Playing Pushdown Parity Games in a Hurry

Joint work with Wladimir Fridman (RWTH Aachen University)

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Playing infinite games in finite time:

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Results hold only for finite arenas. What about infinite ones?

Parity Games

Arena $\mathcal{A} = (V, V_0, V_1, E, v_{\mathrm{in}})$:

- directed (possibly countable) graph (V, E).
- positions of the players: partition $\{V_0, V_1\}$ of V.
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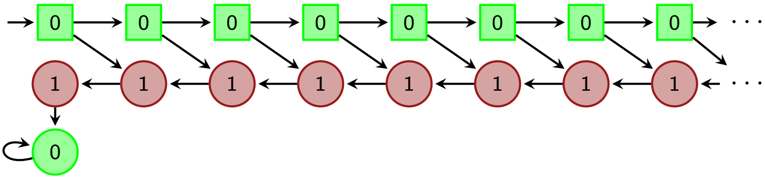

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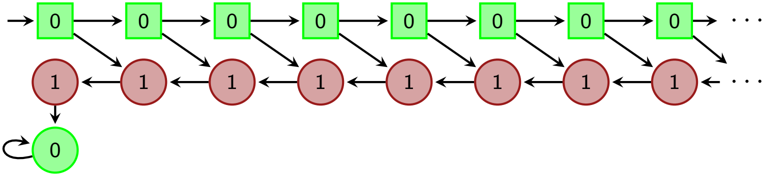

Parity game $\mathcal{G} = (\mathcal{A}, \mathrm{col})$ with $\mathrm{col}\colon V \to \{0, \ldots, d\}$.

- Player 0 wins play $\Leftrightarrow$ minimal color seen infinitely often even.
- (Winning / positional) strategies defined as usual.
- Player i wins $\mathcal{G}$ $\Leftrightarrow$ she has winning strategy from v_{in}.

Scoring Functions for Parity Games

For $c \in \mathbb{N}$ and $w \in V^*$: $\mathsf{Sc}_c(w)$ denotes the number of occurrences of c in the suffix of w after the last occurrence of a smaller color.

Formally: $\mathsf{Sc}_c(\varepsilon) = 0$ and

$$\mathsf{Sc}_c(wv) = \begin{cases} \mathsf{Sc}_c(w) & \text{if } \mathrm{col}(v) > c, \\ \mathsf{Sc}_c(w) + 1 & \text{if } \mathrm{col}(v) = c, \\ 0 & \text{if } \mathrm{col}(v) < c. \end{cases}$$

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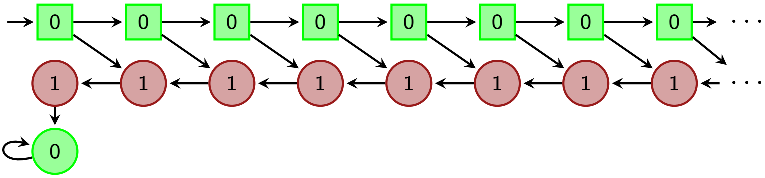
The remark does not hold in infinite arenas:



Pushdown Arenas

Pushdown arena $\mathcal{A} = (V, V_0, V_1, E, v_{\text{in}})$ induced by Pushdown System $\mathcal{P} = (Q, \Gamma, \Delta, q_{\text{in}})$:

- (V, E): configuration graph of $\mathcal{P}$.
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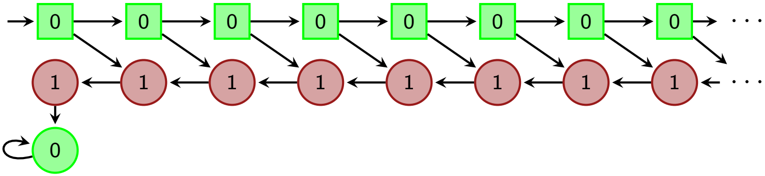

