The First-Order Logic of Hyperproperties

Joint work with Bernd Finkbeiner (Saarland University)

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The system $S$ is input-deterministic: for all traces $t, t'$ of $S$, $t = I t'$ implies $t = O t'$.

Noninterference: for all traces $t, t'$ of $S$, $t = I_{\text{public}} t'$ implies $t = O_{\text{public}} t'$.
The system $S$ is input-deterministic: for all traces $t, t'$ of $S$

$t =_I t' \implies t =_O t'$
The system $S$ is input-deterministic: for all traces $t, t'$ of $S$

$$t =_I t' \quad \text{implies} \quad t =_O t'$$

Noninterference: for all traces $t, t'$ of $S$

$$t =_{\text{public}} t' \quad \text{implies} \quad t =_{\text{public}} t'$$
Hyperproperties

- Both properties are not trace properties, but hyperproperties, i.e., sets of sets of traces.
- A system $S$ satisfies a hyperproperty $H$, if $\text{Traces}(S) \in H$.
- Many information flow properties can be expressed as hyperproperties.
Hyperproperties

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- A system $S$ satisfies a hyperproperty $H$, if $\text{Traces}(S) \in H$.
- Many information flow properties can be expressed as hyperproperties.

Specification languages for hyperproperties [Clarkson et al. ’14]

**HyperLTL:** Extend LTL by trace quantifiers.

**HyperCTL*:** Extend CTL* by trace quantifiers.
HyperLTL

HyperLTL = LTL +

\[ \psi ::= a \mid \neg \psi \mid \psi \lor \psi \mid X \psi \mid \psi U \psi \]

where \( a \in AP \) (atomic propositions)
HyperLTL

HyperLTL = LTL + trace quantification

\[ \varphi ::= \exists \pi. \varphi \mid \forall \pi. \varphi \mid \psi \]

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where \( a \in AP \) (atomic propositions) and \( \pi \in V \) (trace variables).
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where \( a \in AP \) (atomic propositions) and \( \pi \in V \) (trace variables).

Shortcuts as usual:

- \( F \psi = \text{true} U \psi \)
- \( G \psi = \neg F \neg \psi \)
Semantics

\[ \varphi = \forall \pi. \forall \pi'. G(\text{on}_\pi \leftrightarrow \text{on}_{\pi'}) \]

\( T \subseteq (2^{AP})^\omega \) is a model of \( \varphi \) iff
Semantics

$$\varphi = \forall \pi. \forall \pi'. G(\text{on}_\pi \leftrightarrow \text{on}_{\pi'})$$

$T \subseteq (2^{AP})^\omega$ is a model of $\varphi$ iff

$$\{ \} \models \forall \pi. \forall \pi'. G(\text{on}_\pi \leftrightarrow \text{on}_{\pi'})$$
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\[
\{ \} \models \forall \pi. \forall \pi'. G (\text{on}_\pi \leftrightarrow \text{on}_\pi')
\]

\[
\{ \pi \mapsto t \} \models \forall \pi'. G (\text{on}_\pi \leftrightarrow \text{on}_\pi') \quad \text{for all } t \in T
\]
Semantics

\[ \varphi = \forall \pi. \forall \pi'. \mathbf{G} (\text{on}_\pi \leftrightarrow \text{on}_{\pi'}) \]

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\[ \{ \pi \mapsto t, \pi' \mapsto t' \} \models \mathbf{G} (\text{on}_\pi \leftrightarrow \text{on}_{\pi'}) \quad \text{for all } t' \in T \]
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\[ \phi = \forall \pi. \forall \pi'. \mathbf{G}(\text{on}_\pi \leftrightarrow \text{on}_\pi') \]

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\[
\{ \pi \mapsto t[n, \infty), \pi' \mapsto t'[n, \infty) \} \models \text{on}_\pi \leftrightarrow \text{on}_{\pi'} \quad \text{for all } n \in \mathbb{N}
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Semantics

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\[ \text{on} \in t(n) \iff \text{on} \in t'(n) \]
LTL vs. HyperLTL

LTL has many desirable properties.

1. Every satisfiable LTL formula is satisfied by an ultimately periodic trace, i.e., by a finite and finitely-represented model.
2. LTL and FO[<] are expressively equivalent.
3. LTL satisfiability and model-checking are $\text{PSPACE}$-complete.

Only partial results for HyperLTL.

3a. HyperLTL satisfiability [F. & Hahn '16]:
   - alternation-free: $\text{PSpace}$-complete
   - $\exists^* \forall^*$: $\text{ExpSpace}$-complete
   - $\forall^* \exists^*$: undecidable

3b. HyperLTL model-checking is decidable [F. et al. '15].
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3a. HyperLTL satisfiability [F. & Hahn ’16]:
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   - \( \forall^* \exists^* \): undecidable

3b. HyperLTL model-checking is decidable [F. et al. ’15].
The Models of HyperLTL
What about Finite Models?

