

Game Theory

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Literature

- Fun and Games
 - Ken Binmore
- A Course in Game Theory
 - Martin Osborne en Ariel Rubinstein

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What is it all about?

- game = interaction
 - traffic
 - supermarket
 - employee, employer, board, union
 - student / teacher
 - judge and lawyers
 - George en Osama
 - marriage and career

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What is it all about?

- terminology of chess and bridge
- logic and systematics of interaction
- analysis takes you from irrational issues
 - strategic interaction is difficult
 - because reasoning is circular

If J and M play a game, J's strategy will typically depend on his prediction of M's strategy, which, on its turn depends on M's expectation of J's....

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Surprise and Paradox

- does it make sense
 - to vote for a candidate you fancy least? YES!!
 - for a general, to toss a coin? YES!!
 - in poker, place a maximal bid with the worst cards? YES!!
 - to throw some goods away before starting to negotiate about them YES!!
 - to sell your house to the second best bidder? YES!!

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Strategic voting

- Boris, Horace and Maurice determine who can be a member of the Dead Poet Society
 - proposal: allow Alice
 - amendment: allow Bob, rather than Alice
 - first vote over amendment, then over proposal

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Strategic voting

Borice	Horace	Maurice
Alice	Nobody	Bob
Nobody	Alice	Alice
Bob	Bob	Nobody

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Strategic voting

- first between A, B
 - winner Alice
- then between A, N
 - winner Alice
- strategic voting H:
 - first vote for Bob!
 - result ... B, N
- M anticipates: vote for A

Horace	Maurice
Nobody	Bob
Alice	Alice
Bob	Nobody

Borice
Alice
Nobody
Bob

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History

- Von Neumann and Morgenstern
 - The Theory of Games and Economic Behaviour* (1944)
- ideas from economics and mathematics
- initially very optimistic, then draw-back
- revival since 1970's
 - Nash, Aumann, Shapley, Selten, Harsanyi

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Game Theory

- theory of decision makers
 - are rational:
 - aware of alternatives
 - form expectations
 - have preferences
 - optimize after deliberation
- set A of actions;
- set C consequences;
- $g: A \rightarrow C$
 - consequence function
- preference relation $\geq$ on A
 - or: utility function
 - $u: C \rightarrow \mathbb{R}$

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Abstracts from 'emotions'

- suppose you're offered £ 1.000
- you make a deal with the first person you encounter: $(1.000-x, x)$ $x = 1, 2 \dots$
 - if he accepts: (you, person) get $(1.000-x, x)$
 - else (0,0)
- only money counts, and that is known
- both are rational: prefer $y+1$ over y
- what will you offer?

ONE £!!!!!!

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Game Theory

- theory of decision makers
 - are rational
 - reason strategically
- players anticipate on knowledge and expectations about behaviour of other decision makers

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Strategic Games

- Definition $G = \langle N, (A_i), (\succeq_i) \rangle$
 - finite set N (**players**)
 - set A_i (**actions**) for every player i
 - preference relation** $\succeq_i$ for every player i
 - u_i is **utility function**: $A \rightarrow \mathbb{R}$ with
 - $a \succeq_i b \Leftrightarrow u_i(a) \geq u_i(b)$
 - also called **payoff-function**
 - (although not the same)

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Representation strategic games

- $N = \{1, 2\}$
- $A_1 = \{T, B\}$
- $A_2 = \{L, R\}$
- $u_1(T) = w_1$, etc

	L	R
T	w_1, w_2	x_1, x_2
B	y_1, y_2	z_1, z_2

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Representation strategic games

- Interpretation
 - one-shot
 - simultaneous
 - independent
 - utilities are known
 - not the choice of others

	L	R
T	w_1, w_2	x_1, x_2
B	y_1, y_2	z_1, z_2

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Example: BoS

- $N = \{1, 2\}$
- $A_1 = \{B, S\}$
- $A_2 = \{B, S\}$
- u_1, u_2 see figure
 - B: Bach
 - S: Stravinsky
- Battle of the Sexes

	B	S
B	2, 1	0, 0
S	0, 0	1, 2

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Profiles

- $A_1, A_2, \dots, A_n$ are the action sets
- $(a_1, a_2, \dots, a_n) \in A_1 \times A_2 \times \dots \times A_n$ is a **profile**
- notation: (x) , or a^*
- $x_i \in A_1 \times A_2 \times \dots \times A_{i-1} \times A_{i+1} \times \dots \times A_n$
- $(x_{-i}, x_i) = (x)$
- focus on i , given the profile of the others

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Profiles: example

- $A_1, A_2, \dots, A_7$ are bids ($\in \mathbb{R}$)
- $(a_1, a_2, \dots, a_7)$ is a concrete bid
- notation: $(x) = (25, 22, 20, 12, 0, 27, 22) = a^*$
- $x_{-6} \in (25, 22, 20, 12, 0, 22)$
- $(x_{-6}, x_6) = ((25, 22, 20, 12, 0, 22), 27)$
 - $(x_{-6}, x'_6) = ((25, 22, 20, 12, 0, 22), 26)$ would have been better for player 6, given the profiles of the others

