 Lower Approximation: 1-Depleting Modules

We introduce a lower approximation of depleting $\Sigma$-modules and give an algorithm extracting such approximations from arbitrary $\text{ALCQI}$-TBoxes. The results of this section can be easily extended to arbitrary first-order ontologies.

Assume that $T_1$ and $T_2$ are TBoxes and $\Sigma$ a signature. Then $T_1$ and $T_2$ are $1-\Sigma$-inseparable, in symbols $T_1 \equiv_{1-\Sigma} T_2$, if

$$\{I|_{\Sigma} \mid I \models T_1\} = \{I|_{\Sigma} \mid I \models T_2\}$$

If $T_1$ and $T_2$ are $\Sigma$-inseparable, then they are $1-\Sigma$-inseparable. The following example shows that the converse does not hold.

#### Example 2

Let $T$ be the following subset of NCI:

- Thoracic_Cavity $\sqsubseteq \exists has\_Location.Thorax$
- Pleural_Tissue $\sqsubseteq \forall has\_Location.Thoracic\_Cavity$

and let $\Sigma = \{\text{Pleural\_Tissue, has\_Location}.\text{Thoracic\_Cavity}\}$. Then one can show that $T \equiv_{\Sigma} \emptyset$, but $T \not\equiv_{\Sigma} \emptyset$.

In contrast to $\Sigma$-inseparability which is undecidable, $1-\Sigma$-inseparability can be decided by reduction to the validity of quantified Boolean formulas (QBF). For simplicity, we consider $1-\Sigma$-inseparability between the empty TBox and $\text{ALCQI}$-TBoxes. Given $T$ and $\Sigma$, take a propositional variable $p_A$ for each concept name $A \in \Sigma$ and a (distinct) propositional variable $q_X$ for each symbol $X \in \text{sig}(T) \setminus \Sigma$. Translate concepts $D$ in the signature $\text{sig}(T)$ into propositional formulas $D^I$ by setting

$$A^I = p_A, \text{ for all } A \in \Sigma$$

$$A^I = q_X, \text{ for all } A \in \text{sig}(T) \setminus \Sigma$$

$$(D_1 \sqcap D_2)^I = D_1^I \land D_2^I$$

$$(-D)^I = \neg D^I$$

$$(\geq n \cdot r.D)^I = (\geq n \cdot r\.D)^I = \{q_{\leq n \cdot r.D}, \text{ if } n = 1 \text{ and } r \in \text{sig}(T)\}$$

$$\bot, \text{ else}$$

#### Figure 1.

Minimal 1-depleting module extraction in $\text{ALCQI}$

#### Definition 1

Let $M \subseteq T$ be TBoxes and $\Sigma$ a signature. Then $M$ is a depleting $\Sigma$-module of $T$ if $T \setminus M \equiv_{\Sigma, \text{sig}(M)} \emptyset$.

Every depleting module $M$ of $T$ is inseparable from $T$ for its signature, that is, if $M$ is a depleting $\Sigma$-module of $T$ then $T \equiv_{\Sigma, \text{sig}(M)} M$, and, in particular, $T \equiv_{\Sigma} M$. Thus, a TBox and its depleting $\Sigma$-module can be equivalently replaced by each other in applications which concern $\Sigma$ only. Throughout this paper we use the fact that minimal depleting $\Sigma$-modules of a TBox are uniquely determined [11]. For further discussion of the advantages of depleting modules we refer the reader to [3, 9]. Unfortunately, checking if a subset $M$ of $T$ is a depleting $\Sigma$-module of $T$ for some given signature $\Sigma$ is undecidable already for general TBoxes formulated in $\text{EL}$ and for acyclic $\text{ALC}$-TBoxes [9, 12].
(1) Renal_Pelvis_and_U \subseteq \exists\text{partOf.}K_{\text{and}}U

(2) K_{\text{and}}U_{\text{Neoplasm}} \equiv U_{T_{\text{Neoplasm}}} \setminus (\neg\text{hasSite}K_{\text{and}}U)

(3) Malignt_U.T_{\text{Neoplasm}} \equiv U_{T_{\text{Neoplasm}}} \setminus (\neg\text{hasAbnCell}.Malignt_Cell)

(4) Benignt_U.T_{\text{Neoplasm}} \equiv U_{T_{\text{Neoplasm}}} \setminus (\neg\text{excludesAbnCell}.Malignt_Cell)

Figure 2. TBox for Example 6

Now let $T^1 = \bigwedge [\Sigma \subseteq T] \forall X \in \Sigma, \exists q x. D_{\text{insep}}(X) \rightarrow D_1$ and let $\phi$ denote the sequence of variables $p_{\Delta}, A \in \Sigma$, and $q$ denote the sequence of variables $q_X, X \in \Sigma$.

**Lemma 3** $T \equiv \emptyset$ iff the QBF $\forall T := \exists \phi \exists q T^1$ is valid.

We define 1-depleting $\Sigma$-modules in the same way as depleting $\Sigma$-modules except that 1-$\Sigma$-inseparability replaces inseparability:

**Definition 4** Let $M \subseteq T$ be TBoxes and $\Sigma$ a signature. Then $M$ is a 1-depleting $\Sigma$-module of $T$ if $T \setminus M \equiv \Sigma_{\text{insep}(M)} \emptyset$.

Example 2 shows that there are acyclic $\text{ALCQI}$-TBoxes in which not every 1-depleting $\Sigma$-module is a depleting $\Sigma$-module. Note, however, that using results from [9] one can prove that for acyclic $\text{EL}$-TBoxes 1-depleting $\Sigma$-modules coincide with depleting $\Sigma$-modules.

