The First-Order Logic of Hyperproperties

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The system $S$ is input-deterministic: for all traces $t$, $t'$ of $S$, $t = I t'$ implies $t = O t'$.

Noninterference: for all traces $t$, $t'$ of $S$, $t = I public t'$ implies $t = O public t'$. 

Hyperproperties
The system $S$ is input-deterministic: for all traces $t, t'$ of $S$

$t =_I t'$ implies $t =_O t'$
The system $S$ is input-deterministic: for all traces $t, t'$ of $S$

$$t =_I t' \implies t =_O t'$$

Noninterference: for all traces $t, t'$ of $S$

$$t =_{l_{\text{public}}} t' \implies t =_{O_{\text{public}}} t'$$
Both properties are not trace properties, but hyperproperties, i.e., sets of sets of traces.

A system $S$ satisfies a hyperproperty $H$, if $\text{Traces}(S) \in H$.

Many information flow properties can be expressed as hyperproperties.
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Many information flow properties can be expressed as hyperproperties.

Specification languages for hyperproperties [Clarkson et al. ’14]

**HyperLTL**: Extend LTL by trace quantifiers.

**HyperCTL**: Extend CTL* by trace quantifiers.
HyperLTL

HyperLTL = LTL + 

\[ \psi ::= a \mid \neg \psi \mid \psi \lor \psi \mid X \psi \mid \psi U \psi \]

where \( a \in AP \) (atomic propositions)
HyperLTL

HyperLTL = LTL + trace quantification

\[ \varphi ::= \exists \pi. \varphi \mid \forall \pi. \varphi \mid \psi \]
\[ \psi ::= a_{\pi} \mid \neg \psi \mid \psi \lor \psi \mid X \psi \mid \psi U \psi \]

where \( a \in AP \) (atomic propositions) and \( \pi \in V \) (trace variables).
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\[ \psi ::= a_\pi | \neg \psi | \psi \lor \psi | X \psi | \psi U \psi \]

where \( a \in AP \) (atomic propositions) and \( \pi \in V \) (trace variables).

Shortcuts as usual:

- \( F \psi = \text{true} U \psi \)
- \( G \psi = \neg F \neg \psi \)
Semantics

\[ \varphi = \forall \pi. \forall \pi'. \mathbf{G}(\text{on}_\pi \leftrightarrow \text{on}_{\pi'}) \]

\( T \subseteq (2^{2\mathbb{P}})^{\omega} \) is a model of \( \varphi \) iff
Semantics

\[ \varphi = \forall \pi. \forall \pi'. G(on_{\pi} \leftrightarrow on_{\pi'}) \]

\( T \subseteq (2^{AP})^\omega \) is a model of \( \varphi \) iff

\[ \emptyset \models \forall \pi. \forall \pi'. G(on_{\pi} \leftrightarrow on_{\pi'}) \]
\[
\varphi = \forall \pi. \forall \pi'. G\left(\text{on}_\pi \leftrightarrow \text{on}_{\pi'}\right)
\]

\(T \subseteq (2^{AP})^\omega\) is a model of \(\varphi\) iff

\[
\{\} \models \forall \pi. \forall \pi'. G\left(\text{on}_\pi \leftrightarrow \text{on}_{\pi'}\right)
\]

\[
\{\pi \mapsto t\} \models \forall \pi'. G\left(\text{on}_\pi \leftrightarrow \text{on}_{\pi'}\right) \quad \text{for all } t \in T
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\[ \{ \pi \mapsto t, \pi' \mapsto t' \} \models \mathbf{G} (\text{on}_\pi \leftrightarrow \text{on}_{\pi'}) \quad \text{for all } t' \in T \]
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\[ \{ \pi \mapsto t[n, \infty), \pi' \mapsto t'[n, \infty) \} \models \text{on}_\pi \leftrightarrow \text{on}_{\pi'} \quad \text{for all } n \in \mathbb{N} \]
\[ \varphi = \forall \pi. \forall \pi'. G(\text{on}_\pi \leftrightarrow \text{on}_{\pi'}) \]

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\[ \{\pi \mapsto t[n, \infty), \pi' \mapsto t'[n, \infty)\} \models \text{on}_\pi \leftrightarrow \text{on}_{\pi'} \text{ for all } n \in \mathbb{N} \]

\[ \text{on} \in t(n) \iff \text{on} \in t'(n) \]
LTL vs. HyperLTL

LTL has many desirable properties.