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Nash equilibrium

- John Nash
- equilibrium ("solution")
 - every player is rational
 - every player plays optimally
 - no use to divert individually
 - not an algorithmic approach

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Nash equilibrium (definitie)

- Given $G = \langle N, (A_i), (\geq_i) \rangle$
- $a^* \in A = A_1 \times A_2 \times \dots \times A_n$ is Nash equilibrium iff
- $\forall i \in N \forall a_i \in A_i (a^*_{-i}, a_i) \geq_i (a^*_{-i}, a^*_i)$
 - no player i can improve in a^* , if the other players still play a^*_{-i}

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Nash equilibrium (alternative)

- define for every $a_{-i} \in A_{-i}$ the **best response for i** , $B_i(a_{-i})$
- $B_i(a_{-i}) = \{a_i \in A_i \mid \forall a'_i \in A_i (a_{-i}, a_i) \geq_i (a_{-i}, a'_i)\}$
- a^* is N.eq iff $\forall i \in N a^*_i \in B_i(a^*_{-i})$

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Example: BoS (N.eq)

- $N = \{1, 2\}$
- $A_1 = \{B, S\}$
- $A_2 = \{B, S\}$
- u_1, u_2 see figure
 - B: Bach
 - S: Stravinsky
- two equilibria:
 - (bach, bach) and
 - (strawinsky, strawinsky)

	B	S
B	2,1	0,0
S	0,0	1,2

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Example: coordination game

- Mozart or Mahler?
- same preferences
- two equilibria:
 - (Mozart, Mozart) and
 - (Mahler, Mahler)
- N.eq right concept?

	Mo	Ma
Mo	2,2	0,0
Ma	0,0	1,1

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Pareto Efficiency

- (Mozart, Mozart) and
- (Mahler, Mahler)
- N.eq right concept?
- (2,2) is (strongly) Pareto efficient:
 - $\neg \exists x \neg \exists y (x, y) > (2, 2)$

	Mo	Ma
Mo	2,2	0,0
Ma	0,0	1,1

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Pareto Optimality (definition)

- Given $G = \langle N, (A_i), (\geq_i) \rangle$
- $a^* = (a_1, a_2, \dots, a_n) \in A_1 \times A_2 \times \dots \times A_n$ is Pareto optimal iff
- $\forall i \in N \forall b^* \in A_1 \times A_2 \times \dots \times A_n (a^*) \geq_i (b^*)$

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Example: prisoner's dilemma

- C: cooperate with the other, keep silent
- D: justify against the other

	C	D
C	0,0	-2,3
D	3,-2	-1,-1

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Example: prisoner's dilemma

- C: cooperate with the other, keep silent
- D: justify against the other
- Although cooperate would be better, every player has a preference for defeat

	C	D
C	0,0	-2,3
D	3,-2	-1,-1

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Example: hawk-dove

- preference:
 - hawkish if other is dovish
 - dovish if other is hawkish
- N.eq: (Dove, Hawk) and (Hawk, Dove)

	D	H
D	3,3	1,4
H	4,1	0,0

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Example: Matching Pennies

- Head and Tail
- if different, 1 pays a Pound to 2 if the same, 2 pays a Pound to 1
- no equilibrium!
- game is strictly competitive

	H	T
H	1,-1	-1,1
T	-1,1	1,-1

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SCSG

- Strictly Competitive Strategic Game
- if $G = \langle \{1,2\}, (A_i), (\geq_i) \rangle$,
- and $\forall a, b \in A: a \geq_1 b \Leftrightarrow b \geq_2 a$
- also called zero-sum game:
 - with u_1 and u_2 we have
 - $u_1 + u_2 = 0$

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maximizer

- let $G = \langle \{1,2\}, (A_i), (\geq_i) \rangle$ an SCSG
- action $x^* \in A_1$ is maximizer for 1:
- $\forall x \in A_1 \min_{y \in A_2} u_1(x^*, y) \geq_1 \min_{y \in A_2} u_1(x, y)$
- action $y^* \in A_2$ is maximizer for 2:
- $\forall y \in A_2 \min_{x \in A_1} u_2(x, y^*) \geq_2 \min_{x \in A_1} u_2(x, y)$

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maximizer

- action $x^* \in A_1$ is maximizer for 1:
- $\forall x \in A_1 \min_{y \in A_2} u_1(x^*, y) \geq_1 \min_{y \in A_2} u_1(x, y)$
- solves for 1 $\max_x \min_y u_1(x, y)$
- solves for 2 $\max_y \min_x u_2(x, y)$
- 'maximises the minimum that i can guarantee'
- x^* is a *security strategy* for 1

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Equilibria and maximizers

- (x^*, y^*) is N.eq for G , iff:
 - x^* is a maximizer for 1;
 - y^* is a maximizer for 2
 - $\max_x \min_y u_1(x, y) =$
 - $\min_y \max_x u_1(x, y) = u_1(x^*, y^*)$

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maximizers

- solves for 1
- $\max_x \min_y u_1(x, y) =$
 - $\max\{\min\{u_1(x, y) | y \in A_2\} | x \in A_1\} =$