One can show that 1-$\Sigma$-inseparability is a monotone inseparability relation with the replacement property, as defined in [11]. Thus, it follows from [11] that for any TBox $T$ and signature $\Sigma$ there is a unique minimal 1-$\Sigma$-depleting module $M$ of $T$. Moreover, the minimal 1-depleting $\Sigma$-module of $T$ is always contained in the unique minimal depleting $\Sigma$-module of $T$. Thus, it is always a lower approximation of the minimal depleting $\Sigma$-module of $T$.

Definition 4 can be used directly for a naïve minimal 1-depleting module extraction algorithm which goes through all subsets of $T$ to identify a smallest possible $M$ such that $T \setminus M \equiv \Sigma_{\text{insep}(M)} \emptyset$. Instead, in our implementation we consider an approach based on the notion of a 1-separability causing axiom. Call an $\alpha \in T \setminus M$ 1-separability causing if there exists a $W \subseteq T \setminus M$ such that $\alpha \in W; (W \\setminus \\{\alpha\}) \equiv \Sigma_{\text{insep}(M)} \emptyset; W \neq \Sigma_{\text{insep}(M)} \emptyset$.

Clearly, if $T \setminus M \equiv \Sigma_{\text{insep}(M)} \emptyset$ then $T \setminus M$ contains a 1-separability causing axiom. Now one can show the following

**Theorem 5** Given an $\text{ALCQI}$-TBox $T$ and signature $\Sigma$, the algorithm in Figure 1 computes the unique minimal 1-depleting $\Sigma$-module of $T$.

The algorithm is in polynomial time with each call to a QBF solver treated as a constant time oracle call. To reduce the number of calls to the QBF solver, the search for a 1-separability causing axiom can be implemented as binary search. Hence, in the worst case one performs $\log_2(|T \setminus M|)$ inseparability checks to locate a 1-separability causing axiom.

**Example 6** We apply the algorithm in Figure 1 to the fragment $T$ of NCI given in Figure 2 with $\Sigma$ defined as Malignt_U.T_Neoplasm, K_and_U_Neoplasm, and Renal_Pelvis_and_U. Here ‘K’, ‘U’, and ‘T’ abbreviate ‘kidney’, ‘urter’ and ‘tract’, respectively.

The search for 1-separability causing axioms first establishes that $T \not\equiv \emptyset$. An example showing this is $I$ with $\Delta I = \{d\}$ such that Renal_Pelvis_and_U = Malignt_U.T_Neoplasm = $\{d\}$ and K_and_U_Neoplasm = $\emptyset$. Then no $\mathcal{J}$ with $\mathcal{J} |_{\Sigma} = I |_{\Sigma}$ satisfies $T$.

Then the search splits $T$ into two parts, $T_1 = \{(1), (2)\}$ and $T_2 = \{(3), (4)\}$. We have $T_1 \equiv \emptyset$ and the algorithm ‘grows’ $T_1$ with the upper part of $T_2$ to construct $T_2 = \{(1), (3)\}$. We have $T_2 \not\equiv \emptyset$ and so the algorithm identifies (3) as a 1-separability causing axiom and applies the rule (1-insep). It obtains $M = \{(3)\}$.

The binary search proceeds with $T \setminus M = \{(1), (2), (4)\}$ and determines that $T \setminus M \not\equiv \Sigma_{\text{insep}(M)} \emptyset$. Then it proceeds to identify (2) as a 1-separability causing axiom and after applying the rule (1-insep), sets $M = \{(2), (3)\}$. Finally, the algorithm proceeds with $T \setminus M = \{(1), (4)\}$ and identifies (1) as a 1-separability causing axiom, so $M$ is set to $\{(1), (2), (3)\}$.

The rule (1-insep) does not apply any further and the computation finishes with the minimal 1-depleting module $M = \{(1), (2), (3)\}$.

### 4 Upper Approximations

In this section, we provide an upper approximation algorithm for depleting $\Sigma$-modules in acyclic $\text{ALCQI}$-TBoxes that extends and optimises the algorithm presented in [9]. The algorithm is also based on 1-$\Sigma$-inseparability but uses an additional dependency check to ensure that a depleting module is extracted. We also address the problem of extracting depleting modules from ‘acyclic’ TBoxes with multiple CI’s for a single concept name and of combining depleting module extraction algorithms.

Let $T$ be an acyclic TBox and $\Sigma$ a signature. We say that $T$ has a direct $\Sigma$-dependency if there exists $\{A, X\} \subseteq \Sigma$ with $A \dashv\vdash X$; otherwise we say that $T$ has no direct $\Sigma$-dependencies. Although one can construct TBoxes $T$ and depleting $\Sigma$-modules $M$ of $T$ such that $T \setminus M$ contains direct $\Sigma \cup \Sigma_{\text{insep}(M)}$-dependencies (see [9]), for typical depleting $\Sigma$-modules $M$, the set $T \setminus M$ should not contain direct $\Sigma \cup \Sigma_{\text{insep}(M)}$-dependencies because such dependencies indicate a semantic link between two distinct symbols in $\Sigma \cup \Sigma_{\text{insep}(M)}$. We show that $\Sigma$-inseparability reduces to 1-$\Sigma$-inseparability if one does not have direct $\Sigma$-dependencies. In detail, let for an acyclic TBox $T$ and signature $\Sigma$

$$\text{Lhs}_{\Sigma}(T) = \{A \bowtie C \in T \mid A \in \Sigma \lor \exists X \in \Sigma (X \dashv\vdash A)\}.$$ 

The following lemma is proved in [9] for acyclic $\text{ALCQI}$-TBoxes and generalised here to acyclic $\text{ALCQI}$-TBoxes.