1. Every satisfiable LTL formula is satisfied by an **ultimately periodic** trace, i.e., by a finite and finitely-represented model.

2. LTL and FO[<] are **expressively equivalent**.

3. LTL satisfiability and model-checking are PSPACE-complete.

Only partial results for HyperLTL.

3a. HyperLTL satisfiability [F. & Hahn '16]: alternation-free: PSPACE-complete.

3b. HyperLTL model-checking is decidable [F. et al. '15].
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Only partial results for HyperLTL.

3a. HyperLTL satisfiability [F. & Hahn '16]:
   - alternation-free: \( \text{PSPACE} \)-complete
   - \( \exists^*\forall^* \): \( \text{EXPSPACE} \)-complete
   - \( \forall^*\exists^* \): undecidable

3b. HyperLTL model-checking is decidable [F. et al. ’15].
The Models of HyperLTL
What about Finite Models?

Fix $\text{AP} = \{a\}$ and consider the conjunction $\varphi$ of

$$\forall \pi. \ (\neg a_\pi) \ U (a_\pi \land X G \neg a_\pi)$$
What about Finite Models?

Fix $AP = \{a\}$ and consider the conjunction $\varphi$ of

- $\forall \pi. (\neg a_\pi) U (a_\pi \land X G \neg a_\pi)$
- $\exists \pi. a_\pi$

The unique model of $\varphi$ is $\{\emptyset, \{a\}, \emptyset, \{a\}, \emptyset, \ldots\}$.

Theorem

There is a satisfiable HyperLTL sentence that is not satisfied by any finite set of traces.
What about Finite Models?

Fix $AP = \{a\}$ and consider the conjunction $\varphi$ of

- $\forall \pi. (\neg a_\pi) U (a_\pi \land X G \neg a_\pi)$
- $\exists \pi. a_\pi$

$$\{a\} \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \ldots$$
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- \( \forall \pi. (\neg a_{\pi}) U (a_{\pi} \land X G \neg a_{\pi}) \)
- \( \exists \pi. a_{\pi} \)
- \( \forall \pi. \exists \pi'. F (a_{\pi} \land X a_{\pi'}) \)

\( \{a\} \quad \emptyset \quad \emptyset \quad \emptyset \quad \emptyset \quad \emptyset \quad \emptyset \quad \emptyset \quad \emptyset \quad \ldots \)
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- $\forall \pi. (\neg a_\pi) \cup (a_\pi \land X G \neg a_\pi)$
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- $\forall \pi. \exists \pi'. F(a_\pi \land X a_{\pi'})$

\[
\begin{array}{cccccccccc}
\{a\} & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \cdots \\
\emptyset & \{a\} & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \cdots \\
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\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
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Every satisfiable HyperLTL sentence has a countable model.
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**Theorem**

*Every satisfiable HyperLTL sentence has a countable model.*

**Proof**

- W.l.o.g. \( \varphi = \forall \pi_0. \exists \pi'_0. \cdots \forall \pi_k. \exists \pi'_k. \psi \) with quantifier-free \( \psi \).
- Fix a Skolem function \( f_j \) for every existentially quantified \( \pi'_j \).
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\[
\begin{align*}
\text{f}_0(t) & \quad \text{f}_1(t, t) & \quad \cdots & \quad \text{f}_k(t, \ldots, t)
\end{align*}
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The limit is a model of \( \varphi \) and countable.
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What about Regular Models?

