Playing Pushdown Parity Games in a Hurry

Joint work with Wladimir Fridman (RWTH Aachen University)

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Motivation

Playing infinite games in finite time:

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- **Jurdziński**: small progress measures for parity games.
- **Bernet, Janin, Walukiewicz**: permissive strategies for parity games.
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Parity Games

Arena $\mathcal{A} = (V, V_0, V_1, E, v_{in})$:
- directed (possibly countable) graph $(V, E)$.
- positions of the players: partition $\{V_0, V_1\}$ of $V$.
- initial vertex $v_{in} \in V$. 
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Parity game $G = (\mathcal{A}, \text{col})$ with $\text{col}: V \rightarrow \{0, \ldots, d\}$.
- Player 0 wins play $\iff$ minimal color seen infinitely often even.
- (Winning / positional) strategies defined as usual.
- Player $i$ wins $G$ $\iff$ she has winning strategy from $v_{\text{in}}$. 
For $c \in \mathbb{N}$ and $w \in V^*$: $\text{Sc}_c(w)$ denotes the number of occurrences of $c$ in the suffix of $w$ after the last occurrence of a smaller color.

Formally: $\text{Sc}_c(\varepsilon) = 0$ and

$$
\text{Sc}_c(wv) = \begin{cases} 
\text{Sc}_c(w) & \text{if } \text{col}(v) > c, \\
\text{Sc}_c(w) + 1 & \text{if } \text{col}(v) = c, \\
0 & \text{if } \text{col}(v) < c.
\end{cases}
$$
Scoring Functions for Parity Games

For $c \in \mathbb{N}$ and $w \in V^*$: $S_c(w)$ denotes the number of occurrences of $c$ in the suffix of $w$ after the last occurrence of a smaller color.

**Remark**

In a finite arena, a positional winning strategy for Player 0 bounds the scores for all odd $c$ by $|V|$. 

Corollary

In a finite arena, Player 0 wins $\iff$ she can prevent a score of $|V| + 1$ for all odd $c$ (safety condition).

The remark does not hold in infinite arenas:
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For \( c \in \mathbb{N} \) and \( w \in V^* \): \( S_{c}(w) \) denotes the number of occurrences of \( c \) in the suffix of \( w \) after the last occurrence of a smaller color.

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Pushdown Arena 

Pushdown arena $\mathcal{A} = (V, V_0, V_1, E, v_{in})$ induced by Pushdown System $\mathcal{P} = (Q, \Gamma, \Delta, q_{in})$:

- $(V, E)$: configuration graph of $\mathcal{P}$.
- $\{V_0, V_1\}$ induced by partition $\{Q_0, Q_1\}$ of $Q$.
- $v_{in} = (q_{in}, \bot)$. 
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Pushdown Arenas

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- $v_{\text{in}} = (q_{\text{in}}, \bot)$.

Pushdown parity game $G = (A, \text{col})$ where \text{col} is lifting of \text{col}: $Q \rightarrow \{0, \ldots, d\}$ to configurations.
$w$ finite path starting in $v_{in}$:
Stairs and Stair-Scores

\( w \) finite path starting in \( v_{\text{in}} \):
- Stair in \( w \): position s. t. no subsequent position has smaller stack height (first and last position are always a stair).
- \( \text{reset}(w) \): prefix of \( w \) up to second-to-last stair.
- \( \text{lstBmp}(w) \): suffix after second-to-last stair.
Stairs and Stair-Scores

$w$ finite path starting in $v_{in}$:

- **Stair in $w$**: position s. t. no subsequent position has smaller stack height (first and last position are always a stair).
- **reset($w$)**: prefix of $w$ up to second-to-last stair.
- **lstBmp($w$)**: suffix after second-to-last stair.
For every color $c$, define $\text{StairSc}_c : \mathcal{V}^* \to \mathbb{N}$ by $\text{StairSc}_c(\varepsilon) = 0$ and

$$\text{StairSc}_c(w) = \begin{cases} 
\text{StairSc}_c(\text{reset}(w)) & \text{if minCol(\text{lstBmp}(w))} > c, \\
\text{StairSc}_c(\text{reset}(w)) + 1 & \text{if minCol(\text{lstBmp}(w))} = c, \\
0 & \text{if minCol(\text{lstBmp}(w))} < c.
\end{cases}$$
Stairs and Stair-Scores

```
reset(\(w\))
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IstBmp(\(w\))
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Stairs and Stair-Scores

\[ \text{reset}(w) \quad \text{lstBmp}(w) \]

\[ \begin{array}{c}
\text{stack height} \\
\text{col:} \\
\text{StairSc}_0: \quad 2 \\
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\end{array} \]
Stairs and Stair-Scores

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\begin{align*}
\text{stack height} & \quad \text{col:} & 1 & 1 & 1 & 2 & 1 \\
\text{StairSc}_0 & \quad 2 & 2 & 2 & 0 \\
\text{StairSc}_1 & \quad 2 & 3 & 3 & 3 \\
\text{StairSc}_2 & \quad 0 & 0 & 0 & 0
\end{align*}
Main Theorem

Finite-time pushdown game: \((A, \text{col}, k)\) with pushdown arena \(A\), coloring \(\text{col}\), and \(k \in \mathbb{N} \setminus \{0\}\).

Rules:

- Play until \(\text{StairSc}_c = k\) is reached for the first time for some color \(c\) (which is unique).
- Player 0 wins \(\Leftrightarrow c\) is even.
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Let \(d = |\text{col}(V)|\).

Theorem

Let \(G = (A, \text{col})\) be a pushdown game and \(k > |Q| \cdot |\Gamma| \cdot 2^{|Q| \cdot d} \cdot d\). Player \(i\) wins \(G\) if and only if Player \(i\) wins \((A, \text{col}, k)\).

Note: \((A, \text{col}, k)\) is a reachability game in finite arena.
Proof Idea

Walukiewicz (96):

- Reduction from pushdown parity game $G$ to parity game $G'$ in finite arena $A'$ (of exponential size):
- Turn winning strategy $\sigma'$ for $G'$ into winning strategy $\sigma$ for $G$.

One can show more:
For every play prefix $w$ in $G$ consistent with $\sigma$, there exists play prefix $w'$ in $G'$ consistent with $\sigma'$ such that $\text{StairSc}(w) = \text{Sc}(w')$ for every color $c$.

If $\sigma'$ is positional winning strategy for Player $i$ in $G'$, then $\sigma$ bounds the scores of Player $1-i$ in $G$ by $|A'|$.

Hence, Player $i$ wins $(A,\text{col},k)$, provided $k > |A'|$.
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- If $\sigma'$ is positional winning strategy for Player $i$ in $G'$, then $\sigma$ bounds the scores of Player $1 - i$ in $G$ by $|A'|$.
- Hence, Player $i$ wins $(A, \text{col}, k)$, provided $k > |A'|$. 
Lower Bounds

For the first $n$ primes $p_1, \ldots, p_n$: Player 0 has to reach stack height $\prod_{j=1}^n p_j > 2^n$ in the upper row; this cannot prevent the losing player from reaching exponentially high scores (in the number of states).

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Conclusion

Playing pushdown parity games in finite time:

- Adapt scores to stair-scores.
- Exponential threshold stair-score yields equivalent finite-duration game (reachability game in finite tree).
- (Almost) matching lower bounds on threshold stair-score.
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Further research:

- Turn winning strategy for finite-duration game into winning strategy for pushdown game.
- Permissive strategies for pushdown parity games.
- Extensions to more general classes of arenas, e.g., higher-order pushdown systems.