**Lemma 7** Let $T$ be an acyclic $\text{ALCQI}$-TBox. If $T \setminus M$ has no direct $\Sigma \cup \Sigma_{\text{insep}(M)}$-dependencies, then the following conditions are equivalent for every $W \subseteq T \setminus M$:

- $W \equiv \Sigma_{\text{insep}(M)} \emptyset$;
- $\text{Lhs}_{\Sigma}(W) \equiv \Sigma_{\text{insep}(M)} \emptyset$.

The algorithm computing a depleting $\Sigma$-module of acyclic $\text{ALCQI}$-TBoxes is now given in Figure 3. In the algorithm, the extraction of depleting $\Sigma$-modules is broken into the the rule (syn) that checks for direct $\Sigma \cup \Sigma_{\text{insep}(M)}$-dependencies and the rule (1-insep) from the algorithm in Figure 1. It follows from Lemma 7 that if neither (syn) nor...
Input: Acyclic $\mathcal{ALCQI}$ TBox $\mathcal{T}$, Signature $\Sigma$
Output: Minimal Module $\mathcal{M}$ s.t. $\mathcal{T} \setminus \mathcal{M} \equiv_{\Sigma,\sqcap,\sqcup}(\mathcal{M}) \emptyset$ and $\mathcal{T} \setminus \mathcal{M}$ has no direct $\Sigma \cup \operatorname{sig}(\mathcal{M})$-dependencies.
Set $\mathcal{M} := \emptyset$ and apply rules (syn) and (1-insep) exhaustively, preferring rule (syn).

(syn) If an axiom $A \sqsupset C \in \mathcal{T} \setminus \mathcal{M}$ is such that $A \in \Sigma \cup \operatorname{sig}(\mathcal{M})$ and $A \not\sqsubseteq_{\mathcal{M}} X$, for some $X \in (\Sigma \cup \operatorname{sig}(\mathcal{M}))$, then set $\mathcal{M} := \mathcal{M} \cup \{A \sqsupset C\}$

(1-insep) If an axiom $A \sqsupset C \in \mathcal{T} \setminus \mathcal{M}$ is a $1$-separability causing axiom then set $\mathcal{M} := \mathcal{M} \cup \{A \sqsupset C\}$

Figure 3. Module extraction in $\mathcal{ALCQI}$

(1-insep) is applicable then $\mathcal{T} \setminus \mathcal{M} \equiv_{\Sigma,\sqcap,\sqcup}(\mathcal{M}) \emptyset$ and so the output of the algorithm in Figure 3 is a depleting $\Sigma$-module. More precisely, one can show the following characterisation of the extracted module:

Theorem 8: Given an acyclic $\mathcal{ALCQI}$ TBox $\mathcal{T}$ and signature $\Sigma$, the algorithm in Figure 3 computes the unique minimal depleting $\Sigma$-module $\mathcal{T} \setminus \mathcal{M}$ contains no direct $\Sigma \cup \operatorname{sig}(\mathcal{M})$-dependencies.

The algorithm is again in polynomial time with each call to the QBF solver being treated as a constant time oracle call. Note that the minimality condition in the theorem means that for any $\mathcal{M}' \subseteq \mathcal{T}$ such that $\mathcal{T} \setminus \mathcal{M}'$ has no direct $\Sigma \cup \operatorname{sig}(\mathcal{M}')$-dependencies and $\mathcal{T} \setminus \mathcal{M}' \equiv_{\Sigma,\sqcap,\sqcup}(\mathcal{M}') \emptyset$ we have $\mathcal{M} \subseteq \mathcal{M}'$. It is, however, still possible that there exists a $\mathcal{M}'' \subseteq \mathcal{T}$ with $\mathcal{T} \setminus \mathcal{M}'' \equiv_{\Sigma,\sqcap,\sqcup}(\mathcal{M}'') \emptyset$, $\mathcal{M} \not\subseteq \mathcal{M}''$ and such that $\mathcal{T} \setminus \mathcal{M}''$ has some direct $\Sigma \cup \operatorname{sig}(\mathcal{M}'')$-dependencies.

Example 9 (Example 6 continued) We apply the algorithm in Figure 3 to the same TBox and signature as in Example 6. The rule (syn) is not applicable. Therefore, as in Example 6, an application of the rule (1-insep) sets $\mathcal{M} = \{\{3\}\}$ and the rule (syn) immediately adds axioms (1) and (2) to $\mathcal{M}$. Neither (syn) nor (1-insep) apply to $\mathcal{T} \setminus \mathcal{M} = \{\{4\}\}$ and the computation concludes with $\mathcal{M} = \{\{1\}, \{2\}, \{3\}\}$. Thus, the computed depleting $\Sigma$-module of $\mathcal{T}$ coincides with the minimal 1-depleting $\Sigma$-module of $\mathcal{T}$ and our approximation is optimal.

Notice that axiom (4) is neither $\Delta$- nor $\emptyset$-local for $\Sigma \cup \operatorname{sig}(\mathcal{M})$ and so the $\sqsubseteq_{\Delta}$-module of $\mathcal{T}$ w.r.t. $\Sigma$ coincides with $\mathcal{T}$ (see below and [3] for definitions).

It is often the case (e.g., for the NCI Thesaurus) that a real-world ontology satisfies all conditions for acyclic TBoxes with the exception that it contains multiple CIs of the form $A \sqsubseteq C_1, \ldots, A \sqsubseteq C_n$. We call such TBoxes acyclic with repeated CIs and say that $A$ is a repeated concept name. Clearly, one can convert such a TBox into an equivalent acyclic TBox by replacing the repeated CIs with $A \sqsubseteq C_1 \cap \ldots \cap C_n$. However, such an explicit conversion is an unattractive solution for module extraction because if such an axiom is added to a $\Sigma$-module the signature of the module now contains every symbol in the definition of every repeated name instead of the size of the resulting module considerably. The approach we take to handle acyclic TBoxes with repeated CIs is to introduce fresh concept names for different repeated occurrences of a concept name in the left-hand side of concept CIs, extract modules from the resulting acyclic TBox and then substitute away the added concept names.

