Playing Muller Games in a Hurry

Joint work with John Fearnley, University of Warwick

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May 11th, 2010

Gasics Meeting Spring 2010
Aalborg, Denmark
Motivation

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*We believe that infinite games might have an interest for casual living-room recreation.*
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McNaughton suggests a method of keeping score to declare a winner such that

.. if the play were to continue with each [player] playing forever as he has so far, then the player declared to be the winner would be the winner of the infinite play of the game.
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Questions:

- Is there an equivalent finite-duration version of a Muller game?
- How long do finite plays have to be?
- Do short finite plays lead to faster algorithms?
- Can we turn winning strategies for finite games into (small) finite-state winning strategies for infinite games?
A first idea

Consider an infinite game $\mathcal{G}$ played on finite graph.

- Stop a play as soon as a cycle is closed. The winner of the induced infinite play is declared to win the finite play.
- If $\mathcal{G}$ is positionally determined, then the winning regions of both games coincide.
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- This can be extended to games $G$ that are determined with finite-state strategies: wait for a repetition of a memory state (for some fixed memory structure).

Drawbacks (assuming $G$ is a Muller game with $n$ vertices):

- maximal play length: $n!$.
- need to remember $n!$ memory states.

Our goal: improve both bounds.
Outline

1. Muller Games and Scoring Functions

2. Finite-time Muller Games

3. Conclusion
Muller Games

- Arena: $G = (V, V_0, V_1, E)$ with finite, directed graph $(V, E)$, partition $(V_0, V_1)$ of $V$. 
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- Muller game: \((G, \mathcal{F}_0, \mathcal{F}_1)\) with partition \((\mathcal{F}_0, \mathcal{F}_1)\) of \(2^V\).

- Player \(i\) wins play \(\rho\) iff \(\text{Inf}(\rho) = \{v \mid \exists \omega \text{ s.t. } \rho_j = v\} \in \mathcal{F}_i\).
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- Unique play started at $v$ that is played according to $\sigma \in \Pi_i$ and $\tau \in \Pi_{1-i}$: $\text{Play}(v, \sigma, \tau)$.
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- Winning region of Player $i$:

  $$W_i = \{v \in V \mid \exists \sigma \in \Pi_i \forall \tau \in \Pi_{1-i} : \text{Play}(v, \sigma, \tau) \text{ won by Player } i\}$$
For $F \subseteq V$ define $S_{cF} : V^+ \rightarrow \mathbb{N}$:

$$S_{cF}(w) = \max\{k \mid \text{exists suffix } x_1 \cdots x_k \text{ of } w \text{ s.t. } x_i \in V^+ \text{ and } \text{Occ}(x_i) = F \text{ for all } i\}$$

where $\text{Occ}(w) = \{v \mid \exists j \text{ s.t. } w_j = v\}$. 
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For $F \subseteq V$ define $Sc_F : V^+ \rightarrow \mathbb{N}$:

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Example:

| w   | a a b b a a b c a b c a c |
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Scoring Functions

For $\mathcal{F} \subseteq 2^V$ define $\text{MaxSc}_\mathcal{F} : V^+ \cup V^\omega \rightarrow \mathbb{N} \cup \{\infty\}$:

$$\text{MaxSc}_\mathcal{F}(\rho) = \max_{F \in \mathcal{F}} \max_{w \sqsubseteq \rho} \text{Sc}_F(w)$$
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$\mathcal{F} = \{\{a\}, \{a, b\}, \{a, b, c\}\}$:

$$\text{MaxSc}_\mathcal{F}(w) = 3$$
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Results about Scoring

Lemma

Every \( w \in V^* \) with \( |w| \geq k|V| \) satisfies \( \text{MaxSc}_{2V}(w) \geq k \).

“If you play long enough, some score value will be high”

Lower bound: there are words \( w_k \) of length \( k|V| - 1 \) with \( \text{MaxSc}_{2V}(w_k) < k \).
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Lemma (McNaughton 2000)

Let \( k, m \geq 2 \), let \( F, H \subseteq V \), let \( w \in V^* \) and \( s \in V \) such that \( \text{Sc}_F(w) < k \) and \( \text{Sc}_H(w) < m \). If \( \text{Sc}_F(ws) = k \) and \( \text{Sc}_H(ws) = m \), then \( F = H \).

“At most one score value can increase at a time”
Finite-time Muller Games

- Finite-time Muller game: \((G, F_0, F_1, k)\) with threshold \(k \geq 2\).
- Play: path \(w = w_1 \cdots w_n\) with \(\text{MaxSc}_{2V}(w_0 \cdots w_n) = k\), but \(\text{MaxSc}_{2V}(w_1 \cdots w_{n-1}) < k\).
- Previous Lemma yields unique \(F \subseteq V\) such that \(\text{Sc}_F(w) = k\). Player \(i\) wins \(w\) iff \(F \in F_i\).
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McNaughton considered a different definition of a finite-time Muller game: stop play when some \(\text{Sc}_F\) reaches \(|F|! + 1\).

**Theorem (McNaughton 2000)**

The winning regions in a Muller game and in McNaughton’s finite-time Muller game coincide.
Main Theorem

**Theorem**

The winning regions in a Muller game \((G, \mathcal{F}_0, \mathcal{F}_1)\) and in the finite-time Muller game \((G, \mathcal{F}_0, \mathcal{F}_1, 3)\) coincide.
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We prove a stronger statement, which implies the theorem.

Lemma

Player \(i\) has a strategy \(\sigma\) for a Muller game \((G, \mathcal{F}_0, \mathcal{F}_1)\) such that \(\max_{\mathcal{F}_{1-i}} \text{Sc}(\text{Play}(v, \sigma, \tau)) \leq 2\) for every \(v \in W_i\) and every \(\tau \in \Pi_{1-i}\).
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What about 2?

The bound 2 in the lemma is optimal: Player 0 has a winning strategy, but cannot avoid score values of 2 for Player 1.

\[ F_0 = \{\{1, 2, 3\}, \{1\}, \{3\}\} \]
\[ F_1 = 2^{\{1,2,3\}} \setminus F_0 \]

One of the plays 2112 or 2332 is consistent with every winning strategy.
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Consequence:
To show that the finite-time Muller game with threshold 2 is equivalent, we need other proof techniques.
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We have presented a finite-duration version of a Muller game that is equivalent to the original game.

- Reachability game on a tree; hence, simple algorithms are available.
- Maximal play length: $3^n$;
- Space requirement $O(3^n)$, where $n = |G|$.
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Open questions:

- Is the finite-time Muller game with threshold 2 equivalent to the original Muller game?
- Given a winning strategy for a finite-time Muller game, can we turn it into a winning strategy for the Muller game?