Fix $\text{AP} = \{a\}$ and consider the conjunction $\varphi$ of

$$\forall \pi. \ (\neg a_\pi) \ \text{U} \ (a_\pi \land X \ G \neg a_\pi)$$
What about Finite Models?

Fix $AP = \{a\}$ and consider the conjunction $\varphi$ of

- $\forall \pi. (\neg a_\pi) \mathbf{U} (a_\pi \land X G \neg a_\pi)$
- $\exists \pi. a_\pi$

The unique model of $\varphi$ is $\{\emptyset, \{a_n\} \mid n \in \mathbb{N}\}$. Theorem: There is a satisfiable HyperLTL sentence that is not satisfied by any finite set of traces.
What about Finite Models?

Fix $\text{AP} = \{a\}$ and consider the conjunction $\varphi$ of

- $\forall \pi. (\neg a_\pi) \bigcup (a_\pi \land X G \neg a_\pi)$
- $\exists \pi. a_\pi$

$$\{a\} \emptyset \emptyset \emptyset \emptyset \emptyset \emptyset \emptyset \emptyset \emptyset \ldots$$
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- $\forall \pi. (\neg a_\pi) \mathbf{U} (a_\pi \land \mathbf{X} \mathbf{G} \neg a_\pi)$
- $\exists \pi. a_\pi$
- $\forall \pi. \exists \pi'. \mathbf{F} (a_\pi \land \mathbf{X} a_{\pi'})$

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\[
\begin{array}{cccccccccc}
\{a\} & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \ldots \\
\emptyset & \{a\} & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \ldots \\
\end{array}
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\emptyset & \emptyset & \{a\} & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \cdots \\
\emptyset & \emptyset & \emptyset & \{a\} & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\end{array}
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The unique model of $\varphi$ is $\{\emptyset^n \{a\} \emptyset^\omega \mid n \in \mathbb{N}\}$.

**Theorem**

*There is a satisfiable HyperLTL sentence that is not satisfied by any finite set of traces.*
Theorem

Every satisfiable HyperLTL sentence has a countable model.
What about Countable Models?

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Proof

- W.l.o.g. $\varphi = \forall \pi_0. \exists \pi'_0. \cdots \forall \pi_k. \exists \pi'_k. \psi$ with quantifier-free $\psi$.
- Fix a Skolem function $f_j$ for every existentially quantified $\pi'_j$. 
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- Fix a Skolem function $f_j$ for every existentially quantified $\pi'_j$. 

$$f_0(t) \quad f_1(t, t) \quad \cdots \quad f_k(t, \ldots, t)$$
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The limit is a model of $\varphi$ and countable.
What about Regular Models?

Theorem

There is a satisfiable HyperLTL sentence that is not satisfied by any $\omega$-regular set of traces.
What about Regular Models?

Theorem
There is a satisfiable HyperLTL sentence that is not satisfied by any \( \omega \)-regular set of traces.

Proof

Express that a model \( T \) contains..

1. \( (\{a\}\{b\})^n \emptyset^\omega \) for every \( n \).
What about Regular Models?

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*There is a satisfiable HyperLTL sentence that is not satisfied by any $\omega$-regular set of traces.*

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Express that a model $T$ contains

1. $\ldots (\{a\}\{b\})^n \emptyset^\omega$ for every $n$. 

$\{a\} \{b\} \{a\} \{b\} \{a\} \{b\} \emptyset^\omega$

Then, $T \cap \{a\}^* \{b\}^* \emptyset^\omega = \{\{a\} \{b\} \emptyset^\omega | n \in \mathbb{N}\}$ is not $\omega$-regular.
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Proof

Express that a model $T$ contains...

1. $(\{a\}\{b\})^n \emptyset^\omega$ for every $n$.
2. for every trace of the form $x\{b\}\{a\}y$ in $T$, also the trace $x\{a\}\{b\}y$. 

Martin Zimmermann  Saarland University  The First-Order Logic of Hyperproperties
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$$\emptyset^\omega$$
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\[
\begin{array}{c}
\{a\} \quad \{b\} \\
{a} \quad {a} \quad {b} \\
{x\{b\}\{a\}y} \\
{a} \quad {a} \quad {b} \\
{x\{a\}\{b\}y} \\
{a} \quad {a} \quad {a} \quad {b} \\
\{b\} \quad \{b\} \quad \emptyset^\omega \\
\end{array}
\]
Theorem

There is a satisfiable HyperLTL sentence that is not satisfied by any \( \omega \)-regular set of traces.