Input: Acyclic $\mathcal{ALCQI}$ TBox $\mathcal{T}$ with repeated CIs, Signature $\Sigma$
Output: A module $\mathcal{M}$ s.t. $\mathcal{T} \setminus \mathcal{M} \equiv_{\Sigma,\sqcap,\sqcup}(\mathcal{M}) \emptyset$.
Let $\mathcal{T}' := \{B \sqsupset D \in \mathcal{T} \mid B$ is not repeated$\}$. Let $\mathcal{T}^A := \{A' \sqsubseteq C_1 \mid A \sqsubseteq C_1 \in \mathcal{T}\} \cup \{A \in A_1 \cap \ldots \cap A_n\}$. for repeated $A_1, \ldots, A_n$ fresh.
Let $\mathcal{M}'$ be depleting $\Sigma$-module of $\mathcal{T}' \cup \bigcup A$ is repeated $\mathcal{T}'^A$.
Let $\mathcal{M}$ be obtained from $\mathcal{M}'$ by dropping the added axioms of the form $A \sqsubseteq A_1 \cap \ldots \cap A_n$ and replacing every occurrence of the introduced symbols $A_1', \ldots, A_n'$ with $A$.

Figure 4. Module extraction in $\mathcal{ALCQI}$ with repeated CIs

Theorem 10: Let $\mathcal{T}$ be an acyclic TBox with repeated CIs and $\Sigma$ a signature. Then $\mathcal{M}$ computed by the algorithm in Figure 4 is a depleting $\Sigma$-module of $\mathcal{T}$.

We close this section with a result about nested depleting modules which is used in the next section to guarantee that combinations of depleting module extraction algorithms extract a depleting module.

Theorem 11: Let $\mathcal{M} \subseteq \mathcal{M}' \subseteq \mathcal{T}$ be TBoxes and $\Sigma$ a signature such that $\mathcal{M}'$ is a depleting $\Sigma$-module of $\mathcal{T}$ and $\mathcal{M}$ is a depleting $\Sigma$-module of $\mathcal{M}'$. Then $\mathcal{M}$ is a depleting $\Sigma$-module of $\mathcal{T}$.

5 Experiments and Evaluation
To evaluate how close depleting module extraction algorithms can approximate minimal depleting modules we consider the following three implementations of upper approximation algorithms for minimal depleting module extraction:

- Our system AMEX that implements the algorithm in Figure 3 and the refinement for acyclic TBoxes with repeated CIs. AMEX is written in Java aided by the OWL-API library [6] for ontology manipulation. The inseparability check was implemented using the reduction to the validity of QBF (Lemma 3) and uses the QBF solver skizzo [2].
- $\sqsubseteq\sqcap^*$ locality-based module extraction [3, 14] as implemented in the OWL-API library version 3.2.4.1806 (called STAR-modules for ease of pronunciation).
- A hybrid approach in which one iterates AMEX and STAR-module extraction. This results in a depleting module contained in both the AMEX and the STAR-module (Theorem 11).

For the lower approximation, we implemented the algorithm in Figure 1 to compute the minimal 1-depleting module. The inseparability check was again implemented using the reduction to the validity of QBF and uses skizzo. In our experiments we use the NCI Thesaurus version 08.09d taken from the Bioportal [19] containing 98.752 TBox axioms among which 87.934 are CIs of the form $A \sqsubseteq C$ and 10.836 are CIs of the form $A \sqsupseteq C$. In what follows, NCI$^*$ denotes the TBox consisting of all such CIs, NCI$^*$ denotes the TBox consisting of all such CIs, and NCI$^*$ denotes the union of both. All three TBoxes are acyclic (with repeated CIs), so AMEX can be applied to them. NCI$^*$ together with the rest of the ontology (452 axioms) is called NCI and contains, in addition, role inclusions, domain and range restrictions, and disjointness axioms.
Most of NCI\(^*\) (all but 4,588 axioms) are \(\mathcal{EL}\)-inclusions. The non-
\(\mathcal{EL}\) inclusions contain 7,806 occurrences of value restrictions. The
signature of NCI\(^*\) contains 68 862 concept and 88 role names.

**Experiments with NCI\(^*\) and its Fragments.** The results given in
Table 5 show the average sizes of the modules extracted by our four
algorithms from the TBoxes NCI\(^*\), NCI\( \subseteq \mathcal{EL}\), and NCI\( \equiv \mathcal{EL}\) over 200 random signatures for each signature size combination of 100
to 1000 concept names and 0%, 50%, and 100% of role names. In
addition, in each case we give the number of signatures (out of 200)
in which there is a difference between the hybrid module and the
minimal 1-depleting module. It can be seen that

- in NCI\(^*\) and NCI\( \subseteq \mathcal{EL}\) the hybrid module almost always coincides
  with the minimal 1-depleting module. Thus the hybrid module almost
  always coincides with the minimal depleting module.
- in NCI\( \equiv \mathcal{EL}\), the hybrid module coincides with the minimal 1-
depleting module (and therefore the minimal depleting module) for
  approximately 50% of all signatures. Moreover, on average the
  minimal 1-depleting module (and therefore the minimal depleting module)
is less than 0.3% smaller than the hybrid module.
- in all three TBoxes, hybrid modules are only slightly smaller than
  AMEX-modules.
- in NCI\( \equiv \mathcal{EL}\), AMEX-modules are significantly smaller
  than STAR-modules.
- in NCI\( \subseteq \mathcal{EL}\), STAR-modules are slightly smaller than AMEX
  modules.
- in NCI\(^*\), AMEX-modules are still significantly smaller than
  STAR-modules, but less so than in NCI\( \equiv \mathcal{EL}\).