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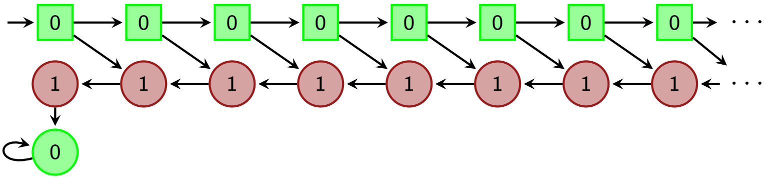
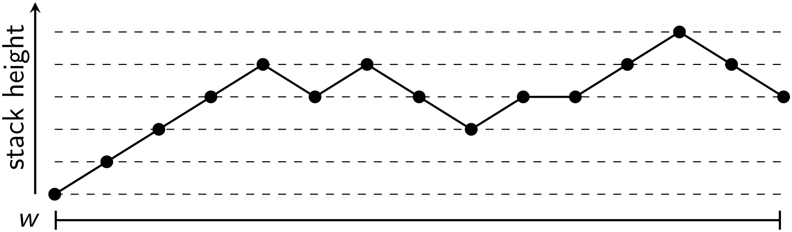


Pushdown parity game $\mathcal{G} = (\mathcal{A}, \mathrm{col})$ where col is lifting of $\mathrm{col}\colon Q \to \{0, \dots, d\}$ to configurations.

Stairs and Stair-Scores



w finite path starting in v_{in}:

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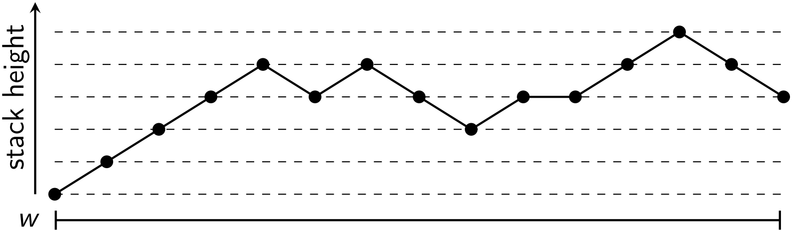

w finite path starting in v_{in}:

- Stair in w: position s. t. no subsequent position has smaller stack height (first and last position are always a stair).
- reset(w): prefix of w up to second-to-last stair.
- lstBmp(w): suffix after second-to-last stair.

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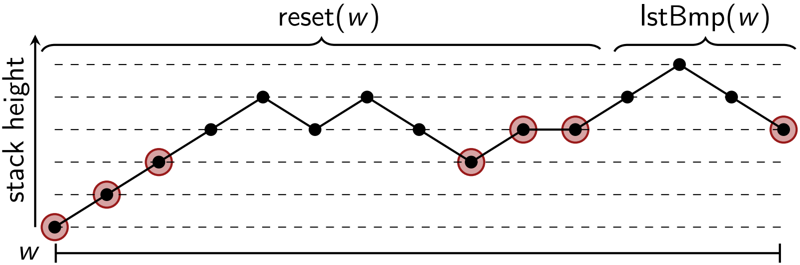

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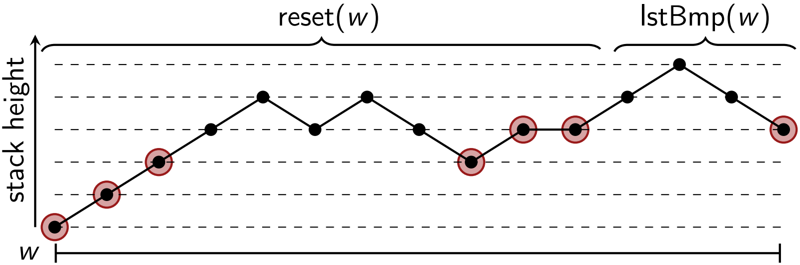