Proof

Express that a model \( T \) contains:

1. \( (\{a\}\{b\})^n\emptyset^\omega \) for every \( n \).
2. for every trace of the form \( x\{b\}\{a\}y \) in \( T \), also the trace \( x\{a\}\{b\}y \).

Then, \( T \cap \{a\}^*\{b\}^*\emptyset^\omega = \{\{a\}^n\{b\}^n\emptyset^\omega \mid n \in \mathbb{N}\} \) is not \( \omega \)-regular.
What about Ultimately Periodic Models?

**Theorem**

There is a satisfiable HyperLTL sentence that is not satisfied by any set of traces that contains an ultimately periodic trace.

One can even encode the prime numbers in HyperLTL!
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One can even encode the prime numbers in HyperLTL!
First-order Logic for Hyperproperties
First-order Logic vs. LTL

FO[<]: first-order order logic over signature \{<\} \cup \{P_a \mid a \in AP\} over structures with universe \(\mathbb{N}\).

**Theorem (Kamp ’68, Gabbay et al. ’80)**

*LTL and FO[<] are expressively equivalent.*
First-order Logic vs. LTL

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*LTL and FO[<] are expressively equivalent.*

Example

\[\forall x (P_q(x) \land \neg P_p(x)) \rightarrow \exists y (x < y \land P_p(y))\]

and

\[G(q \rightarrow Fp)\]

are equivalent.
First-order Logic for Hyperproperties

TE[[<,E]: first-order logic with equality over the signature 
\{<,E\} \cup \{P_a | a \in AP\} over structures with universe T \times N.

\[ \mathbb{N} \]

\[ \vdash \]
First-order Logic for Hyperproperties

\[ \text{TE}: \text{first-order logic with equality over the signature } \{<, E\} \cup \{P | a \in AP\} \text{ over structures with universe } T \times \mathbb{N}. \]
First-order Logic for Hyperproperties

\( T \): first-order logic with equality over the signature \( \{<, E\} \cup \{p_a | a \in AP\} \) over structures with universe \( T \times \mathbb{N} \).
First-order Logic for Hyperproperties

- FO[<, E]: first-order logic with equality over the signature \( \{<, E\} \cup \{P_a \mid a \in \text{AP}\} \) over structures with universe \( T \times \mathbb{N} \).

**Example**

\[
\forall x \forall x' \ E(x, x') \rightarrow (P_{on}(x) \leftrightarrow P_{on}(x'))
\]
**First-order Logic for Hyperproperties**

- **Proposition**

  For every HyperLTL sentence there is an equivalent FO[$<, E]$ sentence.

- **FO[$<, E$]:** first-order logic with equality over the signature \{[$<, E$]$\cup \{P_a \mid a \in AP\}$} over structures with universe $T \times \mathbb{N}$.
Let $\varphi$ be the following property of sets $T \subseteq (2^\{p\})^\omega$:

There is an $n$ such that $p \notin t(n)$ for every $t \in T$.

**Theorem (Bozzelli et al. ’15)**

$\varphi$ is not expressible in HyperLTL.
A Setback

Let $\varphi$ be the following property of sets $T \subseteq (2^\{p\})^\omega$:

There is an $n$ such that $p \notin t(n)$ for every $t \in T$.

Theorem (Bozzelli et al. '15)

$\varphi$ is not expressible in HyperLTL.

But, $\varphi$ is easily expressible in $\text{FO}[<, E]$:

$\exists x \forall y \ E(x, y) \rightarrow \neg p$

Corollary

$\text{FO}[<, E]$ strictly subsumes HyperLTL.
HyperFO

- $\exists^M x$ and $\forall^M x$: quantifiers restricted to initial positions.
- $\exists^G y \geq x$ and $\forall^G y \geq x$: if $x$ is initial, then quantifiers restricted to positions on the same trace as $x$. 
HyperFO

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**HyperFO**: sentences of the form

$$\varphi = Q_1^M x_1 \cdots Q_k^M x_k \cdot Q_1^G y_1 \geq x_{g_1} \cdots Q_{\ell}^G y_{\ell} \geq x_{g_{\ell}} \cdot \psi$$

- $Q \in \{\exists, \forall\}$,
- $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_{\ell}\}$ are disjoint,
- every guard $x_{g_j}$ is in $\{x_1, \ldots, x_k\}$, and
- $\psi$ is quantifier-free over signature $\{<, E\} \cup \{P_a \mid a \in AP\}$ with free variables in $\{y_1, \ldots, y_{\ell}\}$.
Equivalence

Theorem

*HyperLTL and HyperFO are equally expressive.*
Equivalence

Theorem
HyperLTL and HyperFO are equally expressive.