...	y_1	y_2	y_3	y_4	y_5
x_1	2,-2	2,-2	3,-3	1,-1	1,-1
x_2	3,-3	5,-5	4,-4	6,-6	4,-4
x_3	5,-5	2,-2	4,-4	3,-3	3,-3
x_4	6,-6	8,-8	5,-5	7,-7	5,-5
x_5	3,-3	5,-5	4,-4	2,-2	3,-3
x_6	4,-4	3,-3	6,-6	5,-5	4,-4

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maximizers

- solves for 1
- $\max_x \min_y u_1(x, y) =$
 - $\max\{\min\{u_1(x, y) | y \in A_2\} | x \in A_1\} =$
- $x_1: \min_y u_1(x_1, y) = 1$

...	y_1	y_2	y_3	y_4	y_5
x_1	2,-2	2,-2	3,-3	1,-1	1,-1
x_2	3,-3	5,-5	4,-4	6,-6	4,-4
x_3	5,-5	2,-2	4,-4	3,-3	3,-3
x_4	6,-6	8,-8	5,-5	7,-7	5,-5
x_5	3,-3	5,-5	4,-4	2,-2	3,-3
x_6	4,-4	3,-3	6,-6	5,-5	4,-4

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maximizers

- solves for 1
- $\max_x \min_y u_1(x, y) =$
 - $\max\{\min\{u_1(x, y) | y \in A_2\} | x \in A_1\} =$
- $x_1: \min_y u_1(x_1, y) = 1$
- $x_2: \min_y u_1(x_2, y) = 3$
- $\dots \dots \dots \dots = \dots$
- $x_6: \min_y u_1(x_6, y) = 3$

$\max = 5$ for $x^* = x_4$

...	y_1	y_2	y_3	y_4	y_5
x_1	2,-2	2,-2	3,-3	1,-1	1,-1
x_2	3,-3	5,-5	4,-4	6,-6	4,-4
x_3	5,-5	2,-2	4,-4	3,-3	3,-3
x_4	6,-6	8,-8	5,-5	7,-7	5,-5
x_5	3,-3	5,-5	4,-4	2,-2	3,-3
x_6	4,-4	3,-3	6,-6	5,-5	4,-4

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maxminimizers

- solves for 1
- $\max_x \min_y u_1(x,y) =$
 - $\max\{\min\{u_1(x,y) | y \in A_2\} | x \in A_1\} = 5$
- solves for 2
- $\max_y \min_x u_2(x,y) =$
 - $\max\{\min\{u_2(x,y) | x \in A_1\} | y \in A_2\} =$

...	y_1	y_2	y_3	y_4	y_5
x_1	2,-2	2,-2	3,-3	1,-1	1,-1
x_2	3,-3	5,-5	4,-4	6,-6	4,-4
x_3	5,-5	2,-2	4,-4	3,-3	3,-3
x_4	6,-6	8,-8	5,-5	7,-7	5,-5
x_5	3,-3	5,-5	4,-4	2,-2	3,-3
x_6	4,-4	3,-3	6,-6	5,-5	4,-4

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maxminimizers

- solves for 1
- $\max_x \min_y u_1(x,y) =$
 - $\max\{\min\{u_1(x,y) | y \in A_2\} | x \in A_1\} = 5$
- solves for 2
- $\max_y \min_x u_2(x,y) =$
 - $\max\{\min\{u_2(x,y) | x \in A_1\} | y \in A_2\} = -5!$
- Equilibrium (5,-5)

...	y_1	y_2	y_3	y_4	y_5
x_1	2,-2	2,-2	3,-3	1,-1	1,-1
x_2	3,-3	5,-5	4,-4	6,-6	4,-4
x_3	5,-5	2,-2	4,-4	3,-3	3,-3
x_4	6,-6	8,-8	5,-5	7,-7	5,-5
x_5	3,-3	5,-5	4,-4	2,-2	3,-3
x_6	4,-4	3,-3	6,-6	5,-5	4,-4

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security level vs equilibria

- consider cooperative game G
- (2,2) looks like 'the optimal' solution
- security strategy of 1 is r , gives 1!
- Nash equilibria?

	L	R
l	2,2	0,0
r	1,1	1,1

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security level vs equilibria

- consider game G
- (2,2) looks like 'the optimal' solution
- security strategy of 1 is r , gives 1!
- Nash equilibria?

	L	R
l	2,2	0,0
r	1,1	1,1

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bimatrix games

- $m \times n$ matrix
- 1 has strategies s_1 and s_2 , 2 has t_1 , t_2 and t_3
- payoff $\pi_i(s_i, t_j) = ij$
 - $\pi_2(s_i, t_j) = (i-2)(j-2)$

	t_1	t_2	t_3
s_1	1,1	2,0	3,-1
s_2	2,0	4,0	6,0

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bimatrix games

- $m \times n$ matrix
- 1 has strategies s_1 and s_2 , 2 has t_1 , t_2 and t_3
- Nash equilibrium (σ, τ) :
 - $\forall s, t \pi_1(\sigma, \tau) \geq \pi_1(s, \tau)$
 - $\forall s, t \pi_2(\sigma, \tau) \geq \pi_2(\sigma, t)$