The very different behaviour of AMEX-modules and STAR-
modules in NCI\( \subseteq \mathcal{EL}\) and NCI\( \equiv \mathcal{EL}\) can be explained as follows: it is
shown in [9] that for acyclic \(\mathcal{EL}\)-TBoxes without CEs, AMEX-
modules and STAR-modules coincide. This is not the case for the
acyclic A\(\mathcal{CQ}\Xi\)-TBoxes (there can be axioms in STAR-modules that are
not AMEX-modules and vice versa), but since the vast majority of
axioms in NCI\( \subseteq \mathcal{EL}\) are \(\mathcal{EL}\)-inclusions one should not expect any
significant difference between the two types of modules. Thus, it
is exactly these acyclic TBoxes that contain many CEs for which

AMEX-modules are significantly smaller than STAR-modules (see
Example 6 for an illustration).

It would be interesting to know how often (and for which reason)
the minimal 1-depleting module is a depleting module (and thus
coincides with the minimal depleting module). This is the case if the
hybrid module coincides with the minimal 1-depleting module. For
the remaining cases, we can currently only check this “by hand” (the
problem is undecidable). Among the modules extracted in the
experiments we found examples in which the minimal 1-depleting module
is not a depleting module (Example 2 is based on such a module) and
the hybrid module is identical to the minimal depleting module and
we found examples in which the minimal depleting module is identical
to the minimal 1-depleting module and smaller than the hybrid
module. However, we were not able to determine a general pattern.

The experiments were carried out on a PC with an Intel i5 CPU
@ 3.30GHz with 2GB of Java heap space available to the program.
For AMEX the average time taken per extraction was just under 3s
and the maximum time taken was 15s. In 97% of all experiments the
QBF solver was called just once. In those cases the AMEX-modules
were computed purely syntactically and the QBF solver simply pro-
vided an assurance that the extracted axioms indeed constituted a
depleting module. In the remaining 3% of all AMEX extractions the
maximum number of 1-separability causing axioms recorded was 4 and
the maximal number of QBF solver calls themselves was 73.

In the minimal 1-depleting module extraction case, we first
extracted the hybrid module and then applied the algorithm in Figure 1.
The time for minimal 1-depleting module extraction varied considerably
over the input TBoxes: for NCI\( \equiv \mathcal{EL}\) a single 1-depleting module
extraction took no more than 2 minutes, for NCI\(^*\) and NCI\( \equiv \mathcal{EL}\)
a single 1-depleting module extraction took up to 30 minutes. This can
be attributed to the number of QBF checks: for NCI\( \subseteq \mathcal{EL}\) the
maximum number of QBF checks needed in a single extraction was 5,052,
for NCI\(^*\) 193,993, and for NCI\( \equiv \mathcal{EL}\) 433,564 checks were required.

**Experiments with full NCI.** We present preliminary experiments for
module extraction from full NCI. Note that AMEX cannot be directly
applied to full NCI since it is not acyclic. To tackle this problem and
applies to cyclic TBoxes one can split a general TBox \(\mathcal{T}\) into

![Figure 5. Modules of NCI\(^*\) and its fragments](Image)
two parts $T_1$ and $T_2$, where $T_1$ is an acyclic TBox (and as large as possible) and $T_2 := T \setminus T_1$. Then for any signature $\Sigma$ it follows from the robustness properties [8] of the inseparability relation that if $M$ is a depleting $\Sigma \cup \text{sig}(T_2)$-module of $T_1$ (note that $M$ can be computed by AMEX), then $M \cup T_2$ is a depleting $\Sigma$-module of $T$ as well. Such a direct application of AMEX to general TBoxes is unlikely to compute small modules when $T_2$ is large. However, the results below indicate that this approach is beneficial when iterated with STAR-module extraction. By Theorem 11 and since both AMEX and STAR extract depleting $\Sigma$-modules, given a signature $\Sigma$ and TBox $T$ one can extract an AMEX module from the STAR module (and vice versa) and it is guaranteed that the resulting module is still a depleting $\Sigma$-module of $T$. In this way, one can repeatedly extract from the output of one extraction approach again a module using the other approach until the sequence of modules becomes stable.

Preliminary experiments shown in Figure 6 are based on 200 signatures for each concept signature size/role percentage combination and compare the average size of modules extracted using STAR-extraction, hybrid extraction, and 1-depleting module extraction.

The results are very similar to the results for NCI'. Hybrid modules are on average significantly smaller than STAR modules and are often identical to the minimal 1-depleting module (and so the minimal depleting module). Thus, the fact that we found no hybrid module that does not coincide with the corresponding minimal 1-depleting module is consistent with the figures for NCI'.

6 Conclusion

Using a new system, AMEX, for upper approximation and the first algorithm for lower approximation of minimal depleting modules we have shown that for the NCI Thesaurus one can compute efficiently depleting modules that are consistently very close to the minimal depleting module and often coincide with the latter. The experiments also show that for TBoxes with many axioms of the form $A \equiv C$, AMEX-modules can be significantly smaller than STAR-modules and that a hybrid approach can lead to significantly smaller modules than ‘pure’ STAR-modules.