**Theorem**

There is a satisfiable HyperLTL sentence that is not satisfied by any $\omega$-regular set of traces.
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**Theorem**
*There is a satisfiable HyperLTL sentence that is not satisfied by any $\omega$-regular set of traces.*

**Proof**

Express that a model $T$ contains...

1. $\ldots (\{a\}\{b\})^n\emptyset^\omega$ for every $n$. 
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There is a satisfiable HyperLTL sentence that is not satisfied by any $\omega$-regular set of traces.

**Proof**

Express that a model $T$ contains...

1. $\emptyset^\omega$ for every $n$. 

\[
\{a\} \{b\} \{a\} \{b\} \{a\} \{b\} \emptyset^\omega
\]
What about Regular Models?

**Theorem**
There is a satisfiable HyperLTL sentence that is not satisfied by any $\omega$-regular set of traces.

**Proof**
Express that a model $T$ contains...

1. $(\{a\} \{b\})^n \emptyset^\omega$ for every $n$.
2. for every trace of the form $x \{b\} \{a\} y$ in $T$, also the trace $x \{a\} \{b\} y$. 
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\[ \{a\} \{b\} \{a\} \{b\} \{a\} \{b\} \emptyset^\omega \]
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\[
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\{a\} & \{b\} \{a\} \{b\} \{a\} \{b\} \emptyset^\omega \\
\{a\} & \{a\} \{b\} \{b\} \{a\} \{b\} \emptyset^\omega \\
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**Proof**

Express that a model \( T \) contains..

1. .. \( (\{a\}\{}b\})^n \emptyset^\omega \) for every \( n \).

2. .. for every trace of the form \( x\{b\}\{}a\}y \) in \( T \), also the trace \( x\{a\}\{}b\}y \).

Then, \( T \cap \{a\}^*\{b\}^* \emptyset^\omega = \{\{a\}^n\{b\}^n \emptyset^\omega | n \in \mathbb{N}\} \) is not \( \omega \)-regular.
What about Ultimately Periodic Models?

**Theorem**

There is a satisfiable HyperLTL sentence that is not satisfied by any set of traces that contains an ultimately periodic trace.
What about Ultimately Periodic Models?

**Theorem**

There is a satisfiable HyperLTL sentence that is not satisfied by any set of traces that contains an ultimately periodic trace.

One can even encode the prime numbers in HyperLTL!
First-order Logic for Hyperproperties
First-order Logic vs. LTL

FO[<]: first-order order logic over signature \{<\} \cup \{P_a \mid a \in AP\} over structures with universe \(\mathbb{N}\).

**Theorem (Kamp ’68, Gabbay et al. ’80)**

*LTL and FO[<] are expressively equivalent.*
FO[<]: first-order order logic over signature \(\{<\} \cup \{P_a \mid a \in AP\}\) over structures with universe \(\mathbb{N}\).

**Theorem (Kamp ’68, Gabbay et al. ’80)**

\(LTL\) and \(FO[<]\) are expressively equivalent.

**Example**

\[\forall x (P_q(x) \land \neg P_p(x)) \rightarrow \exists y (x < y \land P_p(y))\]

and

\[G(q \rightarrow Fp)\]

are equivalent.
First-order Logic for Hyperproperties

\[ \mathbb{N} \]

\[ \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \]
First-order Logic for Hyperproperties

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First-order Logic for Hyperproperties

\[
T \left\{ \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\vdots
\end{array} \right\} \quad E \quad \langle \quad \mathbb{N}
\]

\[
\bigcup \{ \mathcal{P}_a | a \in \mathcal{AP} \}
\]
First-order Logic for Hyperproperties

- \( \text{FO}[<, E] \): first-order logic with equality over the signature \( \{<, E\} \cup \{P_a \mid a \in \text{AP}\} \) over structures with universe \( T \times \mathbb{N} \).