	t_1	t_2	t_3
s_1	1,1	2,0	3,-1
s_2	2,0	4,0	6,0

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domination

- strategy s_d of 1 dominates s_i strongly:
 - $\forall t \pi_1(s_d, t) > \pi_1(s_i, t)$
- and weakly if:
 - $\forall t \pi_1(s_d, t) \geq \pi_1(s_i, t)$
 - $\exists t \pi_1(s_d, t) > \pi_1(s_i, t)$
- t_1 dominates t_2 weakly

	t_1	t_2	t_3
s_1	1,1	2,0	3,-1
s_2	2,0	4,0	6,0

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Iterated elimination

- s_2 of 1 strongly dominates s_1
- No further (weak) domination: all is left are Nash Equilibria
- this is not generally so

	t_1	t_2	t_3
s_1	1,1	2,0	3,-1
s_2	2,0	4,0	6,0

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Order of elimination

	C	D
AE	2,0	1,1
AF	0,2	1,1
BE	3,3	3,3
BF	3,3	3,3

	C	D
AE	2,0	1,1
AF	0,2	1,1
BE	3,3	3,3
BF	3,3	3,3

lost equilibrium!

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elimination: conclusions

- strict strategies: no problem
- with weakly dominated strategies:
 - some equilibria can get lost
 - order of elimination is important

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Example: BoS

- $N = \{1,2\}$
- $A_1 = \{B, S\}$
- $A_2 = \{B, S\}$
- u_1, u_2 see figure
 - B: Bach
 - S: Stravinsky
- Battle of the Sexes

	B	S
B	2,1	0,0
S	0,0	1,2

no dominant strategies

still (two) Nash equilibria

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Ex: coordination game

- Mozart of Mahler?
- Same preference

	Mo	Ma
Mo	2,2	0,0
Ma	0,0	1,1

No dominant strategy

still two Nash equilibria

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Ex: prisoner's dilemma

- C: cooperate, and be silent
- D: justify against the other
- D dominates C
- D dominates C
- gives Nash equilibrium $(-1, -1)$

	C	D
C	0,0	-2,3
D	3,-2	-1,-1

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Ex: Matching Pennies

- Head and Tail
- if different, 1 pays a Pound to 2; if the same, 2 pays a Pound to 1
- no dominant strategy
- No pure Nash equilibrium

	H	T
H	1,-1	-1,1
T	-1,1	1,-1

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mixed strategies

- don't always bid in the same way with poker
- being unpredictable can be an advantage
- sometimes a strategy is not dominated by another pure strategy, but by a mixed one

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security level and strategy

- maximin for 1: 2, via s_2
- note: (s_2, t_2) is a saddlepoint
- then 2 is also security level of player 1

	t_1	t_2	t_3
s_1	0,1	1,2	7,3
s_2	4,6	2,0	3,2
s_3	9,0	0,3	0,4

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security level and strategy

- maximin for 1: 2 via s_3
- (s_3, t_2) is not a saddlepoint
- security level of 1 is indeed not 2, but $2^{2/3}$
- How?

	t_1	t_2	t_3
s_1	1,0	6,4	0,9
s_2	2,1	0,2	3,0
s_3	3,7	2,3	4,0

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mixed strategies

- remove strictly dominated strategy s_2
- 2 has no pure dominating strategy, but, ...

	t_1	t_2	t_3
s_1	1,0	6,4	0,9
s_2	2,1	0,2	3,0
s_3	3,7	2,3	4,0

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mixed strategies

- 2 has no pure dominating strategy, but, ...
- $q = (1/2, 0, 1/2)$ dominates t_2 strongly!
- $\pi_2(s_1, q) =$
 - $(1/2) \cdot 0 + (1/2) \cdot 9 = 4.5 > 4$
- $\pi_2(s_3, q) =$
 - $(1/2) \cdot 7 + (1/2) \cdot 0 = 3.5 > 3$

	t_1	t_2	t_3
s_1	1,0	6,4	0,9
s_2	2,1	0,2	3,0
s_3	3,7	2,3	4,0

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mixed strategies

- 2 has no pure dominating strategy, but, ...
- $q = (1/2, 0, 1/2)$ dominates t_2 strongly!
- $\pi_2(s_1, q) =$
 - $(1/2) \cdot 0 + (1/2) \cdot 9 = 4.5 > 4$
- $\pi_2(s_3, q) =$
 - $(1/2) \cdot 7 + (1/2) \cdot 0 = 3.5 > 3$

	t_1	t_2	t_3
s_1	1,0	6,4	0,9
s_2	2,1	0,2	3,0
s_3	3,7	2,3	4,0

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mixed strategies

- after iterated elimination
- what is security level of 1?
- suppose 1 plays mixed strategy $(1-r, 0, r)$

s	t_1	t_2	t_3
s_1	1,0	6,4	0,9
s_2	2,1	0,2	3,0
s_3	3,7	2,3	4,0

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mixed strategies

- suppose 1 plays mixed strategy $(1-r, 0, r)$
- let $E_k(r)$ be payoff of 1 if 2 plays t_k :

s	t_1	t_2	t_3
s_1	1,0	6,4	0,9
s_2	2,1	0,2	3,0
s_3	3,7	2,3	4,0

- $E_1(r) = 1(1-r) + 3r = 1 + 2r$
- $E_2(r) = 6(1-r) + 2r = 6 - 4r$
- $E_3(r) = 0(1-r) + 4r = 4r$