This paper is only the first step towards a novel systematic evaluation of the quality of upper approximations of modules using lower approximations. It would be of great interest to compute lower approximations for a more comprehensive set of cyclic ontologies and compare them with the upper approximations given by STAR-modules and by the hybrid approach. We conjecture that for many cyclic ontologies 1-depleting modules will still be a good lower approximation and can therefore provide a suitable tool to estimate the difference between STAR/hybrid-modules and minimal depleting modules. For some ontologies, however, it will be necessary to move to $n$-depleting modules (based on in separability for interpretations of size at most $n$) with $n > 1$. These modules can still be extracted by using QBF solvers and exactly the same algorithm; the cost is much higher, though, since the length of the encoding into a QBF is exponential in $n$.

References

A Proofs

In this section we provide proofs for results which are not a direct consequence of previously published results in [9].

Lemma 3 $T \equiv \frac{1}{2} \emptyset$ iff the QBF $\phi_T := \forall \exists \forall \exists^\ast T$ is valid.

Proof Recall the definition of the translation $\cdot$:

$$
\begin{align*}
A^! &= p, & \text{for all } A \in \Sigma, \\
q^! &= q, & \text{for all } A \in \text{sig}(T) \setminus \Sigma, \\
(D_1 \cap D_2)^! &= D_1^! \wedge D_2^!, & \\
(\neg D)^! &= \neg D^!, & \text{if } n = 1 \\
\end{align*}
$$

Let $I$ be an arbitrary interpretation of $T$. Define a truth assignment $J$ for the propositions in $\vec{q}$ such that the propositional formula $T'$ is true under the assignment $I \cup J$. Define a singleton interpretation $I$ as follows:

- $\Delta^I = \{ d \}$.
- $\Delta^J = \{ d \}$ if $I$ assigns $true$ to $p$, and $\Delta^J = \emptyset$ if $I$ assigns $false$ to $p$ for all $A \in \Sigma$.
- $\Delta^J = \{ d, d \}$ if $I$ assigns $true$ to $q$, and $\Delta^J = \emptyset$ if $I$ assigns $false$ to $q$ for all $A \in \text{sig}(T) \setminus \Sigma$.
- $\Delta^J = \{ d \}$ if $I$ assigns $true$ to $q$, and $\Delta^J = \emptyset$ if $I$ assigns $false$ to $q$ for all $A \in \text{sig}(T) \setminus \Sigma$.

As $T \equiv \frac{1}{2} \emptyset$ there exists an interpretation $J$ such that $J|_{\Sigma} = I|_{\Sigma}$ and $J \vdash T$. We define $J$ by setting

- $J$ assigns $true$ to $q$ if $d \in A \cap J$ and $J$ assigns $false$ to $q$ if $d \notin A \cap J$, for all $A \in \text{sig}(T) \setminus \Sigma$.
- $J$ assigns $true$ to $q$, if $(d, d) \in r^J$ and $J$ assigns $false$ to $q$, if $(d, d) \notin r^J$, for all $A \in \text{sig}(T) \setminus \Sigma$.

It follows from $J \vdash T$ that $T'$ is true under the assignment $I \cup J$.

We now analyse the algorithm given in Figure 3. The algorithm in Figure 1 can be analysed similarly.

Theorem 8 Given an acyclic $\text{ALCQI}$ TBox $T$ and signature $\Sigma$ the algorithm in Figure 3 computes the unique minimal depleting $\Sigma$-module $M$ s.t. $T \setminus M$ contains no direct $\Sigma \cup \text{sig}(M)$-dependencies. The algorithm is in polynomial time with each call to the QBF solver being treated as a constant time oracle call.

Proof It has been proved in [9] that given an acyclic $\text{ALCQI}$ TBox $T$ and signature $\Sigma$ the algorithm given in Figure 7 computes the unique minimal depleting $\Sigma$-module of $T$ such that $T \setminus M$ contains no direct $\Sigma \cup \text{sig}(M)$-dependencies. The proof carries over to the case of $\text{ALCQI}$ in a straightforward way, that is, given an acyclic $\text{ALCQI}$ TBox $T$ and signature $\Sigma$ the algorithm given in Figure 7 computes the unique minimal depleting $\Sigma$-module of $T$ such that $T \setminus M$ contains no direct $\Sigma \cup \text{sig}(M)$-dependencies.

Since the output of the algorithm is uniquely determined, the output does not depend on the selection strategy in line 2 of the algorithm in Figure 7. One can use this fact to prove that the algorithm in Figure 7 and Figure 3 return the same value.

As for the running time bound, we assume w.l.o.g that $|\Sigma| \leq |T|$. Then the total number of rule applications does not exceed $|T|$. Checking if $\text{syn}$ is applicable can be performed by simple reachability analysis in $O(|T|)$ time and checking if the 1-insep rule is applicable can also be done in $O(|T|)$ as follows: starting with $\mathcal{W} = \emptyset$ add axioms from $T \setminus M$ to $\mathcal{W}$ one by one while checking that $\mathcal{W} \equiv \Sigma \cup \text{sig}(M)$ $\emptyset$. The axiom for which the condition is violated is separability causing. Thus, the algorithm runs in $O(|T|^2)$ in the worst case (assuming that each call to the QBF solver is treated as a constant time oracle call).

The search for separability causing axioms can be optimised to run in $O(|\log_2|T|)$ time as shown in Figure 8. We first consider $T \setminus M$ itself as $\mathcal{W}$. If $T \setminus M \equiv \Sigma \cup \text{sig}(M)$ $\emptyset$ then $T \setminus M$ contains no separability causing axioms. Otherwise, we consider $\mathcal{W}$ to be equal to the top half of $T \setminus M$ (we treat $T \setminus M$ as an ordered set). We then check if $\mathcal{W} \equiv \Sigma \cup \text{sig}(M)$ $\emptyset$ and, if this is the case, we grow $\mathcal{W}$ from the bottom and if not, we halve it again, etc.