**Example**

\[
\forall x \forall x' \ E(x, x') \rightarrow (P_{on}(x) \Leftrightarrow P_{on}(x'))
\]
First-order Logic for Hyperproperties

- **FO[<, E]**: first-order logic with equality over the signature \{<, E\} \cup \{P_a \mid a \in AP\} over structures with universe \(T \times \mathbb{N}\).

**Proposition**

For every HyperLTL sentence there is an equivalent FO[<, E] sentence.
Let $\varphi$ be the following property of sets $T \subseteq (2\{p\})^\omega$:

There is an $n$ such that $p \notin t(n)$ for every $t \in T$.

**Theorem (Bozzelli et al. ’15)**

$\varphi$ is not expressible in HyperLTL.
Let $\varphi$ be the following property of sets $T \subseteq (2^{\{p\}})^{\omega}$:

There is an $n$ such that $p \notin t(n)$ for every $t \in T$.

**Theorem (Bozzelli et al. ’15)**

$\varphi$ is not expressible in HyperLTL.

But, $\varphi$ is easily expressible in $\text{FO}[<, E]$:

$$\exists x \forall y \ E(x, y) \rightarrow \neg P_p(y)$$

**Corollary**

$\text{FO}[<, E]$ strictly subsumes HyperLTL.
HyperFO

- $\exists^M x$ and $\forall^M x$: quantifiers restricted to initial positions.
- $\exists^G y \geq x$ and $\forall^G y \geq x$: if $x$ is initial, then quantifiers restricted to positions on the same trace as $x$. 
HyperFO

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HyperFO: sentences of the form

$$\varphi = Q_1^M x_1 \cdots Q_k^M x_k \cdot Q_1^G y_1 \geq x_{g_1} \cdots Q_\ell^G y_\ell \geq x_{g_\ell} \cdot \psi$$

- $Q \in \{\exists, \forall\}$,
- $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_\ell\}$ are disjoint,
- every guard $x_{g_j}$ is in $\{x_1, \ldots, x_k\}$, and
- $\psi$ is quantifier-free over signature $\{<, E\} \cup \{P_a \mid a \in AP\}$ with free variables in $\{y_1, \ldots, y_\ell\}$. 
Equivalence

Theorem

HyperLTL and HyperFO are equally expressive.
Equivalence

Theorem

HyperLTL and HyperFO are equally expressive.

Proof

- From HyperLTL to HyperFO: structural induction.
- From HyperFO to HyperLTL: reduction to Kamp’s theorem.
∀x∀x' E(x, x') → (P_{on}(x) ↔ P_{on}(x'))
∀x∀x' E(x, x') → (P_{on}(x) ↔ P_{on}(x'))

∀^M x_1 ∀^M x_2 ∀^G y_1 ≥ x_1 ∀^G y_2 ≥ x_2 E(y_1, y_2) → (P_{on}(y_1) ↔ P_{on}(y_2))
∀x∀x' E(x, x') → (P_{on}(x) ↔ P_{on}(x'))

∀^M x_1 ∃^M x_2 ∀^G y_1 ≥ x_1 ∀^G y_2 ≥ x_2 E(y_1, y_2) → (P_{on}(y_1) ↔ P_{on}(y_2))
\[
\forall x \forall x' \quad E(x, x') \rightarrow (P_{on}(x) \leftrightarrow P_{on}(x'))
\]

\[
\forall^G y_1 \geq x_1 \forall^G y_2 \geq x_2 E(y_1, y_2) \rightarrow (P_{on}(y_1) \leftrightarrow P_{on}(y_2))
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∀x∀x' \ E(x, x') \rightarrow (P_{on}(x) \leftrightarrow P_{on}(x'))

∀^G y_1 \geq x_1 \forall^G y_2 \geq x_2 \ E(y_1, y_2) \rightarrow (P_{on}(y_1) \leftrightarrow P_{on}(y_2))