Proof
- From HyperLTL to HyperFO: structural induction.
- From HyperFO to HyperLTL: reduction to Kamp’s theorem.
∀x∀x' E(x, x') → (P_{on}(x) ↔ P_{on}(x'))
∀x∀x' E(x, x') → (P_{on}(x) ↔ P_{on}(x'))

∀^Mx_1 ∀^Mx_2 ∀^Gy_1 ≥ x_1 ∀^Gy_2 ≥ x_2 E(y_1, y_2) → (P_{on}(y_1) ↔ P_{on}(y_2))
∀\(x\forall x'\ E(x, x') \rightarrow (P_{\text{on}}(x) \leftrightarrow P_{\text{on}}(x'))\)

∀\(Mx_1 \forall^M x_2 \forall^G y_1 \geq x_1 \forall^G y_2 \geq x_2 E(y_1, y_2) \rightarrow (P_{\text{on}}(y_1) \leftrightarrow P_{\text{on}}(y_2))\)
\( \forall x \forall x' \ E(x, x') \rightarrow (P_{on}(x) \leftrightarrow P_{on}(x')) \)

\( \forall^G y_1 \geq x_1 \forall^G y_2 \geq x_2 E(y_1, y_2) \rightarrow (P_{on}(y_1) \leftrightarrow P_{on}(y_2)) \)
∀x∀x' \quad E(x, x') \rightarrow (P_{on}(x) \leftrightarrow P_{on}(x'))

∀Gy_1 \geq x_1 \forall^Gy_2 \geq x_2 E(y_1, y_2) \rightarrow (P_{on}(y_1) \leftrightarrow P_{on}(y_2))

∀y_1 \forall y_2 (y_1 = y_2) \rightarrow (P_{(on,1)}(y_1) \leftrightarrow P_{(on,2)}(y_2))
∀x∀x' \ E(x, x') \rightarrow (P_{on}(x) \leftrightarrow P_{on}(x'))

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G ((on, 1) \leftrightarrow (on, 2))
\[ \forall x \forall x' \quad E(x, x') \rightarrow (P_{\text{on}}(x) \leftrightarrow P_{\text{on}}(x')) \]

\[ \forall^M x_1 \forall^M x_2 \quad \forall^G y_1 \geq x_1 \forall^G y_2 \geq x_2 E(y_1, y_2) \rightarrow (P_{\text{on}}(y_1) \leftrightarrow P_{\text{on}}(y_2)) \]

\[ \forall y_1 \forall y_2 \quad (y_1 = y_2) \rightarrow (P_{(\text{on}, 1)}(y_1) \leftrightarrow P_{(\text{on}, 2)}(y_2)) \]

\[ G ((\text{on}, 1) \leftrightarrow (\text{on}, 2)) \]

\[
\begin{align*}
\{(\text{on}, 1), & \quad (\text{on}, 2)\} \\
\{(\text{on}, 1)\} & \quad \emptyset \\
\{(\text{on}, 1), & \quad (\text{on}, 2)\} & \quad \ldots
\end{align*}
\]
From HyperFo to HyperLTL

∀x∀x'  E(x, x') \rightarrow (P_{on}(x) \leftrightarrow P_{on}(x'))

\forall^M x_1 \forall^M x_2 \forall^G y_1 \geq x_1 \forall^G y_2 \geq x_2 E(y_1, y_2) \rightarrow (P_{on}(y_1) \leftrightarrow P_{on}(y_2))

∀y_1 ∀y_2 (y_1 = y_2) \rightarrow (P_{(on, 1)}(y_1) \leftrightarrow P_{(on, 2)}(y_2))

G ((on, 1) \leftrightarrow (on, 2))

\forall \pi_1 \forall \pi_2  G (on_{\pi_1} \leftrightarrow on_{\pi_2})

\pi_1 \mapsto \{on\} \{on\} \emptyset \{on\} \ldots

\pi_2 \mapsto \{on\} \emptyset \emptyset \{on\} \ldots
Conclusion

Our Results

- The models of HyperLTL are rather not well-behaved, i.e., in general (countably) infinite, non-regular, and non-periodic.
- FO[$\prec$, $E$] is strictly more expressive than HyperLTL.
- HyperFO is expressively equivalent to HyperLTL.
Conclusion

Our Results

- The models of HyperLTL are rather not well-behaved, i.e., in general (countably) infinite, non-regular, and non-periodic.
- FO[<, E] is strictly more expressive than HyperLTL.
- HyperFO is expressively equivalent to HyperLTL.

Open Problems

- Is there a class of languages \( \mathcal{L} \) such that every satisfiable HyperLTL sentence has a model from \( \mathcal{L} \)?
- Is there a temporal logic that is expressively equivalent to FO[<, E]?
- What about HyperCTL*?