Theorem 10 Let $T$ be an acyclic TBox with repeated Cls and $\Sigma$ a signature. Then $M$ computed by the algorithm in Figure 4 is a depleting $\Sigma$-module of $T$.

Proof Let $T'$ be the set of non-repeated Cls in $T$. For a repeated $A$ let $A \subseteq C_1, \ldots, A \subseteq C_n$, for $n > 1$, be all Cls in $T$ with $A$ on the
left hand side. Then
\[
T^A = \{A \subseteq A'_0 \cap \ldots \cap A'_n \cup \bigcup_{i=1}^{n} \{A'_i \subseteq C_i\},
\]
where \(A'_1, \ldots, A'_n\) are fresh concept names (not occurring in \(T\)). For
\[
T^* = T^* \cup \bigcup A^I = \text{A is repeated}
\]
let \(M'\) be the minimal deleting \(\Sigma\)-module of \(T^*\) such that \(T^* \setminus M'\) has no direct \(\Sigma \cup \Sigma(M')\)-dependencies. We require the following observation.

**Claim:** If \(A \subseteq A'_0 \cap \ldots \cap A'_n \notin M',\) then for every \(A'_i \subseteq C_i \in T^A\) we have \(A'_i \subseteq C_i \notin M'.\)

For a proof by contradiction, assume that the claim does not hold. Then there is some \(i \geq 1\) such that \(A'_i \subseteq C_i \in M'.\) Consider \(M'' = M' \setminus \{A'_i \subseteq C_i\}.\) Then neither \(A \subseteq A'_0 \cap \ldots \cap A'_n\) nor \(A'_i \subseteq C_i\) belong to \(M'\), so \(A'_i \notin \Sigma \cup \Sigma(M'').\)

\(T^* \setminus M''\) either has a direct \(\Sigma \cup \Sigma(M'')\) dependency or \(M''\) is not a deleting \(\Sigma\)-module of \(T^*\). Assume that \(T^* \setminus M''\) has a direct \(\Sigma \cup \Sigma(M'')\) dependency. Then for some \(X \in \Sigma \cup \Sigma(M'')\) either \(X <_{\Sigma \cup \Sigma(M'')} A'_i\) or \(X <_{\Sigma \cup \Sigma(M'')} A\). In either case \(T^* \setminus M''\) has a direct \(\Sigma \cup \Sigma(M'')\)-dependency and we have derived a contradiction.

Now assume that \(M''\) is not a deleting \(\Sigma\)-module of \(T^*.\) Let \(I\) be an interpretation such that there does not exist a model \(J\) of \(T^*\) with \(J_{|\Sigma \cup \Sigma(M'')} = I_{|\Sigma \cup \Sigma(M'')}\). Consider the interpretation \(I'\) defined as follows:

- \(\Delta^{I'} = \Delta^I;\)
- \(X^{I'} = X^I,\) for every symbol \(X \neq A'_i;\)
- \(A'_i^{I'} \neq \emptyset.\)

Since \(M'\) is a deleting \(\Sigma\)-module of \(T^*\) there exists a model \(J\) of \(T^* \setminus M'\) such that \(I_{|\Sigma \cup \Sigma(M')} = J_{|\Sigma \cup \Sigma(M')}\). Since \(A'_i \in \Sigma(M')\) we have \(A'_i^{I'} = \emptyset.\) But then \(J\) is a model of \(T^* \setminus M''\) and since \(A'_i \notin \Sigma \cup \Sigma(M''),\) we have \(J_{|\Sigma \cup \Sigma(M'')} = I_{|\Sigma \cup \Sigma(M'')}\), and we have derived a contradiction. This finishes the proof of the claim.

Let \(M'\) be obtained from \(M'\) by dropping the added axioms of the form \(A \subseteq A'_i \cap \ldots \cap A'_n\) and replacing every occurrence of the introduced symbols \(A'_1, \ldots, A'_n\) with \(A.\) To show that \(T \setminus M' \equiv_{\Sigma \cup \Sigma(M')} \emptyset,\) let \(I\) be an arbitrary interpretation. We need to demonstrate that there exists a model \(J\) of \(T \setminus M'\) such that \(I_{|\Sigma \cup \Sigma(M')} = J_{|\Sigma \cup \Sigma(M')}\).

Define a new interpretation \(I'\) by setting

- \(\Delta^{I'} = \Delta^I;\)
- \(X^{I'} = X^I,\) for every symbol \(X \in \Sigma \cup \Sigma(T);\)
- \(A'_i^{I'} = A^{I'}\) for every \(A'_i\) in an added axiom of the form \(A \subseteq A'_i \cap \ldots \cap A'_n.\)

As \(M'\) is a deleting \(\Sigma\)-module of \(T^*\), there exists an interpretation \(J\) such that \(I_{|\Sigma \cup \Sigma(M')} = J_{|\Sigma \cup \Sigma(M')}\) and \(J \models T^* \setminus M'.\) We show that \(J \not\models T \setminus M'\) also holds.