For every color c, define $\mathsf{StairSc}_c \colon V^* \to \mathbb{N}$ by $\mathsf{StairSc}_c(\varepsilon) = 0$ and

$$\mathsf{StairSc}_c(w) = \begin{cases} \mathsf{StairSc}_c(\mathsf{reset}(w)) & \text{if } \mathsf{minCol}(\mathsf{lstBmp}(w)) > c, \\ \mathsf{StairSc}_c(\mathsf{reset}(w)) + 1 & \text{if } \mathsf{minCol}(\mathsf{lstBmp}(w)) = c, \\ 0 & \text{if } \mathsf{minCol}(\mathsf{lstBmp}(w)) < c. \end{cases}$$

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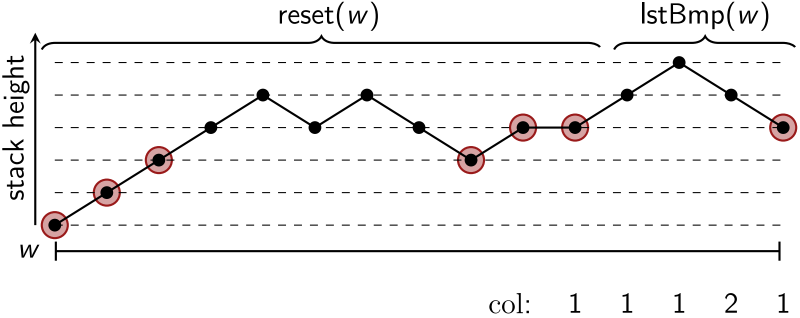

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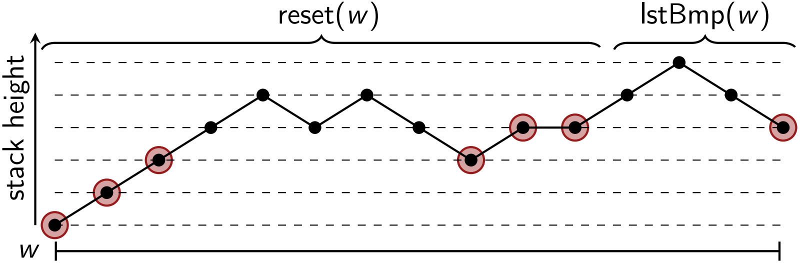

col:	1	1	1	2	1
$\mathsf{StairSc}_0$:	2				
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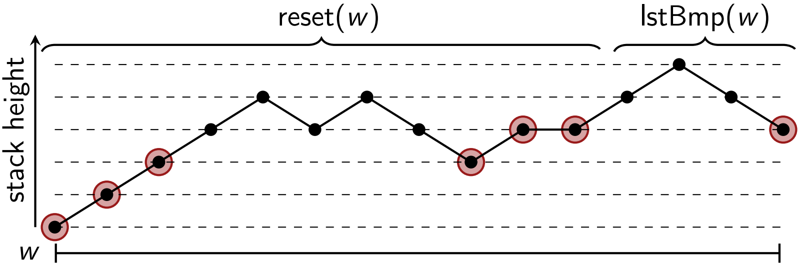

col:	1	1	1	2	1
$\mathsf{StairSc}_0$:	2				2
$\mathsf{StairSc}_1$:	2				3
$\mathsf{StairSc}_2$:	0				0

Main Theorem

Finite-time pushdown game: $(\mathcal{A}, \mathrm{col}, k)$ with pushdown arena $\mathcal{A}$, coloring col, and $k \in \mathbb{N} \setminus \{0\}$.

Rules:

- Play until $\mathsf{StairSc}_c = k$ is reached for the first time for some color c (which is unique).
- Player 0 wins $\Leftrightarrow$ c is even.

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Let $d = |\mathrm{col}(V)|$.

Theorem

Let $\mathcal{G} = (\mathcal{A}, \mathrm{col})$ be a pushdown game and $k > |Q| \cdot |\Gamma| \cdot 2^{|Q| \cdot d} \cdot d$. Player i wins $\mathcal{G}$ if and only if Player i wins $(\mathcal{A}, \mathrm{col}, k)$.

Note: $(\mathcal{A}, \mathrm{col}, k)$ is a reachability game in finite arena.