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mixed strategies

- $E_1(r) = 1(1-r) + 3r = 1 + 2r$
- $E_2(r) = 6(1-r) + 2r = 6 - 4r$
- $E_3(r) = 0(1-r) + 4r = 4r$
- $m(r) = \min\{E_1, E_2, E_3\}$

max for $r = 5/6$
payoff is $E_1(r) = 2\frac{2}{3}$

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Ex: Matching Pennies

- Head and Tail
- if different, 1 pays a Pound to 2; if the same, 2 pays a Pound to 1

	H	T
H	1,-1	-1,1
T	-1,1	1,-1

no dominant strategy

No Nash equilibrium

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Ex: Matching Pennies

- Suppose r plays (0.5,0.5), and
- c plays (0.5,0.5)

	H	T
H	1,-1	-1,1
T	-1,1	1,-1

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Ex: Matching Pennies

- Suppose r plays (0.5,0.5), and
- c plays (0.5,0.5)

■ π_1 would then be:

■ $0.5 \cdot 0.5 \cdot 1 + 0.5 \cdot 0.5 \cdot -1 +$

■ $0.5 \cdot 0.5 \cdot 1 + 0.5 \cdot 0.5 \cdot -1 = 0$

- this is Nash:

■ would r play $(q, 1-q)$ then $\pi_1((q, 1-q), (0.5, 0.5)) =$

■ $(q \cdot 0.5 \cdot 1) + (q \cdot 0.5 \cdot -1) +$

■ $((1-q) \cdot 0.5 \cdot -1) + ((1-q) \cdot 0.5 \cdot 1) = 0$ is not better

	H	T
H	1,-1	-1,1
T	-1,1	1,-1

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Ex: Matching Pennies

- Suppose r plays (0.5,0.5), and
- c plays (0.5,0.5)

■ π_1 would then be:

■ $0.5 \cdot 0.5 \cdot 1 + 0.5 \cdot 0.5 \cdot -1 +$

■ $0.5 \cdot 0.5 \cdot 1 + 0.5 \cdot 0.5 \cdot -1 = 0$

- no other Nash:

■ would r play $(q, 1-q)$ with $q > 0.5$ then c plays T; $\pi_1 =$

■ $(q \cdot 0 \cdot 1) + (q \cdot 1 \cdot -1) +$

■ $((1-q) \cdot 0 \cdot -1) + ((1-q) \cdot 1 \cdot 1) = 1 - (2 \cdot q) < 0$

	H	T
H	1,-1	-1,1
T	-1,1	1,-1

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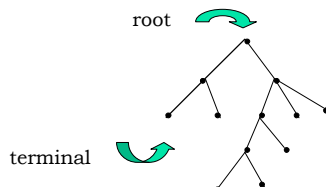
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Game Theory: Part II: Extensive Games

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Rules of the Game

When?



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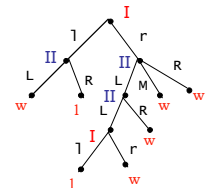
Rules of the Game

When?

What?

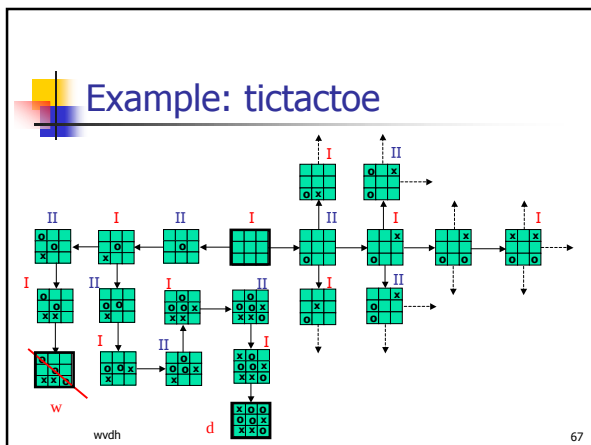
Who?

How much?



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Strategies

- A *pure strategy* for player p specifies for every decision node of p what he will do there
- If all players choose such a strategy, the outcome of the game is determined there

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Strategies

- A *pure strategy* for player p specifies for every decision node of p what he will do there
- Strategies for I:

ll lr rl rr

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Strategies

- A *pure strategy* for player p specifies for every decision node of p what he will do there
- Strategies for II:

LLL, LLR, LML, LMR, LRL, LRR
RLL, RLR, RML, RMR, RRL, RRR

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Strategy profile

Example:

[lr, RMR]

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extensive to strategic

reduced strategic form

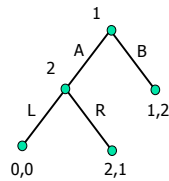
	C	D
AE	a	c
AF	b	c
BE	d	d
BF	d	d

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Equilibria: example

- Nash equilibria?
- via strategic form:

	L	R
A	0,0	2,1
B	1,2	1,2



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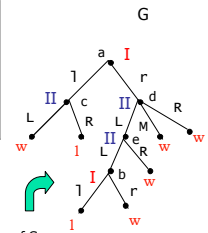
Strategy profile