As \(T^*\) is the set of non-repeated CIs, we have \(T^* \setminus M' = T' \setminus M'\) and so \(J \not\models T'.\) Suppose for some repeated \(A \subseteq C_i \in T \setminus M'\) we have \(J \not\models A \subseteq C_i.\) By definition of \(M'\) we have \(A'_i \subseteq C_i \in T^* \setminus M'.\)

We make a case distinction on whether or not \(A \subseteq A'_i \cap \ldots \cap A'_n \in M'.\) Suppose that \(A \subseteq A'_i \cap \ldots \cap A'_n \in M'.\) Then

**Input:** TBox \(T,\) subset \(M \in T\) and signature \(\Sigma\) such that \(T \setminus M\) contains no direct \(\Sigma \cup \Sigma(M)\)-dependencies and \(T \setminus M \equiv_{\Sigma \cup \Sigma(M)} \emptyset.\)

**Output:** Separability causing axiom \(\alpha\)

1. lastAdded\(=\)topHalf(lhs\(\Sigma \cup \Sigma(M)(T \setminus M))\)
2. if \(W = \text{lastAdded}\) then \(\text{lastAdded} = \text{topHalf(lastRemoved)}\)
3. \(W = \text{W} \cup \text{lastAdded}\)
4. \(\text{lastRemoved} = \text{lastRemoved} \setminus \text{lastAdded}\)
5. \(\text{lastRemoved} = \text{bottomHalf(lastAdded)}\)
6. \(W = W \setminus \text{lastRemoved}\)
7. return \(\text{last axiom of W}\)

**Figure 8.** Finding 1-separability causing axiom

\{\{A, A'_1, \ldots, A'_n\} \subseteq \Sigma(M')\) so \(A'_{\alpha} = A_\alpha^I = A^I = A^I.\) Thus \(J \not\models A \subseteq C_i\) contradicting the assumption that \(J\) is a model of \(T \setminus M'.\)

Suppose that \(A \subseteq A'_1 \cap \ldots \cap A'_n \notin M'.\) Then both \(A \subseteq A'_1 \cap \ldots \cap A'_n \in T \setminus M'\) and \(A'_i \subseteq C_i \subseteq T \setminus M'\) but \(J \not\models A \subseteq A'_i \cap \ldots \cap A'_n\) or \(J \not\models A'_i \subseteq C_i\) since \(J \not\models A \subseteq C_i.\) We have derived a contradiction to the assumption that \(J\) is a model of \(T \setminus M'.\)

We have shown that \(J \not\models T \setminus M.\) By the claim above we have \(\Sigma(M) \subseteq \Sigma(M')\) and so \(J_{|\Sigma \cup \Sigma(M')} = I_{|\Sigma \cup \Sigma(M')}\), as required.

We illustrate Theorem 10 with an example.

**Example 12** Consider an acyclic E\(\mathcal{L}\) TBox with repeated CIs

\(T = \{A \subseteq B, A \subseteq C, B \subseteq C\}\)

and signature \(\Sigma = \{A, B\}.\) First, notice that if we (logically equivalently) rewrite \(T\) as an acyclic E\(\mathcal{L}\) TBox

\(T_{\text{acyclic}} = \{A \subseteq B \cap C, B \subseteq C\}\)

then the minimal deleting \(\Sigma\)-module of \(T_{\text{acyclic}}\) is \(T_{\text{acyclic}}\) itself.

On the other hand, consider \(T' = \{B \subseteq C\}\) and

\(T^A = \{A \subseteq A'_i \cap A'_j \cup \{A'_i \subseteq B \cup \{A'_i \subseteq C\}\}.\)

Then the minimal deleting \(\Sigma\)-module of \(T' \cup T^A\) is

\(M' = \{A \subseteq A'_i \cap A'_j \cup \{A'_i \subseteq B\}\}\)

and, by Theorem 10, \(M' = \{A \subseteq B\}\) is a deleting \(\Sigma\)-module of \(T.\) In fact, it is straightforward to check that \(\{A \subseteq C, B \subseteq C\} \equiv_{\Sigma \cup \Sigma(M')} \emptyset\) and that \(M'\) is the minimal deleting \(\Sigma\)-module of \(T.\)

**Theorem 11** Let \(M \subseteq M' \subseteq T\) be TBoxes and \(\Sigma\) a signature such that \(M'\) is a deleting \(\Sigma\)-module of \(T\) and \(M\) is a deleting \(\Sigma\)-module of \(M'.\) Then \(M\) is a deleting \(\Sigma\)-module of \(T.\)
Assume that $M \subseteq M' \subseteq T$ and $\Sigma$ is a signature such that $M'$ is a depleting $\Sigma$-module of $T$ and $M$ is a depleting $\Sigma$-module of $M'$. To prove that $M$ is a depleting $\Sigma$-module of $T$, consider an interpretation $\mathcal{I}$. We have to show that there exists a model $\mathcal{J}$ of $T \setminus M$ such that $\mathcal{J}|_{\Sigma \cup \text{sig}(M)} = \mathcal{I}|_{\Sigma \cup \text{sig}(M)}$. As $M$ is a depleting $\Sigma$-module of $M'$, there exists an interpretation $\mathcal{J}'$ such that $\mathcal{J}'|_{\Sigma \cup \text{sig}(M)} = \mathcal{I}|_{\Sigma \cup \text{sig}(M)}$ and $\mathcal{J}' \models (M' \setminus M)$. Similarly, as $M'$ is a depleting $\Sigma$-module of $T$, there exists an interpretation $\mathcal{J}$ such that $\mathcal{J}|_{\Sigma \cup \text{sig}(M')} = \mathcal{J}'|_{\Sigma \cup \text{sig}(M')}$ and $\mathcal{J} \models (T \setminus M')$. As $\text{sig}(M) \subseteq \text{sig}(M') \subseteq \text{sig}(T)$ we have $\mathcal{J}|_{\Sigma \cup \text{sig}(M)} = \mathcal{I}|_{\Sigma \cup \text{sig}(M)}$ and $\mathcal{J} \models (M' \setminus M)$. But then $\mathcal{J} \models ((T \setminus M)$ and so $\mathcal{J}$ is as required.