Proof Idea

Walukiewicz (96):

- Reduction from pushdown parity game $\mathcal{G}$ to parity game $\mathcal{G}'$ in finite arena $\mathcal{A}'$ (of exponential size):
- Turn winning strategy σ' for $\mathcal{G}'$ into winning strategy σ for $\mathcal{G}$.

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One can show more:

- For every play prefix w in $\mathcal{G}$ consistent with σ, there exists play prefix w' in $\mathcal{G}'$ consistent with σ' such that

$$\mathsf{StairSc}_c(w) = \mathsf{Sc}_c(w') \quad \text{for every color c.}$$

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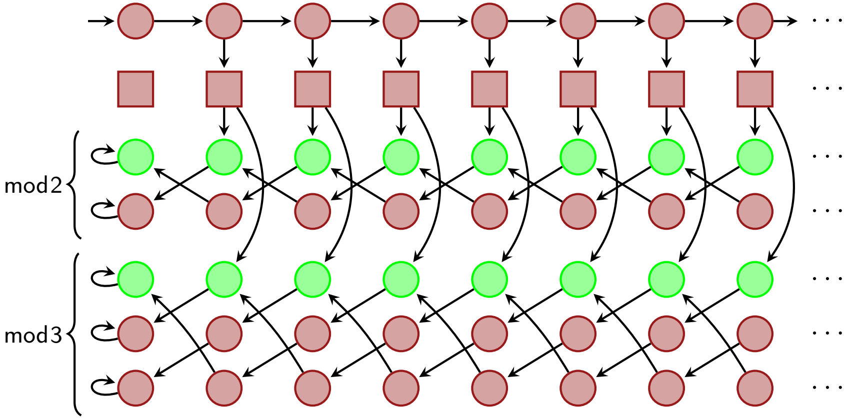
- If σ' is positional winning strategy for Player i in $\mathcal{G}'$, then σ bounds the scores of Player $1 - i$ in $\mathcal{G}$ by $|\mathcal{A}'|$.
- Hence, Player i wins $(\mathcal{A}, \mathrm{col}, k)$, provided $k > |\mathcal{A}'|$.

Lower Bounds



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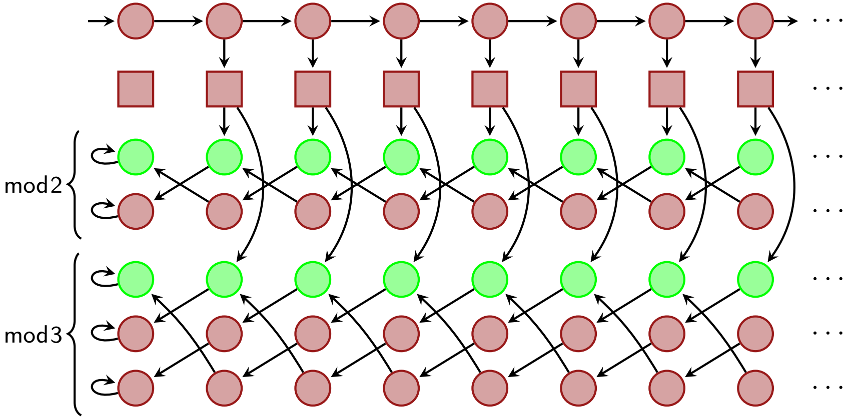

For first n primes $p_1, \ldots, p_n$: Player 0 has to reach stack height $\prod_{j=1}^{n} p_j > 2^n$ in upper row $\Rightarrow$ cannot prevent losing player from reaching exponentially high scores (in the number of states).

Conclusion

Playing pushdown parity games in finite time:

- Adapt scores to stair-scores.
- Exponential threshold stair-score yields equivalent finite-duration game (reachability game in finite tree).
- (Almost) matching lower bounds on threshold stair-score.

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Further research:

- Turn winning strategy for finite-duration game into winning strategy for pushdown game.
- Permissive strategies for pushdown parity games.
- Extensions to more general classes of arenas, e.g., higher-order pushdown systems.