Playing Muller Games in a Hurry

Joint work with John Fearnley, University of Warwick

Martin Zimmermann

RWTH Aachen University

September 21st, 2010

Games Workshop 2010
Oxford, United Kingdom
Motivation


*We believe that infinite games might have an interest for casual living-room recreation.*
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McNaughton suggests a method of keeping score to declare a winner such that

.. if the play were to continue with each [player] playing forever as he has so far, then the player declared to be the winner would be the winner of the infinite play of the game.
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*“Winning regions should be equal”*
Muller Games

A Muller game \((G, \mathcal{F}_0, \mathcal{F}_1)\) consists of an arena \(G = (V, V_0, V_1, E)\) and a partition \((\mathcal{F}_0, \mathcal{F}_1)\) of \(2^V\).

Rules:

- Players move a token through the arena ad infinitum.
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Winning strategy for Player 0 (circles): coming from 1 to 2 move to 3, coming from 3 to 2 move to 1.
Scoring Functions

For $F \subseteq V$ define $Sc_F : V^+ \rightarrow \mathbb{N}$:

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|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
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Finite-time Muller Games

Two properties of the scoring functions (informal versions):

1. If you play long enough, some score value will be high.
2. At most one score value can increase at a time.
Finite-time Muller Games

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Definition

Finite-time Muller game: \((\mathcal{G}, \mathcal{F}_0, \mathcal{F}_1, k)\) with threshold \(k \geq 2\).

Rules:

- Players move a token through the arena.
- Stop play \(w\) as soon as score of \(k\) is reached for the first time.
- There is a unique \(F\) such that \(S_{CF}(w) = k\) (see above).
- Player \(i\) wins \(w\) iff \(F \in \mathcal{F}_i\).

Martin Zimmermann
RWTH Aachen University
Playing Muller Games in a Hurry
McNaughton’s version: stop play when some $S_{CF}$ reaches $|F|! + 1$.

**Theorem (McNaughton 2000)**

*The winning regions in a Muller game and in McNaughton’s finite-time Muller game coincide.*
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Our result:

**Theorem**

The winning regions in a Muller game $(G, F_0, F_1)$ and in the finite-time Muller game $(G, F_0, F_1, 3)$ coincide.
McNaughton’s version: stop play when some $S_{\mathcal{C}F}$ reaches $|F|! + 1$.

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Stronger statement, which implies the theorem:

**Lemma**

*On her winning region in a Muller game, Player $i$ can prevent her opponent from ever reaching a score of 3 for every set $F \in \mathcal{F}_{1-i}$.*
## Conclusion

### Results:

<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>Threshold</td>
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<td>$</td>
<td>F</td>
</tr>
<tr>
<td>Play Length Space</td>
<td>$\leq n \cdot n! + 1$</td>
<td>$\leq (n! + 1)^n$</td>
<td>$\leq 3^n$</td>
</tr>
<tr>
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Open Questions:

- Is the finite-time Muller game with threshold 2 equivalent to the original Muller game?
- Given a winning strategy for a finite-time Muller game, can we turn it into a winning strategy for the Muller game?