∀y_1 \forall y_2 \ (y_1 = y_2) \rightarrow (P_{(on, 1)}(y_1) \leftrightarrow P_{(on, 2)}(y_2))

{(on, 1), (on, 2)}
{(on, 1)}
\emptyset
{(on, 1), (on, 2)}
\ldots
From HyperFO to HyperLTL

\[ \forall x \forall x' \quad E(x, x') \rightarrow (P_{on}(x) \leftrightarrow P_{on}(x')) \]

\[ \forall^G y_1 \geq x_1 \forall^G y_2 \geq x_2 E(y_1, y_2) \rightarrow (P_{on}(y_1) \leftrightarrow P_{on}(y_2)) \]

\[ \forall y_1 \forall y_2 \quad (y_1 = y_2) \rightarrow (P_{(on, 1)}(y_1) \leftrightarrow P_{(on, 2)}(y_2)) \]

\[ G ((on, 1) \leftrightarrow (on, 2)) \]

\[
\begin{align*}
\{(on, 1), (on, 2)\} & \quad \{(on, 1)\} & \quad \emptyset & \quad \{(on, 1), (on, 2)\} & \ldots
\end{align*}
\]
∀x∀x' \ E(x, x') \rightarrow (P_{on}(x) \leftrightarrow P_{on}(x'))

∀^M_{x_1} \forall^M_{x_2} \forall^G_{y_1} \geq x_1 \forall^G_{y_2} \geq x_2 \ E(y_1, y_2) \rightarrow (P_{on}(y_1) \leftrightarrow P_{on}(y_2))

∀y_1 \forall y_2 (y_1 = y_2) \rightarrow (P_{(on, 1)}(y_1) \leftrightarrow P_{(on, 2)}(y_2))

\begin{align*}
\{ (on, 1), & (on, 2) \} \quad \{ (on, 1) \} \quad \emptyset \quad \{ (on, 1), & (on, 2) \} \quad \ldots
\end{align*}
∀x∀x'  \( E(x, x') \rightarrow (P_{\text{on}}(x) \leftrightarrow P_{\text{on}}(x')) \)

∀\(M_{x_1}\)∀\(M_{x_2}\)  \( \forall^G y_1 \geq x_1 \forall^G y_2 \geq x_2 E(y_1, y_2) \rightarrow (P_{\text{on}}(y_1) \leftrightarrow P_{\text{on}}(y_2)) \)

∀y_1 ∀y_2 \( (y_1 = y_2) \rightarrow (P_{(\text{on}, 1)}(y_1) \leftrightarrow P_{(\text{on}, 2)}(y_2)) \)

\( \mathbf{G} ((\text{on}, 1) \leftrightarrow (\text{on}, 2)) \)

∀\(\pi_1\)∀\(\pi_2\)  \( \mathbf{G} (\text{on}_{\pi_1} \leftrightarrow \text{on}_{\pi_2}) \)

\( \pi_1 \mapsto \{ \text{on} \} \quad \{ \text{on} \} \quad \emptyset \quad \{ \text{on} \} \quad \cdots \)

\( \pi_2 \mapsto \{ \text{on} \} \quad \emptyset \quad \emptyset \quad \{ \text{on} \} \quad \cdots \)
Conclusion

Our Results

- The models of HyperLTL are rather **not well-behaved**, i.e., in general (countably) infinite, non-regular, and non-periodic.
- FO[$\leq$, $E$] is strictly **more expressive** than HyperLTL.
- HyperFO is **expressively equivalent** to HyperLTL.
Conclusion

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- The models of HyperLTL are rather not well-behaved, i.e., in general (countably) infinite, non-regular, and non-periodic.
- FO[$\lt$, $E$] is strictly more expressive than HyperLTL.
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Open Problems

- Is there a class of languages $L$ such that every satisfiable HyperLTL sentence has a model from $L$?
- Is there a temporal logic that is expressively equivalent to FO[$\lt$, $E$]?
- What about HyperCTL$^*$?