	L						R					
	LL	LR	ML	MR	RL	RR	LL	LR	ML	MR	RL	RR
ll	w	w	w	w	w	w	1	1	1	1	1	1
lr	w	w	w	w	w	w	1	1	1	1	1	1
rl	1	w	w	w	w	w	1	w	w	w	w	w
rr	w	w	w	w	w	w	w	w	w	w	w	w

a strategy profile: $[rr, RLL]$

strategic form of G

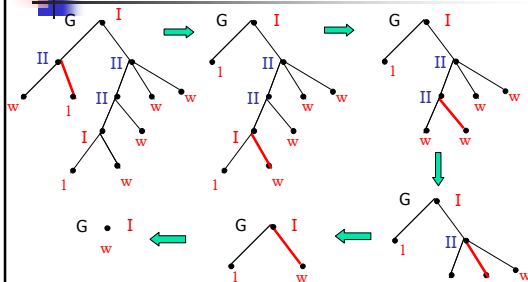
extensive form of G



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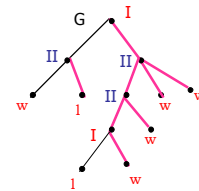
Backward Induction



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backward induction



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I can win

	LLL	LLR	LML	LMR	LRL	LRR	RLL	RLR	RML	RMR	RRL	RRR
ll	w	w	w	w	w	w	1	1	1	1	1	1
lr	w	w	w	w	w	w	1	1	1	1	1	1
rl	1	w	w	w	w	w	1	w	w	w	w	w
rr	w	w	w	w	w	w	w	w	w	w	w	w

again: rr is winning strategy,
since that row only contains a w

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extensive games: definitions

- extensive games: $G = \langle N, H, P, (\succeq_i) \rangle$
 - N : set of players
 - H histories: $\emptyset, (a^k)_{k=1..K}$ (may be infinite)
 - closed under prefixes
 - terminals Z : no successor or infinite
 - P : $H \setminus Z \rightarrow N$ player who is to move
 - $\succeq_i$: preference relation on Z

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extensive games: definitions

- N : set of players
- H histories: $\emptyset, (a^k)_{k=1..K}$ (may be infinite)
 - closed under prefixes
 - terminals Z : no successor or infinite
 - $h \in H, a \text{ action} \Rightarrow (h,a) \in H$
- H is finite $\Rightarrow G$ is **finite**
- H only contains finite $h \Rightarrow G$ has **finite horizon**

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Subgame perfect solutions

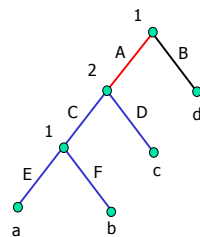
- extensive games: $\Gamma = \langle N, H, P, (\succeq_i) \rangle$
 - N : set of player
 - H histories: $\emptyset, (a^k)_{k=1..K}$ (may be infinite)
 - $P: H \setminus Z \rightarrow N$ player to play
 - $\succeq_i$: preference relation on Z
- subgames: $\Gamma(h) = \langle N, H|_h, P|_h, (\succeq_i|_h) \rangle$
 - all continuations of h

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Subgames

- history h
- subgame $\Gamma(h)$



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subgame perfect N.-eq

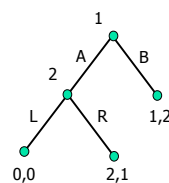
- let $\Gamma = \langle N, H, P, (\succeq_i) \rangle$ extensive
 - s^* is N.-eq if $\forall i \forall s_i O(s_{-i}^*, s_i^*) \succeq_i O(s_{-i}^*, s_i)$
- s^* is subgame perfect N.-eq if
 - $\forall i \forall h \in H \setminus Z (P(h)=i \Rightarrow$
 - $O_h(s_{-i}^*|_h, s_i^*|_h) \succeq_{i,h} O_h(s_{-i}^*|_h, s_i|_h)$
 - for all strategies s_i for i in $\Gamma(h)$
- $s^*|_h$ is N.-eq for all $\Gamma(h)$

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Equilibria: Example

- Nash equilibria?

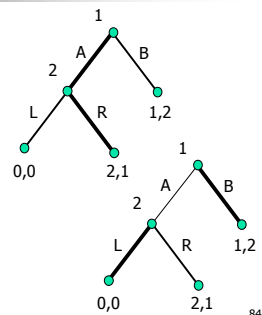


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equilibria (ctd)

- so: (A,R) and (B,L)
- interpretation (B,L):
 - given that 2 plays L after A, 1 better choose B
- intuitive?
- what is optimal for 1?



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equilibria (ctd)

- so: (A,R) and (B,L)
- interpretation (B,L):
- given that 2 plays L after A, 1 better choose B
- AR is the only subgame perfect equilibrium

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equilibria (ctd)

- so: (A,R) and (B,L)
- interpretation (B,L):
- given that 2 plays L after A, 1 better choose B
- AR is the only subgame perfect equilibrium
- not BL!

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shop-chain game

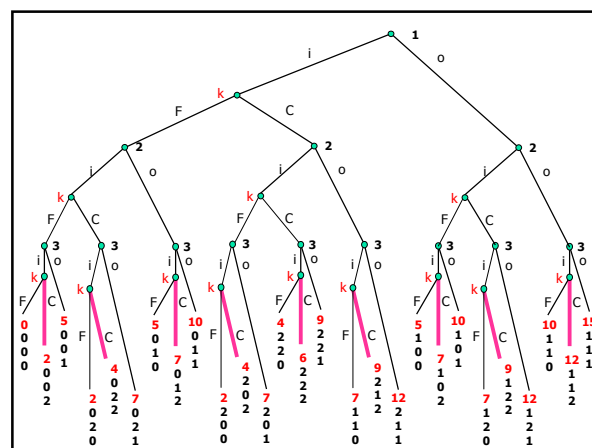
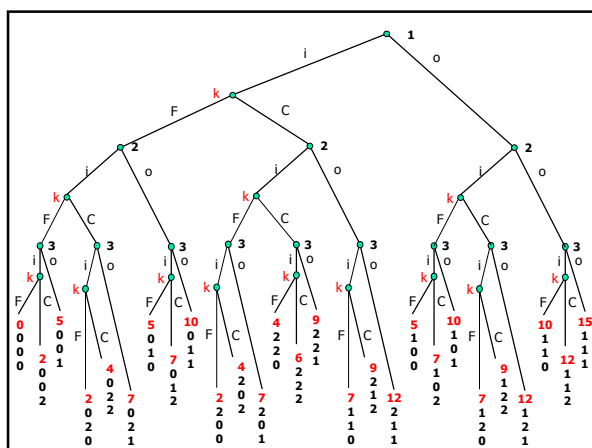
- chain k and n competitors
- every competitor can either enter challenge k (i), or not (o)
- if so, k chooses between cooperate (c) and fight (f)

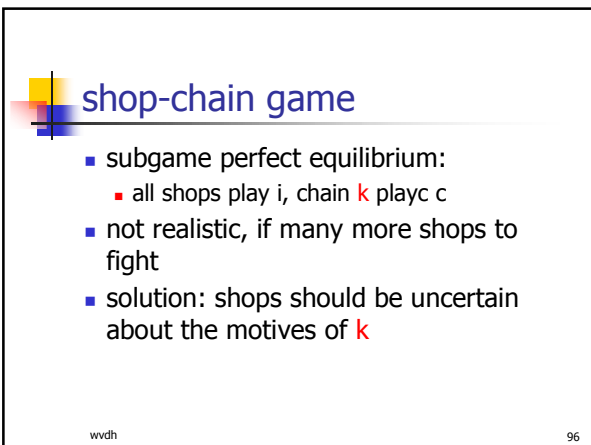
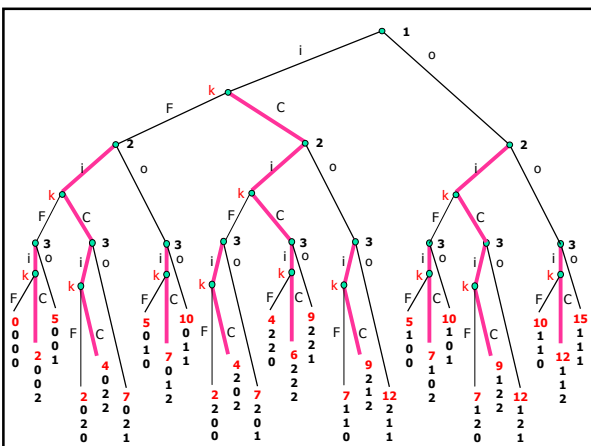
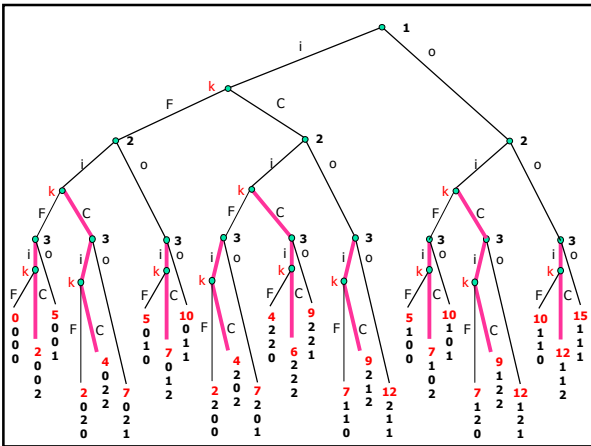
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shop-chain game

- chain k and n competitors
- every competitor can either enter challenge k (i), or not (o)
- if so, k chooses between cooperate (c) and fight (f)

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Backward Induction

1,1 2,2 3,3 0,0

1 2 1

d r d r d r

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Backward Induction

1,1 2,2 3,3 0,0

1 2 1

d r d r d r

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Backward Induction

1,1 2,2 3,3 0,0

1 2 1

d r d r d r

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Backward Induction

1,1 2,2 3,3 0,0

1 2 1

d r d r d r

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Centipede

- 1 and 2 divide n marbles; they choose in turn, if somebody picks two, the game is over

2,0 1,2 3,1 2,3 4,2 3,4 3,3

1 2 1 2 1 2

t e t e t e t e

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Centipede

- Intuitively correct?

2,0 1,2 3,1 2,3 4,2 3,4 3,3

1 2 1 2 1 2

t e t e t e t e

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Strategic voting

- Boris, Horace and Maurice determine who can be a member of the Dead Poet Society
 - proposal: allow Alice
 - counterprop: allow Bob, rather than Alice
 - first vote over counterprop, then over proposal

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Strategic voting

- first between A, B
 - winner Alice
- then between A, N
 - winner Alice
- strategic voting H:
 - first vote Bob!
 - solution... B, N

Maurice		Horace		Borice	
Bob		Nobody		Alice	
Alice		Alice		Nobody	
Nobody		Bob		Bob	

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Strategic voting

- first between A, B
 - winner Alice
- then between A, N
 - winner Alice
- strategic voting H:
 - first vote Bob!
 - solution... B, N
- M anticipates: vote for A

Maurice		Horace		Borice	
Bob		Nobody		Alice	
Alice		Alice		Nobody	
Nobody		Bob		Bob	

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Strategic voting

- first between A, B
 - winner Alice
- then between A, N
 - winnner Alice

utility	B	H	M
a	3	2	2
b	1	1	3
n	2	3	1

Maurice		Horace		Borice	
Bob		Nobody		Alice	
Alice		Alice		Nobody	
Nobody		Bob		Bob	

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Strategic voting: extensive

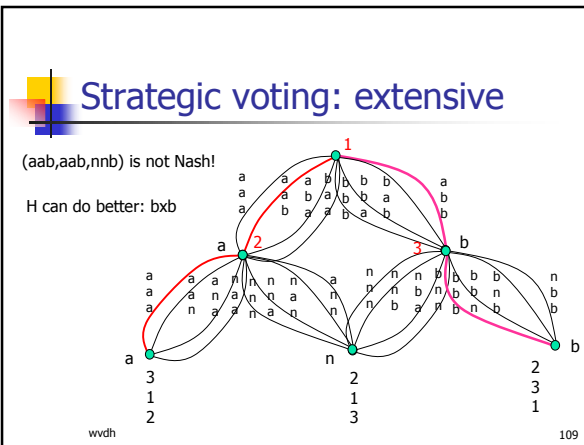
u	B	H	M
a	3	2	2
b	1	1	3
n	2	3	1

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Strategic voting: extensive

(aaa,aaa,xyz) is Nash

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Pirates on an island

- Five pirates $p_1, \dots, p_5$ are on an island
- There is also a bag of 100 diamonds
- And hence, a need to distribute them

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Five Pirates: procedure

player i proposes a division D_i over $p_i, \dots, p_5$

- with a majority for D_i : so be it done
- no majority for D_i : p_i gets shot, we move on to p_{i+1}

Now you are p_1 . What will D_1 be?

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Pirates on an island

- Assumptions: Any pirate**
 - values his life higher than 100 diamonds
 - values 1 diamond higher than another's life
 - votes in favour of a proposal iff others are worse

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Voting agenda paradox

- 1: $x > z > y$; 2: $y > x > z$; 3: $z > y > x$
- 40% type 1, 30% type 2, 30% type 3
- majority rule: x wins
- binary protocol: chair decides!

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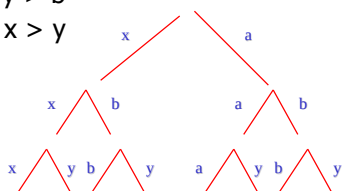
Voting agenda paradox

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- 40% type 1, 30% type 2, 30% type 3
- majority rule: x wins
- binary protocol: chair decides!

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Pareto dominated paradox

- 1: $x > y > b > a$
- 2: $a > x > y > b$
- 3: $b > a > x > y$

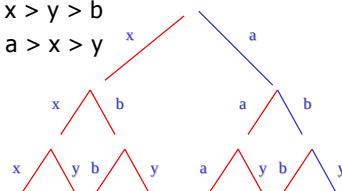


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Pareto dominated paradox

- 1: $x > y > b > a$
 - 2: $a > x > y > b$
 - 3: $b > a > x > y$
- but for all, $x > y$!!



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Borda protocol

- allocate points: 4, 3, 2, 1.
 - 1: $x > c > b > a$ Σ : $x: 22, a: 17, b: 16, c: 15$
 - 2: $a > x > c > b$
 - 3: $b > a > x > c$
 - 4: $x > c > b > a$
 - 5: $a > x > c > b$
 - 6: $b > a > x > c$
 - 7: $x > c > b > a$
- If x withdraws:
 $c: 15, b: 14, a: 13$!!!

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Arrow's theorem

- m agents, each with preference $\leq_i$ over D
- Wanted:
 - $G(\leq_1, \dots, \leq_m, D) = \leq$

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Arrow's theorem

- 1 completeness: $x \leq y$ or $y \leq x$
- 2 transitivity: if $x \leq y \leq z$, then $x \leq z$
- 3 unrestricted domain: all $\leq$ satisfy 1 and 2
- 4 Pareto: if $\forall i, x \leq_i y$, then $x \leq y$
- 5 independence of irrelevant choices
 if $\leq_i$ is as $\leq_i'$ regarding x and y ,
 then $\leq = \leq'$ regarding x and y
- 6 no dictator: no i completely determines $\leq$
 It is impossible to generate such a $\leq$!

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